Abstract

A multiple scales approach is used to approximate the effects of nonparallelism and streamwise curvature on the stability of three-dimensional disturbances in incompressible flow. The multiple scales approach is implemented with the full second-order system of equations. A detailed exposition of the source of all terms is provided.

1 Introduction

Multiple scales analysis is a useful tool in fluid mechanics. It is often used to derive weakly nonlinear and weakly nonparallel corrections to eigenvalue problems that arise in the study of the stability of laminar flows. Although the general procedures are well-known [1, 2], the novice can easily become bogged down in the derivation of the appropriate equations. This note is intended to clarify the detailed steps that are needed in the analysis.

Previous analyses of the weakly nonparallel and curvature effects have been based on a reduction of the governing equations to a first-order system of equations [1, 2, 3]. This reduction in order was accomplished with the introduction of many additional variables; although these additional variables are mathematically well-defined, the physical importance of their adjoint counterparts, which are required for the solution of the problem, is nonintuitive. Historically, the first-order formulation of stability equations was popularized at a time when the numerical methods for the solution of linear eigenvalue problems were first being developed. With the addition of orthonormalization, previously developed marching techniques for first-order systems of differential equations could be applied to eigenvalue problems that were formulated as first-order systems. As computing power has increased, matrix methods, which are more suitable for obtaining the global eigenvalue spectrum, have largely superseded marching methods. However, to compute nonparallel effects, researchers have continued to use the first-order formulation [4, 5]. We show that the multiple scales analysis can also be performed easily in the context of the original equations, with the original, physically meaningful variables.

Masad and Malik [4] previously used multiple scales analysis to include surface curvature for the case of flow over a swept circular cylinder, hence a cylindrical coordinate system was appropriate for their application. Our goal is to extend the analysis to a wider range of boundary layers, hence the analysis is presented in surface-fitted coordinates. The mathematics are presented in section 2; numerical verification is provided in section 3.

2 Mathematical Model

For expository purposes, we focus on the linear stability of a three-dimensional, incompressible boundary layer. However, apart from the specification of the operators involved, most of the analysis is directly applicable to compressible flows as well. All lengths are nondimensionalized by a length scale L^*; all velocities, by a velocity scale U^*; time, by the ratio L^*/U^*; and pressure, by $\rho^* U^{*2}$, where ρ^* is the density. The flow Reynolds number is $R = U^* L^*/\nu^*$, where ν^* is the dimensional kinematic viscosity. The streamwise direction is denoted by x; the wall-normal direction, by y; and the spanwise direction, by z. The base flow

$$\mathbf{Q}_0 = (U_0, \epsilon V_1, W_0) \tag{1}$$

consists of $O(1)$ streamwise and spanwise components and a small $O(\epsilon)$ surface-normal component. For the zero-pressure-gradient boundary layer, only the surface-normal component of mean velocity has an $O(\epsilon)$ term; other velocity component corrections enter at higher orders. To simplify the exposition, we assume that the mean flows considered here can be approximated in a similar manner. All mean-flow components are invariant in both time t and spanwise direction. The mean flow varies rapidly in the surface-normal direction and changes slowly in the streamwise direction. The slow variation in the streamwise direction suggests that a slow scale $x_1 = \epsilon x$ should be introduced. We consider the disturbance quantities

$$\mathbf{q}(x, y, z, t) = [A(x_1)\mathbf{q}_0(x_1, y) + \epsilon \mathbf{q}_1(x_1, y) + ...]e^{i\theta} \tag{2}$$

where $\mathbf{q} = [u, v, w, p]^T$ represents the vector of the streamwise, surface-normal, and spanwise disturbance velocities, and the disturbance pressure, respectively. The variable $A(x_1)$ represents the amplitude of the disturbance, and $\mathbf{q}_0(x_1, y)$ denotes the vector of the normalized quasi-parallel eigenfunctions. The $O(\epsilon)$ term in Eq. (2) denotes the nonparallel correction to the disturbance. The argument θ of the exponential function is defined in terms of the streamwise and spanwise wave numbers and the circular frequency of the disturbance. That is,

$$\frac{\partial \theta}{\partial x} = \alpha_0(x_1), \qquad \frac{\partial \theta}{\partial z} = \beta, \qquad \frac{\partial \theta}{\partial t} = -\omega \tag{3}$$

Logarithmic differentiation of Eq. (2) with respect to x yields the growth rate of the disturbance:

$$-\operatorname{Imag}(\alpha) = -\operatorname{Imag}[\alpha_0(x_1) + \epsilon\alpha_1] + O(\epsilon^2) \tag{4}$$

The term $-\operatorname{Imag}[\alpha_0(x_1)]$ is the quasi-parallel growth rate. The $\epsilon\alpha_1$ term is given by

$$\epsilon\alpha_1 = -i\left[\frac{1}{A}\frac{\partial A(x_1)}{\partial x_1} + \frac{1}{q_0(x_1, y_1)}\frac{\partial q_0(x_1, y_1)}{\partial x_1}\right] \tag{5}$$

of which the negative of the imaginary part represents the nonparallel growth of the amplitude function A and the change in shape of the quasi-parallel eigenfunction. Unlike the quasi-parallel case, the growth rate in the nonparallel context depends on both the particular disturbance quantity q_0 and the surface-normal location y_1. Here, we choose $q_0(x_1, y_1)$ to represent u_0 (i.e., the streamwise velocity component of $\mathbf{q}_0$ at the surface-normal location at which u_0 is at a maximum). This location varies with x_1.

