All minimal prime extensions of hereditary classes of graphs

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Abstract

The substitution composition of two disjoint graphs $G_1$ and $G_2$ is obtained by first removing a vertex $x$ from $G_2$ and then making every vertex in $G_1$ adjacent to all neighbours of $x$ in $G_2$. Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations. Giakoumakis [V. Giakoumakis, On the closure of graphs under substitution, Discrete Mathematics 177 (1997) 83–97] proved that $\mathcal{F}^*$, the closure under substitution of $\mathcal{F}$, can be characterized by a set $\mathcal{Z}^*$ of forbidden configurations — the minimal prime extensions of $\mathcal{Z}$. He also showed that $\mathcal{Z}^*$ is not necessarily a finite set. Since substitution preserves many of the properties of the composed graphs, an important problem is the following: find necessary and sufficient conditions for the finiteness of $\mathcal{Z}^*$. Giakoumakis [V. Giakoumakis, On the closure of graphs under substitution, Discrete Mathematics 177 (1997) 83–97] presented a sufficient condition for the finiteness of $\mathcal{Z}^*$ and a simple method for enumerating all its elements. Since then, many other researchers have studied various classes of graphs for which the substitution closure can be characterized by a finite set of forbidden configurations.

The main contribution of this paper is to completely solve the above problem by characterizing all classes of graphs having a finite number of minimal prime extensions. We then go on to point out a simple way for generating an infinite number of minimal prime extensions for all the other classes of $\mathcal{F}^*$.

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1. Motivation

The substitution composition of graphs has been widely used by researchers in the study of both theoretical as well as practical problems; we refer the interested reader to Brandstäd et al. [4] for a comprehensive discussion. The appeal of the substitution composition is, most certainly, due to the fact that it preserves many of the properties of the composed graphs. For example, Lovász [12] relied on the well known fact that substitution preserves perfection\footnote{For the definition of perfect graphs see Berge [2] or Brandstäd et al. [4].} in order to prove that a graph is perfect if and only if its complement is. We recall that the famous Strong Perfect Graph Conjecture (SPGC, for short) introduced by Berge [1] in 1961 was recently answered in the affirmative becoming the
Strong Perfect Graph Theorem. We refer the reader to Chvátal’s web page [8] for a detailed survey and to Chudnovsky et al. [7] for the proof of the Strong Perfect Graph Theorem.

Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations and let $\mathcal{F}^*$ be the closure of $\mathcal{F}$ under substitution.

**Problem 1.** Find a forbidden induced subgraph characterization of $\mathcal{F}^*$.

Giakoumakis [10] proved that:
1. $\mathcal{F}^*$ can be characterized by a set $\mathcal{Z}^*$ of forbidden configurations;
2. $\mathcal{Z}^*$ is not necessarily a finite set;
3. If no graph $Z$ of $\mathcal{Z}$ contains a module of more than two vertices then $\mathcal{Z}^*$ is finite.

**Problem 2.** Find necessary and sufficient conditions for $\mathcal{Z}^*$ to be finite.

Various researchers investigated Problem 2 and many sufficient conditions have been presented [5,6,10,11,13,16,17]. These, and other similar papers, give forbidden subgraph characterizations of the closure under substitution of various classes of graphs. It is worth noting that such characterizations are very likely to lead to efficient graph optimization algorithms. Indeed, for optimization problems including finding the weighted stability number (see [3] and [9]) and the domination problem (see [15]), efficient solutions can be found when dealing with hereditary classes of graphs.

The main contribution of this paper is to offer a complete answer to Problem 2 by characterizing all classes of graphs defined by a finite set of forbidden configurations, whose closure under substitution can also be defined by a finite number of forbidden subgraphs. For all the other classes of $\mathcal{F}^*$ we give a simple way for generating an infinite number of minimal prime extensions.

2. Notation and previous results

The main goal of this section is to establish notation and terminology and to review a number of known results that will be needed in the subsequent sections of the paper.

2.1. Notation and terminology

For terms not defined here the reader is referred to [2] and [4]. All the graphs in this work are finite, with no loops nor multiple edges. Given a graph $G = (V, E)$, the set $V$ of its vertices will also be denoted by $V(G)$; similarly, the set $E$ of its edges will be denoted by $E(G)$. We also write $n = |V|$ and $m = |E|$ to denote the cardinality of $V$ and $E$.

Let $x$ be a vertex of graph $G$. The neighborhood of $x$ will be denoted by $N(x)$; we let degree$(x)$ stand for $|N(x)|$.

The subgraph of $G$ induced by $V(G) - \{x\}$ will be denoted by $G \setminus x$.

If $N(x) = V(G) - \{x\}$, $x$ is said to be a universal vertex of $G$. The graph induced by a set $X \subseteq V$ will be denoted by $[X]$; $[X]$ is a proper induced subgraph of $G$ if $X$ is strictly contained in $V(G)$. We shall let $N_G(X)$ stand for the set of vertices in $V(G) - X$ adjacent to at least one vertex of $X$.

The set $X$ is said to be stable (resp. complete) if the graph $[X]$ is edgeless (resp. fully connected). A stable (resp. complete) set of $r$ vertices is denoted by $S_r$ (resp. $K_r$). The edgeless graph of $r$ vertices will be denoted by $O_r$. The graph induced by $V(G) - X$ is also written as $G \setminus X$ and the graph induced by $V(G) - \{x\}$ where $x$ a vertex of $G$, will be written as $G \setminus x$. A vertex $x$ is total, indifferent or partial with respect to $X$ if it is, respectively, adjacent to all, to none or to some but not all of the vertices of $X$. A set of vertices $Y$ is total (or universal) with respect to $X$ if every vertex of $Y$ is adjacent to all the vertices of $X$; $Y$ is indifferent with respect to $X$ if no vertex of $Y$ is adjacent to a vertex of $X$; finally, $Y$ is partial with respect to $X$ if at least one of the vertices of $Y$ is partial with respect to $X$.

We shall write $P_k$ (resp. $C_k$) to denote a chordless path (resp. cycle) on $k$ vertices. The complementary graph of a chordless path is referred to as a co-path. A $2K_2$ is the complementary graph of a $C_4$. When no confusion is possible, we shall use the notation $P_k$ to design also the set of vertices of the chordless chain $P_k$.

The notation $G_1 \sim G_2$ signifies that the graph $G_1$ is isomorphic to the graph $G_2$.

Let $\mathcal{Z}$ be a set of graphs. A graph $G$ is said to be $\mathcal{Z}$-free if $G$ contains no induced subgraph isomorphic to a graph of $\mathcal{Z}$. A set of graphs $\mathcal{F}$ is $\mathcal{Z}$-free if every graph of $\mathcal{F}$ is $\mathcal{Z}$-free.

A set $M \subseteq V(G)$ is called a module if every vertex of $G$ outside $M$ is adjacent to all vertices of $M$ or to none of them. The empty set, $V(G)$ and the singletons are trivial modules. A graph $G$ that contains only trivial modules
A graph that contains a non-trivial module is said to be non-trivial or a homogeneous set. A graph that contains a non-trivial module is said to be substitution-decomposable or, simply, decomposable.

Let $M$ be a module of a graph $G$. $M$ is said to be a strong module if for every non-trivial module $M'$ of $G$ either $M \cap M' = \emptyset$ or one of $M$ and $M'$ is included in the other. The decomposition of a graph into its modules was discovered independently by researchers in many seemingly unrelated areas. We refer the reader to Brandstädt et al. [4] for a comprehensive discussion and further references.

The modular decomposition of a graph $G$ is a form of decomposition that associates with $G$ a unique decomposition tree $T(G)$. The set of leaves of $T(G)$ is the set $V(G)$. The set of leaves associated with a subtree of $T(G)$ rooted at a node $f$ of $T(G)$ is $\text{leaves}(f)$. It is well known that for each internal node $f$ of $T(G)$ different from its root, $\text{leaves}(f)$ forms a strong module of $G$ and that $\{\text{leaves}(f)\}$ is the set of all strong modules of $G$. An internal node $f$ is labeled $P$, $S$, or $N$ to denote respectively, parallel, series or neighbourhood modules. The subgraph induced in $G$ by a parallel module is disconnected, the one induced by a series module is connected and has a disconnected complement; and, finally, the one induced by a neighbourhood module is connected both in the graph and the complement.

Let $f_1, \ldots, f_k$ be the set of children of $f$ in $T(G)$ and let $H$ be the subgraph of $G$ whose vertex-set consists of one vertex from each module $\text{leaves}(f_i)$, $i = 1, \ldots, k$. Clearly, $H$ is an edgeless graph whenever $f$ is a $P$-node, a complete graph whenever $f$ is an $S$-node, and a prime graph whenever $f$ is an $N$-node.

Due to its vast array of practical applications the problem of finding efficient algorithms (both sequentially and parallel) for the modular decomposition and for the construction of the corresponding decomposition tree has received a great deal of attention in the recent literature. We refer the reader to the excellent web page [18] for a very informative synopsis of research in this area.

**Definition 2.1.** Let $G$ be a graph. The graph $G'$ is a minimal prime extension of $G$ if the following conditions are satisfied:

- $G'$ is prime,
- $G'$ contains an induced subgraph isomorphic to $G$, and
- $G'$ is minimal with respect to set inclusion and primality.

In other words, if $G'$ is a minimal prime extension of $G$, no proper prime induced subgraph of $G'$ contains an induced subgraph isomorphic to $G$. Observe that if $G$ itself is prime then $G'$ coincides with $G$.

**Notation 2.2.** Let $G$ be a graph and let $\text{Ext}(G)$ denote the set of minimal prime extensions of $G$.

Let $\mathcal{F}$ be a family of graphs defined by a set $\mathcal{Z}$ of forbidden configurations. Giakoumakis [10] proved the following result:

**Lemma 2.3 ([10]).** The closure $\mathcal{F}^*$ of $\mathcal{F}$ under substitution is defined by a set $\mathcal{Z}^*$ of forbidden configurations which is the union of the sets $\text{Ext}(Z)$ where $Z$ is a graph of $\mathcal{Z}$.

### 2.2. Known results

We begin by recalling two results concerning minimal prime extensions of various classes of graphs that we shall need in the following sections.

**Theorem 2.4 ([13]).** The substitution composition of $C_3$-free graphs is defined by the three forbidden configurations depicted in Fig. 1.

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\[^2\text{We warn the reader that in the sequel of this paper ‘homogeneous set’ and ‘non-trivial module’ will be regarded as synonyms and used interchangeably.}\]
Theorem 2.5 ([16]). If every nontrivial module of a graph \( G \) induces a subgraph of a \( P_4 \) then the set of all minimal prime extensions of \( G \) is finite.

Notation 2.6. A graph whose every nontrivial module induces a subgraph of a \( P_4 \) is called \( P_4 \)-homogeneous. We prefer this terminology to that of a simple graph used in [16], in order to avoid any possible confusion with the meaning of the term ‘simple’ used in other contexts in graph theory.

An interesting procedure proposed by Zverovich [14] generating prime extensions of a graph \( G \) is the Reducing Pseudopath Method. We recall its definition using the notation of [14].

Definition 2.7. Let \( G \) be an induced subgraph of a graph \( H \) and let \( W \) be a homogeneous set of \( G \). We define a reducing \( W \)-pseudopath in \( H \) as a sequence \( R = (u_1, u_2, \ldots, u_t) \), with \( t \geq 1 \), of pairwise distinct vertices of \( V(H) \setminus V(G) \) satisfying the following conditions:

1. \( u_1 \) is partial with respect to \( W \);
2. \( \forall i = 2, \ldots, t \), either \( u_i \) is adjacent to \( u_{i-1} \) and indifferent with respect to \( W \cup \{u_1, \ldots, u_{i-2}\} \) or \( u_i \) is total with respect to \( W \cup \{u_1, \ldots, u_{i-2}\} \) and non-adjacent to \( u_{i-1} \) (when \( i = 2 \), \( \{u_1, u_2, \ldots, u_{i-2}\} = \emptyset \));
3. \( \forall i = 1, \ldots, t-1 \), vertex \( u_i \) is total with respect to \( N(W) \) in \( G \) and indifferent with respect to \( V(G) - N(W) - W \) and either \( u_i \) is non-adjacent to a vertex of \( N(W) \) or \( u_i \) is adjacent to a vertex of \( V(G) - N(W) \).

