Assignment 3 – Solutions

Objective: The main objective of your third assignment was for you to practice reasoning about functions and their various properties. In dealing with functions, you were expected to bring to bear set-theoretical concepts that you had studied in the previous chapters.

Statement of your assignment:

Problem 1. [20%] Let $A$ and $B$ be sets and let $f: A \rightarrow B$ be a function from $A$ to $B$. Let $S$ and $T$ be arbitrary subsets of $A$. Find a counter-example to the claim that

$$f(S \cap T) = f(S) \cap f(T).$$

![Figure 1: Illustrating the solution to Problem 1.](image)

Solution. Consider the sets $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$ as well as the subsets $S = \{a, b\}$ and $T = \{b, c\}$ of $A$. Referring to Figure 1, define the function $f: A \rightarrow B$ as follows

- $f(a) = f(c) = y$;
- $f(b) = x$;
- $f(d) = z$.

Visibly,

- $f(S \cap T) = f(b) = y$;
- $f(S) \cap f(T) = \{x, y\} \cap \{x, y\} = \{x, y\}$

confirming that $f(S \cap T) \neq f(S) \cap f(T)$.

Problem 2. [20%] Determine which of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$
2.1. \( f(x) = -x + 1; \)
2.2. \( f(x) = x^2 - 2x + 1. \)

Solution. Problem 2.1 involves a special case of the function \( f(x) = ax + b \) for \( a \neq 0 \) discussed in class.\(^1\) Indeed, in class we had established that the function \( f(x) = ax + b \) was both one-to-one and onto and, consequently, a bijection.

With Problem 2.1 out of the way, we now turn to Problem 2.2. A bit of thought (and, of course, algebra) reveals that \( f(x) = (x - 1)^2 \). In turn, this implies that \( f(0) = f(2) = 1 \), invalidating the claim that \( f(x) \) is one-to-one. Since the function is not one-to-one is cannot be a bijection either.

Problem 3. [20%] Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and let \( f(x) > 0 \). Show that \( f(x) \) is strictly increasing if and only if \( \frac{1}{f(x)} \) is strictly decreasing.

Solution. Consider the function \( g : \mathbb{R} \rightarrow \mathbb{R} \) defined as

\[
g(x) = \frac{1}{f(x)}. \]

In this notation, our job is to show that \( f(x) \) is strictly increasing if and only if \( g(x) \) is strictly decreasing. For this purpose, consider arbitrary reals \( x \) and \( y \) such that \( x < y \). Assume, for now, that \( f(x) \) is strictly increasing. In other words,

\[
f(x) < f(y). \tag{1} \]

Recall that we were told that \( f(x) > 0 \) for all \( x \); in particular, \( f(y) > 0 \). As a result, we can divide the inequality (1) by \( f(x)f(y) > 0 \) to obtain

\[
\frac{1}{f(y)} < \frac{1}{f(x)}
\]

which is the same as \( g(x) > g(y) \). We just proved that if \( f(x) \) is strictly increasing then \( g(x) \) is strictly decreasing. The converse statement proceeds along similar lines and is, therefore, omitted.

Problem 4. [20%] Let \( f \) and \( g \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( f(x) = ax + b \) and \( g(x) = cx + d \) where \( a, b, c, d \) are constants. What relation should exist between \( a, b, c, d \) for \( f \circ g = g \circ f \) to hold?

Solution. Let us evaluate \( f \circ g \) and \( g \circ f \). We write

\[
\bullet \ f \circ g(x) = f(g(x)) = a(cx + d) + b = acx + ad + b;
\]

\[
\bullet \ g \circ f(x) = g(f(x)) = c(ax + b) + d = acx + bc + d.
\]

It follows that in order for \( f \circ g = g \circ f \) to hold, \( a, b, c, d \) must satisfy the relation \( ad + b = bc + d \).

Problem 5. [20%] Give examples of functions from \( \mathbb{N} \) to \( \mathbb{N} \) that are

5.1. One-to-one but not onto;
5.2. Onto but not one-to-one;
5.3. Both one-to-one and onto;
5.4. Neither one-to-one nor onto.

Solution. To solve Problem 5.1, we define the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined as

\[
f(n) = n + 1.
\]

Refer to Figure 2 for an illustration.

\(^1\)To refresh your memory, you may want to refer to your or someone else’s notes.
It is fairly obvious that the function $f$ thus defined associates with every natural number its "successor". Clearly, $f(x)$ is one-to-one. Is is onto? Well, it is not, since the number 0 is the image of no number under $f$: think of it as saying that 0 is the successor of no natural number.

Turning to Problem 5.2, our goal is to find a function $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one. For this purpose, define the function $f(n)$ as

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 1 & \text{if } n > 0 \end{cases}$$

and refer to Figure 3 for an illustration.

It is easy to confirm that the function $f(n)$ defined above is onto; however, by definition it is not one-to-one since 0 and 1 have the same image: to wit, $f(0) = f(1) = 0$.

Moving on, we now address Problem 5.3. In this case we need to exhibit a function $f : \mathbb{N} \to \mathbb{N}$ that is both
one-to-one and onto. The simplest such function that comes to mind is

\[ f(n) = n \]

and is illustrated in Figure 4.

![Figure 4](image_url)

**Figure 4: Illustrating the function \( f(n) \) for Problem 5.3.**

Finally, Problem 5.4 asks us to invent a function \( f : \mathbb{N} \to \mathbb{N} \) that is neither one-to-one nor onto. Of course, several solutions are possible. We chose to define the desired function \( f(n) \) by minimal alterations to the function defined in Figure 4. More formally, we define the function as follows:

\[
f(n) = \begin{cases} 
0 & \text{if } n = 0, 1 \\
n & \text{if } n > 1
\end{cases}
\]

![Figure 5](image_url)

**Figure 5: Illustrating the function \( f(n) \) for Problem 5.4.**

It is clear that the function thus defined is not onto since 1 is the image of no number; it is not one-to-one since, by definition, \( f(0) = f(1) = 0 \).