To obtain the nonparallel equations, the mean and disturbance quantities are substituted into the Navier-Stokes equations. Equations for the mean flow $\mathbf{Q}_0(x_1, y)$ are solved separately. The disturbance equations are obtained by linearizing and equating like powers of ϵ. The $O(1)$ set of equations can be written as a multivariable second-order form of the Orr-Sommerfeld equation:

$$\mathbf{L}_{os}\mathbf{q}_0 = \mathbf{0} \tag{6}$$

with boundary conditions,

$$u_0 = v_0 = w_0 = 0 \quad \text{at } y = 0 \tag{7}$$

and

$$u_0, v_0, w_0, p_0 \to 0 \quad \text{as } y \to \infty \tag{8}$$

With the notation $D = d/dy$, $k^2 = (\alpha_0^2 + \beta^2)$, and $S = k^2 + iR(\alpha_0 U_0 - \omega) - D^2$, the Orr-Sommerfeld operator for incompressible flow is

$$\mathbf{L}_{os} = \begin{pmatrix} S & RDU_0 & 0 & i\alpha_0 R \\ 0 & S & 0 & RD \\ 0 & 0 & S & -i\beta R \\ i\alpha_0 & D & i\beta & 0 \end{pmatrix} \tag{9}$$

Note that $\mathbf{L}_{os}$ is a matrix operator that can be expressed as

$$\mathbf{L}_{os} = \mathbf{L}_0 + (i\alpha_0)\mathbf{L}_1 + (i\alpha_0)^2\mathbf{L}_2 \tag{10}$$

where $\mathbf{L}_0$, $\mathbf{L}_1$, and $\mathbf{L}_2$ are independent of α_0 and depend only on the mean flow and the stability parameters ω and β. The operator $\mathbf{L}_0$ can be thought of as arising from those terms in the primitive-variable equations that do not contain any x derivatives. Similarly, the operators $\mathbf{L}_1$ and $\mathbf{L}_2$ are the coefficient matrices for terms that contain only first and second x derivatives respectively. The multiple scales analysis indicates that the first derivative of the assumed zeroth-order solution $A\mathbf{q}_0 e^{i\theta}$ with respect to x can be written as

$$\frac{\partial(A\mathbf{q}_0 e^{i\theta})}{\partial x} = \left[i\alpha_0 A\mathbf{q}_0 + \epsilon \left(\frac{\partial A}{\partial x_1}\mathbf{q}_0 + A\frac{\partial \mathbf{q}_0}{\partial x_1} \right) \right] e^{i\theta} \tag{11}$$

and the second derivative with respect to x can be written as

$$\frac{\partial^2(A\mathbf{q}_0 e^{i\theta})}{\partial x^2} = (i\alpha_0)^2 A\mathbf{q}_0 e^{i\theta}$$

$$+ \epsilon \left[2i\alpha_0 \left(\frac{\partial A}{\partial x_1}\mathbf{q}_0 + A\frac{\partial \mathbf{q}_0}{\partial x_1} \right) + A\mathbf{q}_0 \frac{\partial(i\alpha_0)}{\partial x_1} \right] e^{i\theta} + O(\epsilon^2) \tag{12}$$

The $O(\epsilon)$ problem is obtained by substituting Eqs. (2), (11), and (12) into the Navier-Stokes and continuity equations. Thus we find,

$$\mathbf{L}_{os}\mathbf{q}_1 = -\frac{\partial A}{\partial x_1}\mathbf{M}_0\mathbf{q}_0 - A\left(\mathbf{M}_0 \frac{\partial \mathbf{q}_0}{\partial x_1} + \frac{\partial i\alpha_0}{\partial x_1}\mathbf{M}_1\mathbf{q}_0 + \mathbf{N}\mathbf{q}_0 \right) \tag{13}$$

where $\mathbf{M}_0 = \mathbf{L}_1 + 2i\alpha_0\mathbf{L}_2$; $\mathbf{M}_1 = \mathbf{L}_2$; and $\mathbf{N}$ contains nonparallel mean-flow terms:

$$\mathbf{N} = \begin{pmatrix} -R(\frac{\partial U_0}{\partial x_1} + V_1 D) & 0 & 0 & 0 \\ 0 & -R(DV_1 + V_1 D) & 0 & 0 \\ -R\frac{\partial W_0}{\partial x_1} & 0 & -RV_1 D & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