We refer the reader to Fig. 2 in Section 3.2 for an illustration of a reducing \( W \)-pseudopath.

Theorem 2.8 ([14]). Let \( H \) be a prime extension of its induced subgraph \( G \) and let \( W \) be a homogeneous set of \( G \). Then there exists a reducing \( W \)-pseudopath with respect to every induced copy of \( G \) in \( H \).

The remainder of the paper is organized as follows: in Section 3 we give two methods for constructing a minimal prime extension of a graph \( G \). In Section 4 we discuss necessary and sufficient conditions for finiteness of \( \text{Ext}(G) \). Finally, Section 5 offers concluding remarks and ideas for possible extensions of the results presented in this paper.

3. Two basic constructions

The main goal of this section is to introduce two basic constructions that provide the framework for our main result given in Section 4. Both these constructions build minimal prime extensions of a decomposable graph.

3.1. Constructing the basic extension of a decomposable graph

Let \( G = (V, E) \) be a connected graph and let \( T(G) \) be the corresponding modular decomposition tree.

Notation 3.1. Let \( \pi(G) = \{H_1, \ldots, H_t\} \) be the partition of \( V \) obtained by the following equivalence relation \( R \) on \( V \): for vertices \( x \) and \( y \) in \( V \) we write \( x \sim y \) if and only if \( x \) and \( y \) have the same parent in \( T(G) \).

Throughout the remainder of this section we shall assume that \( G \) is not prime.

Remark 3.2. If \( G \) is decomposable, at least one of the \( H_i \)’s in \( \pi(G) \) is a non-trivial module of \( G \).

Notation 3.3. Let \( \rho(G) = \{M_1, \ldots, M_k\} \) be the subset of \( \pi(G) \) consisting of all the non-trivial modules in \( G \). Thus, for \( i = 1, \ldots, k \), every \( M_i \) is a non-trivial module in \( G \).

Remark 3.4. Let \( f(M_i) \) be the parent in \( T(G) \) of the vertices of \( M_i \) in \( \rho(G) \). Obviously if \( M_i \) is stable then \( f(M_i) \) is a \( P \)-node, if \( M_i \) is a complete set then \( f(M_i) \) is an \( S \)-node and if \( M_i \) induces a prime graph then \( f(M_i) \) is an \( N \)-node. Furthermore, if \( M_i = \text{leaves}(f(M_i)) \) then \( M_i \) is a strong nontrivial module of \( G \), minimal with respect to set inclusion. In particular, this case occurs whenever \( f(M_i) \) is an \( N \)-node (i.e. if \( f(M_i) \) is an \( N \)-node then \( M_i = \text{leaves}(f(M_i)) \)). It is worth noting that since \( G \) is connected, every vertex of module \( M_i \in \rho(G) \) has a neighbour in \( V - M_j \).

Let us associate with every module \( M_i \) of \( \rho(G) \) a set \( V'_i \) of new vertices (i.e. \( V'_i \cap V = \emptyset, V'_i \cap V'_j = \emptyset \), \( i, j = 1, \ldots, k, i \neq j \)) and a set \( E'_i \) of edges connecting the vertices of \( M_i \) with the vertices of \( V'_i \) in the following manner:

1. If \( M_i \) is a stable set or a complete set \( \{x_1, \ldots, x_r\} \) then \( V'_i = \{y_1, \ldots, y_{r-1}\} \) and \( E'_i \) is the set of edges \( x_jy_j, j = 1, \ldots, r-1 \).
2. If \( M_i \) induces in \( G \) a prime graph then \( V'_i \) is a singleton \( \{y\} \) and \( E'_i \) is the edge \( xy \) where \( x \) is a vertex of \( M_i \).
Let $G'$ be the graph whose vertex set is $V \cup V'$, where $V' = V'_1 \cup \cdots \cup V'_k$ and whose edge set is $E \cup E'$, where $E' = E'_1 \cup \cdots \cup E'_k$. Clearly $V'$ is a stable set in $G'$ and each vertex of this set has exactly one neighbour in $G'$, this neighbour being its own ‘private’ neighbour.

We propose to show that $G'$ is a minimal prime extension of $G$. For this purpose, however, we need the following result:

**Lemma 3.5.** Let $H$ be a connected graph, let $x \in V(H)$ be a vertex of degree 1, and let $M$ be a non-trivial module of the graph $H \setminus x$ containing the unique neighbour, say $y$, of $x$ in $H$. If $x$ is contained in a non-trivial module $Q$ of $H$ then $y$ is a universal vertex of $H$.

**Proof.** Suppose not. Since $H$ is connected and since $\text{degree}(x) = 1$, the neighbourhood of any vertex of module $Q$ outside this module must be a singleton and, thus, $N(Q) = \{y\}$. Let $Q_1$ be the set $N(y) \setminus Q$ and let $Q_2$ be the set of the remaining vertices of $H$. Then, no vertex of $Q$ can be adjacent to a vertex of $Q_1 \cup Q_2$. Since, by assumption, $Q_2 \neq \emptyset$, the connectedness of $H$ implies that $Q_1 \neq \emptyset$. Let $z$ be a vertex of $M$ different from $y$. If $z \in Q \setminus x$, every vertex of $Q_1$ would be adjacent to $z$ and if $z \in Q_1 \cup Q_2$ any vertex of $Q \setminus x$ would be adjacent to $z$, a contradiction. □

**Proposition 3.6.** The graph $G'$ is a prime graph.

**Proof.** Assume to the contrary that there exists a nontrivial module $M$ in $G'$.

**Claim 1.** $M$ contains no vertices of $V'$.

**Proof.** If $M$ contains a vertex $x$ of $V'$ then by Lemma 3.5 the unique neighbour $y$ of $x$ in $G'$ is a universal vertex of $G'$. Since every vertex of $V'$ has its own private neighbour in $G'$, it follows that $V'$ contains only the vertex $x$. Let $y$ be the private neighbour of $x$ and let $f$ be the parent of $y$ in $T(G)$. Clearly, $f$ is neither a $P$-node nor a $N$-node, for otherwise $y$ would not be universal in $G$, a contradiction. Consequently, $f$ must be a $S$-node.

Assume first that $f$ is the root of $T(G)$. If $f$ has a child that is an internal node or if $f$ has more than two children distinct from $y$ that are leaves, then $V'$ contains more than one vertex, a contradiction. It follows that $f$ contains exactly two children $y$ and $z$, implying that $G$ is prime, since it is isomorphic to a $K_2$. It follows that $V' = \emptyset$, a contradiction.

Thus, $f$ cannot be the root of $T(G)$. Since $f$ is a $S$-node, the parent of $f$ in $T(G)$ is either a $P$-node or a $S$-node, implying that $y$ is not a universal vertex of $G$, a contradiction. □

**Claim 1** guarantees that $M$ is a nontrivial module of $G$. Let $f$ be the least common ancestor in $T(G)$ of all vertices of $M$ and let $U = \{f_1, \ldots, f_t\}$ be the set of children of $f$ in $T(G)$.

**Claim 2.** If $\text{leaves}(f_j) \cap M \neq \emptyset$ then $\text{leaves}(f_i) \subset M$, $f_i \in U$.

**Proof.** Indeed, if $f_i \in U$ is a leaf then we are done and if not, $\text{leaves}(f_i)$ is a strong module of $G$ and since by definition $M$ is not entirely contained into the set $\text{leaves}(f_i)$, $M$ strictly contains $\text{leaves}(f_i)$ as claimed. □

**Claim 3.** No vertex of $V'$ is adjacent to a vertex of $M$.

**Proof.** The result follows from the fact that every vertex of $V'$ is of degree 1. □

Let $U' \subseteq U$ be the set of $f_i \in U$ such that $\text{leaves}(f_i) \subseteq M$. If $f_i \in U'$ is an internal node of $T(G)$, then by construction there must exist a vertex of $V'$ adjacent to $\text{leaves}(f_i)$, contradicting Claim 3. Hence, every $f_i \in U'$ is a leaf of $T(G)$ that is, $M$ contains only vertices whose least common ancestor $f$ in $T(G)$ is a parent of all of them and, thus, $M$ is entirely contained into a module of $\rho(G)$. Hence, there must exist a vertex of $V'$ which distinguishes the vertices of $M$, a contradiction. It follows that $G'$ is a prime graph as claimed. □

**Proposition 3.7.** Every proper subgraph $G''$ of $G'$ that contains $G$ as induced subgraph is not prime.

**Proof.** Suppose not and consider an arbitrary vertex $x$ of $V(G') \setminus V(G'')$. Clearly $x$ is a vertex of $V'$. Let $M$ be the module of $\rho(G)$ containing the unique neighbour, say $y$, of $x$ in $G$. If $M$ induces a prime graph in $G$, then $x$ is the unique vertex in $G'$ that is partial for $M$ and consequently $M$ is a module in $G''$, a contradiction.

If $M$ is a stable or a complete set in $G$ then the vertex $y$ together with the vertex of $M$ that has no neighbour in $V'$ forms a nontrivial module in $G''$, a contradiction. □
**Proposition 3.8.** If \( G' \) contains a subgraph \( H \neq G \) isomorphic to \( G \) by an isomorphism \( \sigma \), then \( V(G') - V(H) \) is a stable set whose vertices have degree 1 in \( G' \) and have a private neighbour in \( V(G) \cap V(H) \).

**Proof.** Observe first that \( V(H) \) cannot be a subset of \( V' \) since \( |V'| < |V(H)| \) and, additionally, \( V' \) is a stable set while \( H \) is connected. Thus, it must be that \( V(H) \cap V(G) \neq \emptyset \) and \( V(H) \cap V' \neq \emptyset \). Let \( G'_1 \) be the graph induced by \( V(H) \cap V(G) \). Write \( X = V(G) - V(H) \), \( Y = V(H) - V(G) \) and \( Z = V' - Y \). It is obvious that \( G'_1 \) is not the empty graph, \( X, Y \neq \emptyset \), \( X \cap Y = \emptyset \) and \( Y \subseteq V' \). Let \( x \) be a vertex of \( Y \) and \( y \) its private neighbor in \( G'_1 \). By the definition of \( G' \) there must be a nontrivial module \( M \) in \( G \) containing \( y \). Moreover \( M \) is either a stable set or a complete set or it induces a prime graph in \( G \). We recall that if \( M \) is a stable or a complete set there is a vertex \( z \) of \( M \) having no neighbour in \( V' \) while all the others vertices of \( M \) have their private neighbor in \( V' \). Let \( M'_0 \) be a submodule of \( M \) formed as follows:

1. if \( M \) is a stable or a complete set then \( M'_0 \) contains \( z \) and all vertices of \( M \) having a neighbour in \( Y \).
2. if \( M \) induces a prime graph in \( G \) then \( M'_0 = M \).

Let \( Y_0 \) be the subset of \( Y \) which is the neighborhood of \( M'_0 \) in \( Y \). Let \( M_1 \) be the nontrivial module of \( H \) isomorphic to \( M'_0 \) by \( \sigma \). It is clear that no vertex of \( M_1 \) can belong to \( Y \) and consequently \( M_1 \) is entirely contained in \( G'_1 \). Now, if \( M_1 \) is not an homogeneous set in \( G \) there must be a set of vertices \( X'_1 \) outside \( M_1 \) that distinguishes the vertices of \( M_1 \). Since \( M_1 \) is a non-trivial module of \( H \), \( X'_1 \) must be a subset of \( X \). Let \( M'_1 \) be a maximal submodule of \( M_1 \) in \( H \). Let \( \mu = (M_1, M'_1), \ldots, (M_l, M'_l) \) be the longest sequence of pair of sets in \( G'_1 \) such that for \( 0 \leq i \leq l \) and \( l \geq 1 \) we have:

1. \( M_i = \sigma(M'_{i-1}) \)
2. \( M_i \) is a non trivial module of \( H \)
3. \( M'_i \) is a maximal submodule of \( M_i \) which is a non trivial module of \( G \)
4. \( M_l \) is not a nontrivial module of \( G \).

Let \( X_i \) be the set of vertices of \( X \) that distinguishes the vertices of \( M_i \), \( i \in [1, l] \). It is clear that if \( X_i = \emptyset \) then \( M_i = M'_i \). Since \( M'_0 \neq \emptyset \) \( M'_0 \) contains the vertex \( y \) and \( M'_0 \) is not an homogeneous set of \( H \) we deduce that for all \( i, j \in [0, l], i \neq j \) \( M_i \cap M_j = \emptyset \) and that \( l \leq |V(G'_1)| \). It is easy to see also that if \( M_i \) is a stable or a complete set then the number of edges between \( X_i \) and \( M_i \) is at least \( |M_i - M'_i| \). Since \( |Y_0| = |M'_0| - 1 \) whenever \( M'_0 \) is a stable or a complete set and \( |Y_0| = 1 \) whenever \( M'_0 \) induces a prime graph, we can easily verify that the number of edges between \( X_1 \cup \cdots \cup X_l \) and \( G'_1 \) is at least \( |Y_0| \). Now, we proceed in an analogous way by considering the set \( Y_1 = Y - Y_0 \), then the set \( Y_2 = Y - Y_1 \) and so on until obtaining \( Y_r = Y - Y_{r-1} = \emptyset \). Let \( X' \) be a minimal with respect to set inclusion subset of \( X \) such that each vertex of \( X' \) ‘breaks’ a module of \( G'_1 \) during the previous process. We can easily see that the number of edges between \( X' \) and \( G'_1 \) is at least \( |Y'| \).

Since \( G \) and \( H \) are isomorphic we have that \( |E(G'_1)| = |Y| = |E(G'_1)| = |X| \). Putting together the facts that \( G \) is connected, \( |X| = |Y| \) and the number of edges between \( X' \) and \( G'_1 \) is at least \( |Y'| \), we deduce that \( X' = X \) and that \( X \) is a stable set whose every vertex is of degree 1 in \( G \). It follows that since \( X' \) is a minimal subset of \( X \) whose every vertex ‘breaks’ a non trivial module in \( G'_1 \), every vertex of \( X \) is of degree 1 in \( G' \) and has a private neighbour in \( G'_1 \), as claimed.

**Theorem 3.9.** The graph \( G' \) is a minimal prime extension of \( G \).

**Proof.** If the only subgraph of \( G' \) which is isomorphic to \( G \) is the graph \( G \) itself, then by Proposition 3.7 we deduce that \( G' \) is a minimal prime extension of \( G \), as claimed. Assume then that there exists a subgraph \( H \neq G \) of \( G' \) which is isomorphic to \( G \) and let \( Q \) be a subgraph of \( G' \) which is prime and contains \( H \). Write \( X = V(G) - V(H) \), \( Y = V(H) - V(G) \) and \( Z = V' - Y \). Let \( \rho_1 = \{M_1, \ldots, M_l\} \) and \( \rho_2 = \{M'_1, \ldots, M'_l\} \) be a bipartition of \( \rho(H) \) such that every module of \( \rho_1 \) is a stable or a complete set and every module of \( \rho_2 \) induces a prime graph in \( H \). Clearly, the set \( \rho(H) \) is isomorphic to the set \( \rho(G) \).

Letting \( R \) stand for the set \( V(Q) - V(H) \) we have \( R \subseteq X \cup Z \). Consider an arbitrary \( M \in \rho(H) \). If \( M \in \rho_1 \) then since \( M \) is either a stable or a complete set, every pair of vertices in \( M \) forms a non-trivial module in \( H \).

Consequently, since by Proposition 3.8 every vertex of \( M \) has at most one neighbour in \( R \) and this neighbour is private, at least \( |M| - 1 \) vertices of \( R \) are needed for ‘breaking’ every submodule of two vertices of \( M \). If \( M \in \rho_2 \) then since \( M \) induces a prime graph in \( H \), at least one vertex which is partial for \( M \) is needed for ‘breaking’ the module \( M \) in
The minimal prime extension $G'$. It follows that $| R | \geq \sum_{i=1}^{s} (| M_i | - 1) + t$. But since $| X | = | Y |$ and $| Y | + | Z | = | V' | = \sum_{i=1}^{s} (| M_i | - 1) + t$ we deduce that $Q$ is precisely the graph $G'$ and the result follows. □

**Definition 3.10.** The minimal prime extension $G'$ of $G$ described in this section, will be called henceforth the basic extension of $G$ and will be noted $\text{basic}(G)$.

### 3.2. The path extension of a decomposable graph

The main goal of this subsection is to present a method for constructing an infinite number of minimal prime extensions of a connected decomposable graph $G$ satisfying the following condition: there exists a nontrivial module $M$ of size at least three in $G$ such that $[M]$, the subgraph of $G$ induced by $M$, is connected and non-isomorphic to a chordless path $P_k$, $k \geq 3$. Importantly, this construction constitutes the framework for characterizing all cases where a graph possesses an infinite number of minimal prime extensions.

Let us now recall the following result of Giakoumakis [10].

**Proposition 3.11 ([10]).** $Q$ is a minimal prime extension of a graph $G$ if, and only if, $\overline{Q}$ is a minimal prime extension of $\overline{G}$.

**Corollary 3.12.** A graph $G$ has an infinite number of minimal prime extensions if, and only if, $\overline{G}$ has an infinite number of minimal prime extensions.

**Corollary 3.12** allows us to restrict ourselves to the case where $G$ is a connected graph.

Assume now that $G$ contains a nontrivial module $M$ of at least three vertices such that $[M]$ is connected and distinct from $P_k$, $k \geq 3$. We may assume without loss of generality that $M$ is maximal with respect to set inclusion, connectivity and that $[M]$ is not isomorphic to a chordless path. Let $A$ be the neighbourhood of $M$ in $G$ and let $B$ stand for its neighbourhood in the complement of $G$. Since $G$ is connected, it follows that $A \neq \emptyset$.

Consider the basic extension $G' = \text{basic}(G)$ of $G$ and denote by $Q$ the set of vertices of $V(\text{basic}(G)) - V(G)$ such that $N(Q) \subseteq M$ and denote by $D$ the vertices of $V(\text{basic}(G)) - V(G)$ such that $N(D) \subseteq (A \cup B)$. Put differently, $Q$ is the set of new vertices that break the module $M$ in $G$ and any nontrivial module of $[M]$, while $D$ is the set of new vertices that break any nontrivial module of $[A \cup B]$ in $G$. It is easy to see that $Q \cup D$ is stable and that every vertex in $Q \cup D$ has exactly one neighbour in $G$ and this neighbour is private in the sense defined above.

Finally, write $D_A = N(A) \cap D$ and $D_B = N(B) \cap D$.

**Lemma 3.13.** $[M \cup Q]$ is a prime graph.

**Proof.** Assume not and let $M'$ be a nontrivial module of $[M \cup Q]$. Observe that $M'$ is neither entirely contained in $Q$ (because every vertex of $Q$ has exactly one neighbour in $\text{basic}(G)$ which is in $M$ and it is private) nor entirely contained in $M$, for otherwise $M'$ would be a module in $\text{basic}(G)$, a contradiction.

It follows that $M'$ contains vertices from both $M$ and $Q$. Let $x$ be a vertex of $M' \cap Q$ and $y$ its neighbour in $M$ (this neighbour is unique in $[M \cup Q]$). Because $\text{basic}(G)$ is prime, $y$ belongs to a non-trivial module of $[M \cup Q] \setminus x$, a contradiction. Notice that by Lemma 3.5, $y$ must be a universal vertex of $[M \cup Q]$ and, consequently, $Q$ must be a singleton.

Let $Q'$ be the neighbourhood of $y$ in $[M]$. Clearly, $Q'$ must be a singleton for otherwise $Q'$ would be a non-trivial module in $G'$. It follows that $[M]$, which by assumption is different from a chordless chain, is isomorphic to a $K_2$, a contradiction. Thus, $[M \cup Q]$ is a prime graph, as claimed. □

**Notation 3.14.** Let $G^+$ be the graph obtained from $\text{basic}(G)$ in the following way: $V(G^+) = V(\text{basic}(G))$ and $E(G^+) = E(\text{basic}(G)) \cup \{(x, y) \mid x \in Q, y \in A\}$. In other words, every vertex of $Q$ is adjacent in $G^+$ to every vertex of $A$, which implies that $M \cup Q$ is a non-trivial module of $G^+$.

**Lemma 3.15.** If $M \cup Q$ is not the unique nontrivial module of $G^+$ then $G^+$ contains exactly a second nontrivial module formed involving vertices of $M \cup Q \cup \{w\}$, where $w$ is a vertex of $B$.

**Proof.** Consider the subgraph $H$ of $G'$ induced by $(V(G^+) \setminus (M \cup Q)) \cup \{h\}$, where $h$ is a vertex of the module $M \cup Q$. In other words, $H$ is obtained by ‘contracting’ $M \cup Q$ to a single vertex. Observe that $H$ is also a proper subgraph of $\text{basic}(G)$. If $H$ is prime then there is nothing to prove since in this case the only non-trivial module in $G^+$ is the set $M \cup Q$. [41x145]
If \( H \) contains a nontrivial module \( \{ w, h \} \) then, again, there is nothing to prove since \( w \) cannot be adjacent to \( h \), for otherwise it would be total for \( M \) and, consequently, \( M \) would not be maximal with respect to set inclusion, a contradiction.

Finally, assume that \( H \) contains a module \( M' \) such that \( M' - \{ h \} \) contains at least two vertices. Then, since no vertex of \( M \cup Q \) can distinguish the vertices of \( M' - \{ h \} \) neither in \( G^+ \) nor in \( \text{basic}(G) \), this set would be also a nontrivial module of \( \text{basic}(G) \), a contradiction. □

**Corollary 3.16.** The subgraph \( H \) of \( \text{basic}(G) \) induced by \( \{ x, y \} \cup A \cup B \cup D \) where \( x \in M \) and \( y \) is the private neighbour of \( x \) in \( Q \), is a prime graph.

**Proof.** Let \( H' \) be the subgraph of \( G^+ \) such that \( V(H') = V(H) \). Lemma 3.15 guarantees that \( H' \) contains at most two non-trivial modules \( \{ x, y \} \) and \( \{ x, y, w \} \) where \( w \) is a vertex of \( B \) nonadjacent to \( \{ x, y \} \). Since in \( H \) the vertex \( y \) has exactly one neighbour, namely \( x \), the result follows. □

**Notation 3.17.** Let \( G \otimes P_k \) be the graph obtained from \( G^+ \) in the following way: \( V(G \otimes P_k) = V((G^+) \) induces a chordless chain \( P_k = x_1, \ldots, x_k \) such that

- \( x_1 \) is adjacent to exactly one vertex of \( Q \),
- every vertex of \( \{ x_1, \ldots, x_{k-1} \} \) is total with respect to \( A \) and adjacent to no vertices of \( M \cup B \cup D \),
- no vertex in \( \{ x_2, \ldots, x_{k-1} \} \) is adjacent to a vertex of \( Q \) and,
- \( x_k \) is adjacent to no vertices of \( G^+ \).

The structure of \( G \otimes P_k \) is illustrated in Fig. 2.

**Proposition 3.18.** The graph \( G \otimes P_k \) is prime.

**Proof.** Since \( [M \cup Q] \) is prime (see Lemma 3.13) and since \( x_1 \) is partial with respect to \( M \cup Q \), this set cannot be a module in \( G \otimes P_k \) and this is the case as well whenever there exists the module \( M \cup Q \cup \{ w \} \) described in Lemma 3.15. It is easy to verify that the addition of the chain \( P_k \) to \( G^+ \) does not create any nontrivial modules and, hence, the resulting graph \( G \otimes P_k \) must be prime, as claimed. □

**Notation 3.19.**
1. \( G^* \) is a minimal prime extension of \( G \) contained, as an induced subgraph, in \( G \otimes P_k \);
2. \( G^* \) is said to be of type 1 if it contains \( P_k \) and of type 2 otherwise;
3. \( G_1 \) is an induced subgraph of \( G^* \) isomorphic to \( G \) by an isomorphism \( \sigma \);
4. \( (M_1, A_1, B_1) \) denotes the partition of \( V(G_1) \) for which \( M_1, A_1, B_1 \) are isomorphic by \( \sigma \) to \( M, A, B \), respectively.

**Lemma 3.20.** \( M_1 \cap D = \emptyset \).
Proof. Indeed, since the degree in $G$ of every vertex in $M$ is at least two and the degree in $G \otimes P_k$ of every vertex of $D$ is exactly one, no vertex of $D$ can belong to $M_1$. □

Proposition 3.21. $M_1 \cup A_1$ is not entirely contained in $P_k \cup Q \cup M$.

Proof. Assume not; since $[M_1]$ is connected and distinct from a chordless chain, $M_1$ is not entirely contained in $P_k \cup Q$ and hence $M_1 \cap M \neq \emptyset$. Since every vertex of $M$ is total with respect to $A$ and indifferent with respect to $B \cup D$, the connectedness of $G_1$ implies that $B_1 \cap (B \cup D) = \emptyset$ and, consequently, $V(G_1)$ is entirely contained in $P_k \cup Q \cup M$.

Since no vertex of $P_k$ is adjacent to a vertex of $M$, no vertex of $A_1$ can be in $P_k$ and consequently $A_1$ is entirely contained in $Q \cup M$.

Observe that in the graph $[P_k \cup Q \cup M]$ every vertex $x \in Q$ has either degree 1 or degree 2 precisely when $x$ is the unique vertex of $Q$ adjacent to a vertex of $P_k$. Now, since the degree of every vertex of $A$ in $G$ is at least $|M|$ and since $M$ contains, by assumption, at least three vertices, no vertex of $A_1$ can be in $Q$. It follows that $A_1$ is entirely contained in $M$. Then, since $M_1$ is total with respect to $A_1$ and no vertex of $P_k$ is adjacent to $M$ we have that $M_1 \cap P_k = \emptyset$. Therefore since $M_1 \cap M \neq \emptyset$, $M_1$ contains vertices from $M$ and $Q$. Furthermore, the connectedness of $[M_1]$ together with the fact that $M_1$ is not entirely contained in $M$ (otherwise, $A_1 = \emptyset$) implies that $M_1 \cap Q \neq \emptyset$ and that the unique neighbour of every vertex $x \in M_1 \cap Q$ belongs to $M_1$. It follows that $M_1$ cannot be total for $A_1$ in $G_1$, a contradiction. □

Proposition 3.22. $V(G^*) \cap A \neq \emptyset$ and $V(G^*) \cap P_k \neq \emptyset$.

Proof. Assume first that $V(G^*) \cap A = \emptyset$. Since no vertex in $M \cup Q \cup P_k$ is adjacent to a vertex of $B \cup D$ and since $M_1$ is total with respect to $A_1$, the connectedness of $G_1$ and Proposition 3.21, combined, imply that $V(G_1)$ is entirely contained in $B \cup D$, $G_1$ is proper subgraph of $G^*$ and since by assumption $V(G^*) \cap A = \emptyset$, the connectedness of $G^*$ implies that $V(G^*)$ is contained in $B \cup D$, which is in contradiction with the fact that $\text{basic}(G)$ is a minimal prime extension of $G$. It follows that $V(G^*) \cap A \neq \emptyset$, as claimed.

Assume next that $V(G^*) \cap P_k = \emptyset$. Since $V(G^*) \cap A \neq \emptyset$, the set $V(G^*) \cap (M \cup Q)$ must contain at most one vertex; otherwise $G^*$ would contain a homogeneous set, a contradiction. It follows that $G^*$ is isomorphic to a proper subgraph of $\text{basic}(G)$, a contradiction. □

Lemma 3.23. If $G^*$ is of type 2 then the set $P_k \cap V(G^*)$ is

1. either a subchain $P' = x_i, x_{i+1}, \ldots, x_{k-1}, x_k$ with $1 < i < k$ of $P_k$
2. or $P' \cup \{x_j\}$ with $1 \leq j < i - 1$.

Proof. The conclusion follows immediately from Proposition 3.22 and the fact that the graph $G^*$ is prime. □

Proposition 3.24. If $G^*$ is of type 2 then $T = V(G^*) \cap (M \cup Q)$ contains at most one vertex.

Proof. Assume to the contrary that $|T| > 1$. Since the vertices of $T$ have the same neighbourhood in $V(G^*) \cap A$ and since $G^*$ is a prime graph, there must exist a set $T' \subseteq V(G^*)$ containing $T$ which is not a homogeneous set of $G^*$. It is easy to see that $T'$ and, consequently, $G^*$ must contain the whole chain $P_k$, a contradiction. □

In Lemma 3.15 we proved that in addition to the module $M \cup Q$, the graph $G^+$ may also contain the module $M \cup Q \cup \{w\}$ where $w$ is a vertex of $B$ (and hence nonadjacent to $M$), whose neighbourhood in $G$ is the set $A$.

To simplify the notation, we shall let $M \cup Q \cup \{w\}$ refer to the set $M \cup Q$ when $w$ does not exist. Now, Notation 3.14, Lemma 3.23 and Proposition 3.24, combined, suggest the following result.

Corollary 3.25. The set $V(G^*) \cap (P_k \cup M \cup Q \cup \{w\})$ equals either $P'$, or $P' \cup \{x_j\}$, or $P' \cup \{w\}$, or $P' \cup \{h\}$ where $h$ is a vertex of $M \cup Q$.

Proof. The conclusion follows directly by observing that we cannot have both $w$ and $h$ or both $w$ and $x_j$ or both $h$ and $x_j$ in $V(G^*) \cap (P_k \cup M \cup Q)$, for otherwise the set of these two vertices would be a homogeneous set in $G^*$, a contradiction. □

Notation 3.26. To simplify the notation, the set $V(G^*) \cap (P_k \cup M \cup Q \cup \{w\})$ will be denoted by $P^*$. 
At this point it is easy to verify the result below which turns out to be a valuable tool in some of the proofs in the sequel of this section.

**Lemma 3.27.** Let $X$ be a subset of $G \otimes P_k$ such that $|X|$ is connected and distinct from a chordless chain. Let $X_1$ be the set $X \cap P_k$ and let $T$ be the set of vertices of $P_k$ that are total with respect to $X$. If $X_1 = \emptyset$ then $|T| \leq 2$ and, moreover:

1. $|T| = 2$ implies that $|T \cup X_1|$ is isomorphic to a $P_3 = abc$ with $a, c \in T$
2. $|T| = 1$ implies that $X_1 = 1$ or that $X_1$ contains exactly two nonadjacent vertices.

**Proposition 3.28.** If $G^*$ is of type 2 then $M_1 \cap P^* = \emptyset$.

**Proof.** Assume not. Since the graph $[M_1]$ is connected and non-isomorphic to a chordless chain, it cannot be entirely contained in $P^*$. Write $X_1 = M_1 \cap P^*$, $X_2 = M_1 - X_1$, $Y_1 = A_1 \cap P^*$ and $Y_2 = A_1 - Y_1$.

Assume first that $X_2 \cap B \neq \emptyset$. Since the graph induced by $M_1$ is connected, we have $X_2 \cap A \neq \emptyset$. Since $P^* - \{x_k\}$ is total with respect to $A$, $x_k$ is indifferent with respect to $A$, $P^*$ is indifferent for $B \cup D$ and $M_1$ is indifferent for $B_1$, it follows that $A_1 \cap P^* = \emptyset$ and $A_1 \cap D = \emptyset$. Consequently, $A_1$ must be entirely contained in $A - \{X_2 \cap A\}$, a contradiction.

Thus, $X_2$ must be entirely contained in $A$. Since no vertex of $P^*$ is adjacent to a vertex of $B \cup D$, it must be the case that $Y_2 \cap (B \cup D) = \emptyset$ and consequently $Y_2$ must be entirely contained in $A$. This implies that $Y_1 \neq \emptyset$. Since, by assumption, $X_1 \neq \emptyset$, Lemma 3.27 guarantees that $|Y_1| \leq 2$.

Assume that $Y_1$ contains two vertices, say $x$ and $y$. Now, Lemma 3.27 guarantees that these vertices are nonadjacent and $X_1$ is a singleton. Since no vertex of $P^*$ is adjacent to a vertex of $B \cup D$ and since $X_1 \neq \emptyset$, it must be that $Y_2 \subset A$. Since $[M_1]$ is not isomorphic to a chordless chain, $X_2$ contains exactly two adjacent vertices, say $z$ and $t$, that is, $[M_1]$ is isomorphic to a $C_3$.

Let $\theta$ stand for the number of edges of the graph induced by $Y_2$ and let $\delta_1$ denote the number of edges of the graph induced by $A_1$. We have that $\delta_1 = \theta + |Y_2| + |Y_2| (x \text{ and } y \text{ are total with respect to } Y_2)$. But the number of edges of $[A]$ is at least $\theta + |Y_2| + |Y_2| + 1 (z \text{ and } t \text{ are total with respect to } Y_2$ and $\{z, t\} \text{ is an edge of } [X_2])$, a contradiction. Hence, $Y_1$ is a singleton and either $X_1$ is a singleton or it contains two non-adjacent vertices. In either case, the set $X_2$ must contain at least two vertices and hence $|A_1| < |A|$, a contradiction. \(\square\)

**Proposition 3.29.** If $G^*$ is of type 2 then either $M_1$ is entirely contained in $A$ or $M_1$ is entirely contained in $B$.

**Proof.** By Lemma 3.20 and Proposition 3.28, it must be the case that $M_1$ is entirely contained in $A \cup B$. Assume for the sake of contradiction that $X_1, X_2 \neq \emptyset$ where $X_1 = M_1 \cap A$ and $X_2 = M_1 \cap B$. Since every vertex of $P^*$ is total with respect to $A$ and indifferent for $B \cup D$, no vertex of $A_1 \cup B_1$ can be in $P^*$ and, as a consequence, $A_1 \cup B_1$ is entirely contained in $A \cup B \cup D$.

Consider the proper subgraph $H$ of basic($G$) induced by $X \cup Y \cup A \cup B \cup D$ where $x$ is a vertex of $M$ and $y$ is the private neighbour of $x$ in $Q$. Since $G_1$ is a subgraph of $H$ and since by Corollary 3.16 $H$ is prime, $H$ must contain an extension of $G_1$, contradicting the fact that $H$ is a proper subgraph of basic($G$). \(\square\)

**Theorem 3.30.** $G^*$ is of type 1.

**Proof.** Assume not. Clearly, we can write $V(G^*) \subseteq A \cup B \cup D \cup P^*$. Let $R$ be the set of vertices of $G$ that are partial with respect to $M_1$ in $V^*(G) - V(G_1)$. Proposition 3.29 guarantees that every vertex of $M$ is either total or indifferent with respect to $M_1$. Thus, it must be that $R \cap M = \emptyset$ and, consequently, $R \subseteq A \cup B$.

Let $S \subseteq A \cup B$ be the set of vertices that are total or indifferent with respect to $M_1$ in the graph induced by $V(G) - V(G_1)$. Let $\gamma$ be a sequence of vertices $\gamma = x_0 x_1 \cdots x_s$ such that:

1. $x_0$ is a vertex of $M$ or a vertex of $R \cup S$;
2. Every vertex of $\gamma \setminus x_0$ belongs to $A \cup B$;
3. $x_i = \sigma(x_{i-1})$, $1 \leq i \leq s$;
4. $\gamma$ is as long as possible with the above properties.

We shall call the path $\gamma$ a special path. In this context, we denote by init($\gamma$) the vertex $x_0$ and by term($\gamma$) the vertex $x_s$. If init($\gamma$) is a vertex of $M$, then $\gamma$ will be a special path of type 1 and if init($\gamma$) is a vertex of $R \cup S$ it will be a
special path of type 2. Let $\Gamma$ be the set of special paths in $G$. We denote by $\Gamma^1$ the set of special paths of type 1 and by $\Gamma^2$ the set of special paths of type 2. Finally, we shall let $\mu(x)$ denote the special path to which vertex $x$ belongs.

Let $\mu$ be a path of $G$. It is easy to see that:

1. $\sigma(\text{term}(\mu))$ belongs to $D \cup P^*$;
2. No two special paths $\mu_1$ and $\mu_2$ share common vertices and $\sigma(\text{term}(\mu_1)) \neq \sigma(\text{term}(\mu_2))$.

**Claim 1.** Let $x$ be a vertex of a special path $\mu$ distinct from $\text{term}(\mu)$ such that the neighbourhood of $x$ in $G$ is not a stable set. Then neither the neighbourhood of $\sigma(x)$ in $G$ nor the neighbourhood of $\sigma(\text{term}(\mu))$ in $G_1$ can be stable sets.

**Proof.** If $\sigma(x)$ belongs to $A$ the result is obvious since $M$ belongs to $G$. Assume then that $\sigma(x)$ belongs to $B$ and consider two adjacent vertices $a$ and $b$ of $G$ which are adjacent to $x$. Since $a$ and $b$ are both in $G$, none of $\sigma(a)$ and $\sigma(b)$ can be a vertex of $D$. Since none of the vertices of $P^*$ can be adjacent to $\sigma(x) \in B$, it follows that $\sigma(a)\sigma(b)$ is an edge of $[A \cup B]$ which proves that the neighbourhood of $\sigma(x)$ in $G$ is not a stable set, as claimed.

Finally, it is clear that the neighbourhood in $G_1$ of $\sigma(\text{term}(\mu))$ is not a stable set if and only if the neighbourhood of $x$ in $G$ is not a stable set. \hfill \Box

In Corollary 3.25 it was shown that $P^*$ is formed by the subchain $P' = x_1, \ldots, x_k$ of $P_k$ such that $1 < i < k$ and possibly by a vertex of $\{x_j, w, h\}$ with $x_j \in P_k$, $j < i - 1$, $w \in B$ and $h \in M \cup Q$. In the following, we shall assume that $P^*$ is formed by the vertices of $P' \cup \{h\}$ where $h$ is a vertex of $Q$. It is an easy task to verify that the claimed result of this theorem holds when considering all the other possibilities concerning $P^*$.

Let $U = \{M_0, \ldots, M_i, \ldots, M_q\}$ be the set satisfying the following conditions:

1. $M_0 = M$.
2. $M_i = \sigma(M_{i-1})$, $i = 1, \ldots, q$.
3. $M_i \subseteq A \cup B$, $i = 1, \ldots, q$.
4. $U$ is the largest set with respect to the above properties.

**Property of $U$.** Since $M_0 \cap V(G^*) = \emptyset$, it is easy to verify that for every choice of $M_i$ and $M_j$, $1 \leq i \neq j \leq q$, in $U$, $M_i \cap M_j = \emptyset$.

**Claim 2.** Let $x$ be a vertex of $M_i \in U$, $i = 0, \ldots, q$ and let $\mu \in \Gamma^1$ be the special path containing $x$. Then $\sigma(\text{term}(\mu)) \in P^*$.

**Proof.** The maximality of $\mu$ implies that $\sigma(\text{term}(\mu)) \in P^* \cup D$ and since the neighbourhood of any vertex of $M_0$ is not a stable set, the result follows from Claim 1. \hfill \Box

**Claim 3.** $M_{q+1} = \sigma(M_q)$ contains vertices from $P^*$ and $A \cup B$.

**Proof.** If not, by the previous claim $\sigma(M_q)$ would be entirely contained in $P^*$ and, consequently, $M_q$ would induce a graph isomorphic to a chordless chain, a contradiction. \hfill \Box

The reader can easily verify the two following claims.

**Claim 4.** Let $M_i$ be a set of $U$ and $X$ a set of vertices of $G$ outside $M_i$. Then $\sigma(X)$ is partial (resp. total, indifferent) with respect to $\sigma(M_i)$ if and only if $X$ is partial (resp. total, indifferent) with respect to $M_i$.

**Claim 5.** The number of partial (resp. total, indifferent) vertices of $M_i \in U$ in $G$ is equal to the number of partial (resp. total, indifferent) vertices of $M_{i+1} \in U$ in $G_1$, $i = 1, \ldots, q - 1$.

Let $H = \{H_0, \ldots, H_i, \ldots, H_r\}$ be a set satisfying the following properties.

1. $H_0$ is a module of $M$.
2. $H_i = \sigma(H_{i-1})$, $i = 1, \ldots, r$.
3. $H_i \subseteq A \cup B$, $i = 1, \ldots, r$.
4. $|H_i| > 1$, and
5. $H$ is the largest set with respect to the above properties.
Claim 6. Let \( H_i \), \( 1 \leq i \leq r \), be a set in \( \mathcal{H} \) and let \( x \) be a vertex of \( G \) which is partial with respect to \( H_i \). The following two conditions are satisfied:

1. \( H_i \) is either entirely contained in \( A \) or else entirely contained in \( B \);
2. \( x \) belongs to a path \( \mu \) of type 2, i.e., \( \mu \in \Gamma^2 \).

**Proof.** The proof is by induction on \( i \). By Proposition 3.29, \( M_1 \) is entirely contained in \( A \) or is entirely contained in \( B \) and, hence, every vertex of \( M \) is either total or indifferent with respect to \( M_1 \) and, consequently, with respect to \( H_1 \). It follows that every partial vertex with respect to \( H_1 \) in \( G \) belongs to \( R \) and hence the result holds for \( i = 1 \).

Assume that the result holds for \( H_i \), \( 1 \leq t < r \), and consider \( H_{i+1} \). If \( H_{i+1} \) contains vertices from both \( A \) and \( B \), then every vertex of \( P^* \) is partial with respect to this set in \( G_1 \). By the induction hypothesis the set, say \( J \), of partial vertices with respect to \( H_i \) consists of vertices belonging to special paths in \( \Gamma^2 \). Since \( \mathcal{J} \) belongs to \( A \cup B \cup D \cup P^* \), it follows that every vertex \( y \) of \( P^* \) must satisfy \( \sigma(y) \in J \), contradicting Claim 2. Hence \( H_{i+1} \) is entirely contained in \( A \) or entirely contained in \( B \), as claimed.

Consider now a vertex \( x \) of \( G \) which is partial for \( H_{i+1} \). If \( x \in V(G_1) \cap V(G) \) then \( x \in \Gamma^2 \) for otherwise \( \sigma^{-1}(x) \notin \Gamma^2 \), contradicting the induction hypothesis. If \( x \in V(G) - V(G_1) \) which equals \( M \cup R \cup S \) then since \( M \) is total or indifferent in \( G \) for \( M_{i+1} \) and consequently for \( H_{i+1} \), \( x \) must be a vertex of \( R \cup S \) which belongs to \( \Gamma^2 \) as claimed.

**Observation.** Since the set \( H_0 \) of \( \mathcal{H} \) is not necessarily a nontrivial module of \( M_0 = M \), \( H_0 \) can be \( M \) itself and, consequently, Claim 6 holds for every set \( M_i \) of \( \mathcal{U} \).

Denote by \( \Omega \) the set \( \Omega = \{ \sigma(\text{term}(\mu(x))) \mid x \in M \} \). Observe that by virtue of Claim 2, \( \Omega \) is entirely contained in \( P^* \). Let \( Y \) be the set \( M_{q+1} \cap P^* \), let \( T \) be the set of vertices of \( P^* \) which are total with respect to \( Y \) and let \( \Omega' \) be the set \( \Omega - (T \cup Y) \).

Assume first that \( M \) contains at least four vertices. By Lemma 3.27, if \( T \neq \emptyset \), \( T \cup Y \) contains at most three vertices and, consequently, \( \Omega' \neq \emptyset \). Let \( J \) be the set of partial vertices of \( M_j \) in \( G \). By Claim 6, for \( x \in J \) we have \( \sigma(\text{term}(\mu(x))) \notin \Omega \) and the number of partial vertices of \( M_{q+1} \) in \( G_1 \) is at least \( |J| + |\Omega'| \), a contradiction.

Assume now that \( M \) contains fewer than four vertices. Since \( [M] \) is connected and distinct from a chordless path, \([M]\) must be isomorphic to a \( C_3 \), say \( abc \), induced, in the obvious way, by vertices \( a \), \( b \), and \( c \). If \( Y \) contains two adjacent vertices then \( T = \emptyset \) and, consequently, \( |\Omega'| = 1 \) which implies that the number of partial vertices of \( M_{q+1} \) in \( G_1 \) is at least \( |J| + 1 \), a contradiction.

Hence \( Y \) contains exactly one vertex, say, \( a \). Clearly if \( J = \emptyset \) or if \( \sigma(J) \subseteq P^* \) then \( \{b, c\} \) is a module in \( G_1 \) and, consequently, in \( G^* \) since there is no vertex of \( V(G_1) \) that “breaks” the module \( \{b, c\} \), a contradiction. Let \( b^* \) and \( c^* \) be, respectively, the vertices \( b^* = \sigma(\text{term}(\mu(b))) \) and \( c^* = \sigma(\text{term}(\mu(c))) \) which by Claim 2 belong to \( P^* \). If at least one of these vertices is not total with respect to \( abc \), the number of partial vertices of \( abc \) in \( G_1 \) is at least \( |J| + 1 \), a contradiction. Hence \( Y = \{b^*, c^*\} \).

Let \( \theta = b_0 c_0, b_1 c_1, \ldots, b_r c_r \) be the longest sequence of edges in \( G \) such that \( b_0 = b, c_0 = c \) and \( b_i = \sigma(b_{i-1}), c_i = \sigma(c_{i-1}) \), \( i = 1, \ldots, r \). In other words \( b_i \in \mu(b_{i-1}) \) and \( c_i \in \mu(c_{i-1}) \), \( i = 1, \ldots, r \). Since by Claim 6 the set of partial vertices of \( b_r c_r \) in \( G \) belong to paths in \( \Gamma^2 \), and \( b^* \) is nonadjacent to \( c^* \) then either \( \sigma(b_r) = b^* \) or \( \sigma(c_r) = c^* \) but not both.

Assume, without loss of generality, that \( \sigma(b_r) = b^* \) and write \( c_{r+1} = \sigma(c_r) \). Since \( b_r c_r \) is an edge of \( G \), it follows that \( b^* c_{r+1} \) is an edge of \( G_1 \) and so \( c_{r+1} \) belongs to \( A \). Thus, \( c^* \) distinguishes in \( G_1 \) the vertices of \( \{b^*, c_{r+1}\} \). Let \( I \) denote the set of partial vertices with respect to \( \{b_r, c_r\} \) in \( G \). By Claim 6 every vertex of \( I \) belongs to a path of \( \Gamma^2 \). By Claim 2, \( c^* \notin \sigma(I) \), a contradiction, since the number of partial vertices in \( G_1 \) of \( \{b^*, c_{r+1}\} \) is larger than the number of partial vertices of \( \{b_r, c_r\} \) in \( G \).

We are now in a position to state the main result of this section.

**Theorem 3.31.** Let \( G \) be a connected graph containing a maximal nontrivial module \( M \) such that \([M]\) induces a connected graph with at least three vertices non-isomorphic to \( P_k \), \( k \geq 3 \). Then, the set of minimal prime extensions \( \text{Ext}(G) \) of \( G \) is infinite.

**Proof.** The result follows from the fact that every prime extension of \( G \) obtained by the path construction \( G \otimes P_k \) is of type 1 and the fact that the chain \( P_k \) is of arbitrary length. \( \Box \)
4. All minimal prime extensions: The finite and infinite cases

The main goal of this section is to characterize all classes of graphs whose closure under substitution closure can be defined by a finite set of forbidden subgraphs. Our result will be obtained by an exhaustive examination of the structure of non-trivial modules of a connected graph $G$. The cases that may arise are illustrated in the Fig. 3.

Theorem 4.1. Let $G$ be a connected graph which is not $P_4$-homogeneous and such that every module of $G$ that induces a connected graph is isomorphic to a chordless chain. If $\overline{G}$ is disconnected then it contains exactly two connected components.

Proof. Suppose not. If $\overline{G}$ contains at least four components, say $F_1$, $F_2$, $F_3$, and $F_4$ then $F_1 \cup F_2 \cup F_3$ is a module in $G$ whose induced graph is not isomorphic to a chordless chain since it contains a $C_3$, a contradiction.

Assume, next, that $\overline{G}$ contains three connected components $F_1$, $F_2$, and $F_3$. If two of these components, say $F_1$ and $F_2$, are not single vertices, then the module $F_1 \cup F_2$ of $G$ contains a $C_4$, a contradiction. Hence two of these components, say $F_2$ and $F_3$, are single vertices, while $|F_1| > 2$, for otherwise $G$ would be $P_4$-homogeneous, a contradiction. It is easy to see now that the nontrivial module $F_1 \cup F_2$ of $G$ cannot be isomorphic to a chordless chain, a contradiction. □

4.1. $2P_3$-homogeneous graphs

We shall now present a class of graphs whose set of minimal prime extensions is finite.

Definition 4.2. Let $G$ be a connected graph which is not $P_4$-homogeneous and having a universal vertex $u$. We shall call $G$ a pseudo-gem if $G' = G \setminus u$ is a $P_4$-homogeneous graph which is isomorphic to a subgraph of a chordless chain.

The following result clarifies the structure of a pseudo-gem.

Lemma 4.3. Let $G$ be a pseudo-gem and $u$ a universal vertex of $G$, then $G' = G \setminus u$ is one of the following types of graphs:

1. $G'$ is isomorphic to a chordless chain $P_l$, $l \geq 5$;
2. $G'$ is the disjoint union of two chordless chains $P_i$ and $P_j$ such that
   - $i = 1$ and $3 \leq t \leq 4$, or
   - $i = 2$ and $2 \leq t \leq 4$, or
   - $i = 3$ and $1 \leq t \leq 4$, or
   - $i = 4$ and $1 \leq t \leq 4$;
3. $G'$ is isomorphic to an $O_3$ or is the union of an $O_2$ and a $P_2$.

Proof. Indeed, since $u$ is a universal vertex of $G$ and $G$ is not a $P_4$-homogeneous graph, $G'$ cannot be a subgraph of a $P_4$. Also, since $G'$ is a $P_4$-homogeneous graph isomorphic to a subgraph of a chordless chain, then either $G'$ is prime and, consequently, isomorphic to a chordless chain $P_l$, $l \geq 5$ (Case 1 above) or every module of $G'$ induces a subgraph of a $P_4$. The conclusion follows. □

Definition 4.4. Let $G$ be a connected graph which is not $P_4$-homogeneous such that $\overline{G}$ contains exactly two connected components $C_1$ and $C_2$. Then, $G$ and $\overline{G}$ are said to be $2P_4$-homogeneous graphs if $[C_1]$ and $[C_2]$ are subgraphs of a chordless chain and one of the following conditions holds:

- $G$ is isomorphic to a pseudo gem
- $[C_2]$ is isomorphic to a subgraph of a $P_4$ and $[C_1]$ is a $P_4$-homogeneous graph which is $2K_2$-free.

The following result clarifies the structure of a $2P_4$-homogeneous graph.

Lemma 4.5. Let $G$ be a connected $2P_4$-homogeneous graph which is not isomorphic to a pseudo gem. Let $C_1$ and $C_2$ be the two connected components of $\overline{G}$ with $[C_2]$ a subgraph of a $P_4$. Then

1. $[C_2]$ is isomorphic to a $P_4$, $\overline{P}_3$ or a $\overline{P}_2$
2. $[C_1]$ is isomorphic to a $P_1 \cup P_3$ or to a $P_1 \cup P_4$ or to an $O_3$ or to an $O_2 \cup P_2$.
Fig. 3. Illustration of the various cases that may occur for graph $G$ and module $M$. For each case we indicate whether $\text{Ext}(G)$ is finite or infinite.

**Proof.** Since $G$ is not isomorphic to a pseudo gem, $C_2$ is not a singleton and since $[C_2]$ is connected and isomorphic to a subgraph of a $P_4$, $[C_2]$ must be isomorphic to a $P_4$ or to a $P_3$ or to a $P_2$ as claimed. Let us prove now the second assertion of the lemma. Indeed, since $[C_1]$ is 2$K_2$-free, $[C_1]$ cannot contain a subgraph isomorphic to either a chordless chain $P_l$, $l \geq 5$ or to the disjoint union of two chordless chains which are both distinct from a $P_1$. Since also $G$ is not a $P_4$-homogeneous graph, $[C_1]$ cannot be isomorphic to a chordless chain having at most 4 vertices. Finally, since $[C_1]$ is a $P_4$-homogeneous graph which is a subgraph of a chordless chain, $[C_1]$ is isomorphic to a $P_1 \cup P_3$ or to a $P_1 \cup P_4$ or to an $O_2 \cup P_2$ to $O_3$ as claimed. □

We shall prove now that the set of minimal prime extensions of a $2P_4$-homogeneous graph is finite. Before this we need some preliminaries results.

**Theorem 4.6 ([16]).** If $W$ is a nontrivial module of a graph $G$ and $W$ induces a subgraph of a $P_4$ then in any minimal prime extension of $G$ there exists a $W$-pseudopath $P$ having at most two vertices. Moreover

1. if $[W]$ is not isomorphic to a $P_4$, $P$ has exactly one vertex.
2. If $W$ is isomorphic to a $P_4$ abcd and $P$ has two vertices, the first vertex of $P$ is adjacent to the two middle vertices $b$ and $c$ of $[W]$ and misses the two other vertices $a$ and $c$.

**Notation 4.7.** Let $Q$ be a minimal prime extension of its induced connected subgraph $G$ and let $W$ be a nontrivial module of $G$ such that $W$ induces a subgraph of a $P_4$. Then there exists in $Q$ a $W$-pseudopath $P$ having at most two vertices.

We can derive from Theorem 4.6 the following result:

**Proposition 4.8.** Let $Q$ be a minimal prime extension of its induced connected subgraph $G$ and let $W$ be a nontrivial module of $G$ such that $W$ induces a subgraph of a $P_4$. Then there exists in $Q$ a $W$-strong pseudopath $P$ having at most two vertices.

**Proof.** If $W$ contains two or four vertices, the result is directly obtained from Theorem 4.6. Assume then that $[W]$ is isomorphic to a $P_3$ or to a $P_3$. Let $x_1$ be the first vertex of $P$ which as we recall, is partial for $W$. If there exists an homogeneous set $W' \subseteq W$ in the graph induced by $G \cup P$, then $W'$ has exactly two vertices. Consequently, by Theorem 4.6 there must be a $W'$-pseudopath $P'$ in $Q$. Clearly $P'$ is a strong pseudopath and is also a $W$-pseudopath. We can easily verify that no subset of $W$ is an homogeneous set in the graph $G \cup P'$ and hence we are done. □

**Proposition 4.9.** Let $Q$ be a minimal prime extension of its induced connected subgraph $G$ and let $W$ be a maximal homogeneous set of $G$ such that $W$ induces a subgraph of a $P_4$. If $W$ is the unique maximal homogeneous set of $G$ then $|V(Q)| < |V(G)| + 2$. 


Proof. Indeed, by Proposition 4.8 there exists in $Q$ a $W$-strong pseudopath $P$ having at most two vertices. Clearly since $M$ is the unique maximal homogeneous set in $G$ the graph $G \cup P$ is prime. Since $Q \in \text{Ext}(G)$, the result follows. \hfill $\square$

We are now in position to present the main theorem of this subsection.

**Theorem 4.10.** If $G$ is a $2P_4$-homogeneous graph, then $\text{Ext}(G)$ is a finite set.

**Proof.** Assume that $G$ is connected and let $\mathcal{E}$ be the set of minimal prime extensions of $G$. Let $C_1$ and $C_2$ be the two connected components of $G$ and assume w.l.o.g. that $[C_2]$ is a subgraph of a $P_4$.

Consider the bi-partition of $\mathcal{E}$ into the following sets:
1. $E_1$ is the set of graphs belonging to $\text{Ext}([C_1]) \cap \mathcal{E}$;
2. $E_2$ is the set of graphs in $\mathcal{E} - E_1$.

**Claim 1.** $E_1$ is finite.

**Proof.** Since, by assumption, $[C_1]$ is a $P_4$-homogeneous graph, Theorem 2.5 guarantees that $\text{Ext}([C_1])$ is finite and, consequently, the same must hold for $E_1$. \hfill $\square$

Our next task is to prove that $E_2$ is finite. We shall distinguish the two complementary cases: $C_2$ is a singleton (i.e. $G$ is a pseudo gem) and $C_2 \neq$ singleton.

**Case 1** $C_2$ is a singleton

Let $Q$ be an arbitrary graph in $E_2$. Let $H$ be a subgraph of $Q$ isomorphic to $G$ and $H' = H \setminus v$ where $v$ is a universal vertex of $H$. Since $Q$ is prime there must be a subgraph $Q'$ in $Q$ containing $H'$ as induced subgraph such that $Q' \in \text{Ext}(H')$.

**Claim 2.** $|V(Q)| \leq |V(Q')| + |V(G)|$.

**Proof.** Assume first that the vertex $v$ is not adjacent to all vertices of $Q'$. Then since $v$ is adjacent to all vertices of $H'$, $v$ is partial with respect to $V(Q')$. Since $Q'$ is a prime graph, the graph formed by $Q'$ and the vertex $v$ is also prime and consequently this graph is the graph $Q$. Assume now that the vertex $v$ is total with respect to $V(Q')$ that is, $V(Q')$ is a nontrivial module of the graph induced by $V(Q') \cup \{v\}$. Then by Theorem 2.8 there must be in $Q$ a $V(Q')$-pseudopath $P = y_1, \ldots, y_r$. Let $P' = y_1, \ldots, y_s, 1 \leq s \leq r$ be the longest sequence of vertices of $P$ inducing a chordless chain. We show now that $s < |V(H')| + 1$. Assume the contrary, then since $H'$ is isomorphic to a subgraph of a chordless chain, $P$ would contain a subgraph isomorphic to $H'$. It follows that $P$ together with the vertex $v$ would form a prime graph containing a minimal prime extension of $H$ strictly contained in $Q$, a contradiction. We shall show now that $P'$ is exactly $P$. Assume the contrary and consider the graph $Q''$ induced by $V(Q') \cup P' \cup y_{s+1}$. By the definition of $P'$, $y_{s+1}$ is adjacent to all vertices of $Q'$ and all vertices of $P' \setminus y_s$. We can easily verify that the subgraph of $Q''$ formed by the vertices of $H'$ and the vertex $y_{s+1}$, is isomorphic to $G$. Consequently, $Q''$ is a prime graph it contains a minimal prime extension of $G$. Since $Q''$ is a proper subgraph of $Q$ ($Q$ contains also the vertex $v$) we obtain a contradiction. \hfill $\square$

Since $H'$ is a $P_4$-homogeneous graph, by Theorem 2.8 $\text{Ext}(H')$ is a finite set. Therefore since each minimal prime extension of $H$ is obtained from a graph of $\text{Ext}(H')$ by adding at most $s < |V(G)| + 1$ vertices, we deduce that whenever $G$ is isomorphic to a pseudo-gem, $E_2$ is a finite set.

**Case 2** $C_2 \neq$ singleton

Let $Q$ be an arbitrary graph in $\text{Ext}(G)$ and $G'$ be a subgraph of $Q$ isomorphic to $G$. Denote by $[C_1']$ and $[C_2']$ the subgraphs of $G'$ isomorphic respectively to $[C_1]$ and $[C_2]$. Clearly, there is a subgraph $H_1$ in $Q$ containing $[C_1']$ as proper subgraph such that $H_1 \in \text{Ext}([C_1'])$.

In [17] all the minimal prime extensions for $[C_1']$ are given. The reader can easily verify that there are 10 minimal prime extensions whenever $[C_1'] \sim O_2 \cup P_2$, 3 minimal prime extensions whenever $[C_1'] \sim P_1 \cup P_3$ or $[C_1'] \sim O_3$ and 9 minimal prime extensions whenever $[C_1'] \sim P_1 \cup P_4$. Furthermore, there is exactly one of these minimal prime extensions whose number of vertices is $|C_1'| + 3$ while the number of vertices of all others is $|C_1'| + 2$. Finally, there are exactly three of them having a universal vertex $x$ with respect to $[C_1']$. In Fig. 4 of this paper we give these three minimal prime extensions using the same notations as in [17].

It is easy to see now that we have the following result:
Fig. 4. Three minimal prime extensions: $A \in \text{Ext}(O_3) = \text{Ext}(P_1 \cup P_3)$, $G_5 \in \text{Ext}(O_2 \cup P_2)$, $L_9 \in \text{Ext}(P_1 \cup P_4)$.

**Claim 3.** $|V(H_1)| \leq |C'_1| + 3$ and there is at most one vertex of $H_1$ which is total with respect to $C'_1$ and not adjacent to any vertex of $V(H_1) - C'_1$.

**Claim 4.** If $V(H_1)$ is not an homogeneous set of the graph $H_1 \cup [C'_2]$, then $|V(Q)| < |V(G)| + 5$.

**Proof.** Let $M$ be a maximal nontrivial module of $H_1 \cup [C'_2]$. If $M$ is entirely contained in $C'_2$, then since $H_1$ is a prime graph, $M$ will be the unique maximal homogeneous set of $H_1 \cup [C'_2]$. The result follows by Proposition 4.9 and Claim 3. Assume then that $M$ is not entirely included in $C'_2$. Then since $H_1$ is a prime graph, there is exactly one vertex of $M$, say $x$ belonging to $V(H_1)$. Consequently $M' = M - \{x\}$ is entirely contained in $C'_2$. Since every vertex of $C'_2$ is total with respect to $C'_1$, we deduce that $x$ is adjacent to every vertex of $C'_1$ which implies that $H_1$ is isomorphic to one of the graphs $A$, $G_5$, $L_9$ depicted in Fig. 4. Since the vertex $x$ is not adjacent to any vertex of $V(H_1) - C'_1$, the same holds for any vertex of $M'$. Consequently the graph $H'_{1}$ induced by $\{V(H_1) \cup \{y\}\} - \{x\}$, with $y$ a vertex of $M'$, is isomorphic to $H_1$. It follows that $H'_{1} \cap C'_2$ contains a unique maximal homogeneous set inducing a subgraph of a $P_4$ and the result follows by Proposition 4.9 and Claim 3. □

Assume now that $V(H_1)$ is a nontrivial module of $[V(H_1) \cup C'_2]$ which is the last case to be examined. By Proposition 4.8 there is a $C'_1$, strong pseudopath $P$ in $Q$, having at most two vertices. If the last vertex of $P$, say $z$, is partial with respect to $V(H_1)$ then the graph $[V(H_1) \cup C'_2] \cup P$ is prime and consequently by Proposition 4.8 and Claim 3 we have that $|V(Q)| < |V(G)| + 5$. Assume then that $z$ is not partial with respect to $V(H_1)$ then since the nonneighbourhood of $C'_2$ in $[V(H_1) \cup C'_2]$ is the empty set, $z$ cannot be total with respect to the set $V(H_1)$. It follows that $z$ is indifferent with respect to $V(H_1)$. Consequently, $V(H_1)$ is the unique maximal homogeneous set of $[V(H_1) \cup C'_2] \cup P$. By Theorem 2.8 there must be in $Q$ a $V(H_1)$- pseudopath $R_k = (x_1, \ldots, x_k)$. Since $V(H_1)$ is the unique maximal homogeneous set of $[V(H_1) \cup C'_2] \cup P$, we deduce that the graph $[V(H_1) \cup C'_2] \cup P \cup R_k$ is prime.

Our next task is to prove that $k \leq c$, where $c$ is a constant.

Let $A = \{A_1, \ldots, A_l\}$ be the largest set of chordless chains obtained from $R_k$ in the following manner:

1. $A_l$ is the longest chordless chain formed by consecutive vertices of $R_k$ and having as first vertex $x_1$.
2. $A_i$ is the longest chordless chain formed by consecutive vertices of $R_k \setminus A_1 \cup \cdots \cup A_{i-1}$ and whose first vertex is the first vertex of $R_k \setminus A_1 \cup \cdots \cup A_{i-1}$, $1 < i < l$.

Let $a_i$ be the first vertex and $b_i$ be the last vertex of $A_i$, $i = 1, \ldots, l$. Clearly if $A_i \sim P_1$, $a_i = b_i$. If $A_i \sim P_r$ such that $r > 1$ we shall note $e_i$ the last but one vertex of $A_i$, $i = 1, \ldots, l$.

**Claim 5.** $A_i$, $i = 1, \ldots, l$ contains at most six vertices.

**Proof.** Assume on the contrary that there exists $A_i \in A$ having more than six vertices, then $A_i$ contains an induced graph $[C''_1]$ isomorphic to $[C'_1]$. Since every vertex of $R_k \setminus x_k$ is total with respect to $C'_2$, the graph $[C''_1] \cup [C'_2]$ is isomorphic to $G$. But since the graph $A_i \cup \cdots \cup A_l \cup P \cup [C'_2]$ is prime, it contains a subgraph isomorphic to $G$ as induced subgraph and is strictly contained in $Q$, we obtain a contradiction. □

We shall show now that $l < 10$. Assume the contrary and consider the set $A' = A \setminus A_1 \cup A_l$.

**Claim 6.** There are not in $A'$ three chains $A_i, A_{i+1}, A_{i+2}$, $1 < i < l - 3$ such that each one is isomorphic to a $P_1$.

**Proof.** If not, $A_i, A_{i+1}, A_{i+2}$ together with the first vertex of $A_{i+3}$ would induce a copath containing a subgraph isomorphic to $[C''_2]$. Since every vertex of this copath is total with respect to $H_1$, the graph $H_1 \cup R_k$ which is prime and is strictly contained in $Q$, would contain an induced subgraph isomorphic to $G$, a contradiction. □
Since by assumption \( l > 9 \), Claim 6 implies that there exists three chordless chains \( \mathcal{A}', \mathcal{A}_r, \mathcal{A}_t \) and \( \mathcal{A}_t, r < s < t \) such that none of them is isomorphic to a \( P_4 \). Assume w.l.o.g. that \( s \) is as small as possible that is, there is no \( A_2, r < i < s \) which is not isomorphic to a \( P_1 \). If \( s = r + 1 \) then the set of vertices \( \{c_1, b_r, c_s, b_s, b_1\} \) induces a subgraph isomorphic to a \( P_1 \cup P_4 \). If \( s > r + 1 \) then since by assumption \( s \) is as small as possible, \( A_{s-1} \) is isomorphic to a \( P_1 \). It follows that the set of vertices \( \{a_i, a_{i-1}, c_s, b_s, b_1\} \) induces a subgraph isomorphic to a \( P_1 \cup P_4 \). Consequently in both cases there exists in \( R_4 \) a subgraph isomorphic to \([C_1']\). It follows that the graph \( R_4 \cup [C_2'] \cup P \) which is prime and is strictly contained in \( \mathcal{Q} \), contains an induced subgraph isomorphic to \( G \) a contradiction.

Since \( |V(H_1)| \leq |C_1'| + 3 \) (Claim 3), \( P \) has at most 2 vertices (Proposition 4.9) and \( l < 10 \), it is easy to see now that every minimal prime extension of \( G \) has \( |V(G)| + \epsilon \) vertices, where \( \epsilon \) is a constant and this completes the proof of the theorem. \( \Box \)

4.2. The main theorem

**Theorem 4.11.** Given a decomposable graph \( G \), \( \text{Ext}(G) \) is finite if and only if \( G \) is \( P_4 \)-homogeneous or a \( 2P_4 \)-homogeneous graph.

**Proof.** The ‘if’ part follows from Theorem 2.5 and Theorem 4.10, combined.

We shall now turn to the ‘only if’ part. For this purpose, assume that \( G \) is connected and non-isomorphic to a \( P_4 \)-homogeneous or to a \( 2P_4 \)-homogeneous graph. Our goal is to show that \( \text{Ext}(G) \) is an infinite set for the different cases illustrated in Fig. 3.

If there exists a module \( M \) in \( G \) such that \( |M| \) is connected and nonisomorphic to a chordless chain \( P_k, k \geq 3 \), the conclusion follows from Theorem 3.31.

Assume, next, that every module \( M \) of \( G \) that induces a connected graph is isomorphic to a chordless chain \( P_k \), \( k \geq 1 \). If \( \overline{G} \) is connected, then since \( \overline{G} \) is not \( P_4 \)-homogeneous, it must contain a module \( M \) maximal with respect to set inclusion and non-isomorphic to a chordless chain \( P_r \) with \( r > 2 \). By Theorem 3.31, \( \text{Ext}(\overline{G}) \) is infinite and by Proposition 3.11 this must also be the case for \( \text{Ext}(G) \).

Therefore, in the remainder of the proof we assume that \( \overline{G} \) is disconnected.

Recall that by Theorem 4.1, \( \overline{G} \) contains exactly two connected components \( C_1 \) and \( C_2 \). We shall distinguish the two complementary cases:

1. Neither \( C_1 \) nor \( C_2 \) induce a chordless chain in \( \overline{G} \).
2. At least one of \( C_1 \) or \( C_2 \) induces a chordless chain in \( \overline{G} \).

**Case 1** Neither \( C_1 \) nor \( C_2 \) induce a chordless chain in \( \overline{G} \).

Let \( Q_1 \) (resp. \( Q_2 \)) be the set of new vertices that need to be added to \([C_1]\) (resp. to \([C_2]\)) in order to obtain its basic extension \( U_1 \) (resp. \( U_2 \)).

We construct a connected graph \( H \) by joining \( U_1 \) with \( U_2 \) with a chordless chain \( P_r = x_1x_2 \ldots x_r \) with \( r > 2 \mid V(G) \mid \), and such that

- \( x_1 \) is adjacent to all but one vertex of \( U_1 \),
- \( x_r \) is adjacent to all but one vertices of \( U_2 \), and
- no vertex of \( \{x_2, \ldots, x_{r-1}\} \) is adjacent to any vertex of \( V(U_1) \cup V(U_2) \).

Clearly the graph \( H \) constructed above is prime and, therefore, it contains a minimal prime extension \( H' \) of \( \overline{G} \). We claim that:

\( H' \) contains the whole chain \( P_r \). \hspace{1cm} (1)

In order to argue for (1) observe that neither \( U_1 \) nor \( U_2 \) can be an extension of \( \overline{G} \). Indeed, assume that one of \( U_1 \) or \( U_2 \), say \( U_1 \), is an extension of \( \overline{G} \) and let \( F_2, F_1 \) be two vertex-disjoint subgraphs of \( U_1 \) isomorphic, respectively, to \([C_2]\) and to \([C_1]\). Since \([C_2]\) is connected its vertex set cannot be entirely included in the stable set \( Q_1 \). Therefore, since there is no edge between \([C_2]\) and \([C_1]\) in \( \overline{G} \), we can easily deduce that \( F_1 \neq C_1 \). Let \( F'_2 \) be the set \( V(F_2) \cap C_1 \) and let \( F'_1 \) be the set \( V(F_1) \cap C_1 \). By Proposition 3.8 \( F'_2 \) is a stable set and every vertex of this set has his private neighbour in \( F'_1 \) which contradicts the fact that there is no edge between \( F_1 \) and \( F_2 \). Since neither \([C_1]\) nor \([C_2]\) is isomorphic to a chordless chain, the vertex-set of any induced copy of \([C_1]\) in \( H' \) is formed by a subset of \( C_1 \) and some of the vertices
of the subchain \( P_1 = x_1 \ldots x_t, t < r/2 \) and the vertex set of any induced copy of \( \overline{C_2} \) in \( H' \) is formed by a subset of \( C_2 \) and some of the vertices of the subchain \( P_2 = x_2 \ldots x_r \), with \( z > r/2 \). Since \( H' \) must be connected, it contains the whole chain \( P_r \). Since \( P_r \) has arbitrary length larger than \( |V(G)| \) the proof of (1) is complete.

**Case 2** At least one of \( C_1 \) or \( C_2 \) induces a chordless chain in \( \overline{G} \).

Assume, without loss of generality, that \( C_2 \) induces a chordless chain in \( \overline{G} \). Clearly, \( \overline{C_2} \) is isomorphic to a \( P_1 \) or to a \( P_2 \) or to a \( P_4 \).

**Claim 1.** If \([C_1]\) is not a \( P_4\)-homogeneous graph then \( \text{Ext}(G) \) is an infinite set.

**Proof.** We shall prove that \( \text{Ext}(\overline{G}) \) is an infinite set. Since by assumption \([C_1]\) is not a \( P_4 \)-homogeneous graph, \([C_1]\) is a disjoint union of a set of chordless chains \( R_1, R_2, \ldots, R_l, l > 1 \). Assume w.l.o.g. that \( \text{length}(R_i) > \text{length}(R_j), 1 < i < l \). Consider now the graph \( R_1 \cup \ldots \cup R_{l-1} \); clearly the set of vertices of this graph forms a nontrivial module \( M \) inducing in \([C_1]\) a subgraph different from a subgraph of a \( P_4 \). More precisely \([M]\) is either isomorphic to a chordless chain \( P_r \), \( r > 4 \) or is the disjoint union of a set of chordless chains. It follows that \( M \) induces in \([C_1]\) a connected graph different from a chordless chain and hence we can use our construction in Section 3.2 for obtaining a graph \( H \) isomorphic to a path extension \([C_1] \otimes P_k \) where \( P_k = v_1 \ldots v_k \) is a chordless path of \( k \) vertices. We assume w.l.o.g. that \( P_k \) contains at least 4 vertices. Let \( H^* \) be a minimal prime extension of \([C_1]\) contained in \( H \).

**Fact 1.** If \( H \) contains a subgraph isomorphic to \( \overline{G} \), then \( \text{Ext}(G) \) is an infinite set.

**Proof.** Indeed, by Proposition 3.18 \( H \) is a prime graph and consequently \( H \) contains a minimal prime extension \( F \) of \( \overline{G} \). We claim that \( F \) contains the whole chain \( P_k \). Indeed, in Theorem 3.30 we proved that any minimal prime extension of \([C_1]\) contained in \( H \), contains the whole chain \( P_k \). Since \( F \) contains a minimal prime extension of \([C_1]\) as induced subgraph, \( F \) contains the whole chain \( P_k \), as claimed. Now, since \( P_k \) is of arbitrary length, the result follows.

**Fact 2.** If \( C_2 \) is a singleton then \( \text{Ext}(G) \) is an infinite set.

**Proof.** Since in \( H \) the vertex \( x_k \) is not adjacent to any vertex of \([C_1]\), \( H \) contains an induced subgraph isomorphic to \( \overline{G} \). The result follows from Fact 1.

We assume then in the following that \( H \) does not contain a subgraph isomorphic to \( \overline{G} \) and that \( \overline{C_2} \) is not isomorphic to a \( P_1 \). Let \( y \) be one of the extremities of the chordless chain \( \overline{C_2} \) and let \( Q \) be the graph whose vertex set is \( V(H^*) \cup C_2 \cup \{v\} \), where \( v \) is a new vertex; the edge set of \( Q \) is \( E(Q) = E(H^*) \cup E(\overline{C_2}) \cup \{\{v, z\}\} \), with \( z \neq x_k, y \). In other words \( Q \) is obtained by adding edges between a new vertex \( v \) with all vertices of \( H^* \) except the vertex \( x_k \) and all vertices of the chain \( \overline{C_2} \) except one of its extremities \( y \).

It is easy to see that \( Q \) is a prime graph containing a subgraph isomorphic to \( \overline{G} \).

Let \( Q' \) be a prime extension of \( G \) contained in \( Q \). Let \( G' \) a subgraph of \( Q' \) isomorphic to \( G \), and let \( U_1 \) and \( U_2 \) be two subgraphs of \( G' \) isomorphic to \([C_1]\) and respectively to \( \overline{C_2} \). Since we assumed that \( H \) does not contain a subgraph isomorphic to \( \overline{G} \), \( G' \) is not a subgraph of \( H^* \).

**Fact 3.** \( G' \) does not contain the vertex \( v \).

**Proof.** Assume first that \( v \) belongs to \( U_2 \) then since \( U_1 \) contains more than two vertices \( v \) misses at most two vertices in \( Q \), there would be an edge between \( U_1 \) and \( U_2 \), a contradiction. Assume now that \( v \) belongs to \( U_1 \), then if \( C_2 \) contains more than two vertices there would be an edge between \( U_1 \) and \( U_2 \), a contradiction. Assume now that \( C_2 \) contains exactly two vertices. Since there is no edge between \( U_1 \) and \( U_2 \), \( U_2 \) would be formed by the two non adjacent vertices \( x_k \) and \( y \) which contradicts the connectedness of \( U_2 \). Since we assumed that \( U_2 \) is not isomorphic to a \( P_1 \), we deduce that \( G' \) does not contain the vertex \( v \), as claimed.

Fact 3 implies that since \( U_1 \) is not isomorphic to a subgraph of a \( P_4 \) and \( U_1 \) is connected, \( U_1 \) is entirely contained in \( H^* \).

**Fact 4.** \( Q' \) contains \( H^* \) as induced subgraph.

**Proof.** Assume the contrary, then since any subgraph of \( Q' \) isomorphic to \([C_1]\) is contained in \( H^* \), there must be a subgraph \( H_1 \) of \( Q' \) strictly contained in \( H^* \) which is connected, it contains a subgraph isomorphic to \([C_1]\) and is maximal with respect to set inclusion and the above properties. The maximality of \( H_1 \) implies that it contains the
vertex $x_k$ for otherwise the neighbourhood of $H_1$ in $Q'$ would be the vertex $v$ and consequently $V(H_1)$ would be a nontrivial module of $Q'$, a contradiction. Also, $H_1$ can not be a prime graph for otherwise $H^*$ would not be a minimal prime extension of $[C_1]$, a contradiction. Let $W$ be a maximal nontrivial module of $H_1$, then if $W$ does not contain the vertex $x_k$, $W$ would be a non trivial module of $Q'$, a contradiction. But since $x_k$ has a unique neighbour in $H^*$ which is the vertex $x_{k-1}$, $W$ can not contain the vertex $x_{k-1}$, for otherwise a neighbour of $W$ in $H_1$ would not be adjacent to $x_k$, a contradiction. It follows that the neighbourhood of $W$ in $H_1$ is the vertex $x_{k-1}$. Now, $W$ must be formed by $x_k$ and exactly one other vertex, for otherwise $W \setminus \{x_k\}$ would be a non trivial module in $Q'$, a contradiction. Let $z$ be the second vertex of $W$. Since $x_{k-1}$ is the neighbourhood of $W$ in $H_1$, $z$ is either the vertex $x_{k-2}$ or a vertex belonging to the neighbourhood of $M$ in $[C_1]$. Assume first that $z$ is the vertex $x_{k-2}$, then the vertex $x_{k-3}$ which distinguishes the vertices $x_{k-2}$ and $x_k$ does not belong to $H_1$. It follows that since $H_1$ contains more than three vertices ([$C_1$] contains at least four vertices), there must be a neighbour, say $u$, of $x_{k-1}$ belonging to $H_1$. The vertex $u$ must belong to $V(R_i)$ which is the neighbourhood of $M$ in $[C_1]$ and we obtain a contradiction since $x_{k-2}$ would be also adjacent to $u$. Consequently, $z$ belongs to $R_i$. It follows that no vertex of $M \cup \{x_1, \ldots, x_{k-2}\}$ belongs to $H_1$ since each one of these vertices is adjacent to $z$ and not adjacent to $x_k$. It is easy to see now that $R_i$ cannot be isomorphic to a $P_1$ for otherwise the graph $H_1$ would be isomorphic to a $P_3$, a contradiction. Consequently, $R_i$ contains at least two vertices and hence since $H_1$ is assumed to be connected, the only vertices of $H$ that can belong to $H_1$ are $x_k, x_{k-1}$, the vertices of $R_i$ and the vertex $w$ which is the only vertex needed in the basic extension of $[C_1]$ for ‘breaking’ the nontrivial module $V(R_i)$ of $[C_1]$. But since $R_i$ contains at least two vertices, it is easy to see that $M$ must contain at least four vertices and we obtain a contradiction. □

Since $H^*$ contains the whole chain $P_k$ which is of arbitrary length, we deduce that Ext($G$) is an infinite set, as claimed. □

Assume now that $[C_1]$ is a $P_4$-homogeneous graph then since $G$ is not a $2P_3$-homogeneous graph, $[C_2]$ cannot be a singleton, for otherwise $G$ would be a pseudo-gem, a contradiction. It follows that $[C_1]$ cannot be $2K_2$-free.

Let $L$ be the shortest chordless chain containing $[C_1]$ as induced subgraph and $G_1$ be the graph obtained by adding all missing edges between $L$ and $[C_2]$. Let $H$ be a prime graph containing $G$ obtained by adding to $G_1$ an $L$-pseudopath $P_k = x_1, x_2, \ldots, x_k, k \geq 4$ and $k$ even, that satisfies:

1. for $i = 1, \ldots, k - 1$, if $i$ is odd then $x_i, x_{i+1}$ is an edge of $H$ and if $i$ is even then $x_i, x_{i+1}$ is a non edge of $H$
2. $x_1$ is adjacent to all but one vertices of $[C_1]$
3. the vertex $x_k$ is adjacent to $[C_2]$ as follows:
   (a) If $[C_2]$ is isomorphic to a $P_3 = ab$ then $x_k$ is adjacent to $a$ and not adjacent to $b$
   (b) If $[C_2]$ is isomorphic to a $P_3 = abc$ with $bc$ the unique edge of $[C_2]$, $x_k$ is adjacent to $a$ and $b$ and not adjacent to $c$
   (c) If $[C_2]$ is isomorphic to a $P_4 = abcd$, $x_k$ is adjacent to the middle vertices $b$ and $c$ of $[C_2]$ and not adjacent to $a$ and $d$.

Clearly, since $P_k$ is an $L$-pseudopath, every vertex of $P \setminus x_k$ is adjacent to every vertex of $L \cup [C_2]$, $x_{2i+1}$ is adjacent to any vertex $x_j$, $j < 2i$, and $x_{2i}$ is not adjacent to any vertex $x_j$, $j < 2i - 1$, $i = 1, \ldots, \frac{k}{2}$.

Claim 2. $P_k$ is $2K_2$-free and there are not two nonadjacent vertices in $P_k$ that are both adjacent to a vertex of $[C_1]$.

Proof. Indeed assume first that there exists a $2K_2$ in $P_k$. Let $x_r x_s$ and $x_t x_v$ be the two edges of this $2K_2$ then one of $r, s$, say $r$ and one of $t, v$, say $t$ is odd and we obtain a contradiction since $x_t$ is adjacent to $x_r$. Let $x_j$ and $x_l$ be two nonadjacent vertices of $P_k$, then at least one of $j, l$, say $j$ is even and consequently $x_j$ is not adjacent to any vertex of $P_k$, a contradiction. □

Since $H$ is a prime graph it contains a minimal prime extension $H'$ of $G$. We claim that $H'$ contains entirely the $L$-pseudopath $P_k$. Indeed, since $[C_1]$ contains a $2K_2$ by Claim 2 there cannot be a subgraph of $P_k$ isomorphic to $[C_1]$ and hence $[C_1]$ is not entirely contained into $P_k$. From the other hand since $[C_2]$ contains two nonadjacent vertices, Claim 2 implies that $[C_2]$ cannot be a subgraph of $P_k$. The reader can easily verify now that $H'$ must contain the whole pseudopath $P_k$. Since this pseudopath is of arbitrary length, we deduce the claimed result. □
5. Concluding remarks

First of all we may observe that the proofs given in the previous section suggest a general method for enumerating in the finite case all minimal prime extensions of a graph $G$. Consider for example the case where $G$ has two connected components, one being an isolated vertex and the second inducing a $P_4$ (i.e. $G$ is the complementary graph of a $Gem$). It is easy to see that from the different cases examined in the proof of the Lemma 4.3, we can derive all extensions of $G$.

It must be pointed out here that, since no general result had been available concerning the set of minimal prime extensions in the finite case, it was necessary for obtaining this set to examine separately each particular case of the graphs under consideration — see, for example, [6,17]. Hence, now it becomes interesting to enumerate by a systematic way derived from the results given in this paper, all the minimal prime extensions in the finite case. If the number of minimal prime extensions is large, instead of exhibiting all these extensions we could propose a simple algorithm for it. In this way for instance, we could characterize all the new classes of perfect graphs which are the substitution-composite of subclasses of $P_4$-homogeneous and $2P_4$-homogeneous graphs already been showed to be perfect.

It would also be interesting to search for different methods generating infinite sets of extensions which could be for instance beneficial to a better understanding of the structure of prime graphs that, to this day, are not well understood. Both of these directions are for us an exciting area for further work.

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