The boundary conditions on the $O(\epsilon)$ problem are

$$u_1 = v_1 = w_1 = 0 \quad \text{at } y = 0 \tag{15}$$

and

$$u_1, v_1, w_1, p_1 \to 0 \quad \text{as } y \to \infty \tag{16}$$

Because the homogeneous part of the $O(\epsilon)$ problem is the same as the $O(1)$ problem, the $O(\epsilon)$ problem has a solution if and only if a special condition (known as the solvability condition) is satisfied [6]. For the solvability condition to be satisfied, the right-hand side of Eq. (13) must be orthogonal to every member within the null space of the adjoint of $\mathbf{L}_{os}$. Hence, if we dot-multiply the adjoint solution $\mathbf{q}_0^H$ with the right-hand side of Eq. (13), integrate over the domain, and set the result equal to zero, we obtain the expression

$$\frac{1}{A}\frac{\partial A}{\partial x_1} = -\frac{\int_0^\infty \mathbf{q}_0^H \cdot \left(\mathbf{M}_0 \frac{\partial \mathbf{q}_0}{\partial x_1} + \frac{\partial i\alpha_0}{\partial x_1}\mathbf{M}_1\mathbf{q}_0 + \mathbf{N}\mathbf{q}_0\right) dy}{\int_0^\infty \mathbf{q}_0^H \cdot (\mathbf{M}_0\mathbf{q}_0)\, dy} \tag{17}$$

The imposition of the solvability condition ensures that no secular terms exist in the solution. (For a discussion of secular terms, see, for example, Ref. [7].)

Typically, the adjoint problem is obtained by first transforming Eq. (6) into a set of first-order differential equations [1, 2]. However, here we show that the adjoint problem is easily obtained from the governing equations in their natural form. First, premultiply Eq. (6) by $\mathbf{q}_0^H$ and then integrate over the surface-normal direction. The adjoint operator is obtained by using integration by parts and the linear-algebra identity, $(\mathbf{y}\mathbf{P}\mathbf{x})^T = \mathbf{x}^T\mathbf{P}^T\mathbf{y}^T$, where the superscript T denotes transpose. The adjoint operator can be written as

$$\mathbf{L}_{os}^H = \mathbf{A}^T D^2 - \mathbf{B}^T D + (\mathbf{C} - D\mathbf{B}^T) \tag{18}$$

where

$$\mathbf{L}_{os} = \mathbf{A}D^2 + \mathbf{B}D + \mathbf{C} \tag{19}$$

The component matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are defined as

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{20}$$

$$\mathbf{C} = \begin{pmatrix} k^2 + iR(\alpha_0 U_0 - \omega) & RDU_0 & 0 & i\alpha_0 R \\ 0 & k^2 + iR(\alpha_0 U_0 - \omega) & 0 & 0 \\ 0 & 0 & k^2 + iR(\alpha_0 U_0 - \omega) & -i\beta R \\ i\alpha_0 & 0 & i\beta & 0 \end{pmatrix}$$

Note that $\mathbf{A} = \mathbf{A}^T$; and for incompressible flow, $D\mathbf{B}^T = 0$ so that the adjoint operator can be simplified.

Two terms on the right-hand side of Eq. (13) must be determined before the solvability condition can be applied. These terms are the x_1 derivatives of $i\alpha_0$ and $\mathbf{q}_0$. Note that the partial derivative of the $O(1)$ equation Eq. (6) with respect to x_1 must satisfy

$$\frac{\partial \mathbf{L}_{os}\mathbf{q}_0}{\partial x_1} = \frac{\partial \mathbf{L}_{os}}{\partial \mathbf{Q}_0}\frac{\partial \mathbf{Q}_0}{\partial x_1}\mathbf{q}_0 + \frac{\partial \mathbf{L}_{os}}{\partial(i\alpha_0)}\frac{\partial(i\alpha_0)}{\partial x_1}\mathbf{q}_0 + \mathbf{L}_{os}\frac{\partial \mathbf{q}_0}{\partial x_1} = 0 \tag{21}$$

After the known terms are moved to the right-hand side, Eq. (21) can be rewritten as

$$\mathbf{L}_{os}\frac{\partial \mathbf{q}_0}{\partial x_1} = -\left[\frac{\partial \mathbf{L}_{os}}{\partial \mathbf{Q}_0}\frac{\partial \mathbf{Q}_0}{\partial x_1}\mathbf{q}_0 + \frac{\partial \mathbf{L}_{os}}{\partial (i\alpha_0)}\frac{\partial (i\alpha_0)}{\partial x_1}\mathbf{q}_0\right] \tag{22}$$

As in Eq. (13), this expression has a nontrivial solution if and only if the right-hand side is orthogonal to every solution of the homogeneous adjoint problem. The solvability condition that determines $\partial(i\alpha_0)/\partial x_1$ is obtained in the same manner as described previously to obtain the result in Eq. (17). With $\partial(i\alpha_0)/\partial x_1$ being known, equation (22) can then be solved for $\partial \mathbf{q}_0/\partial x_1$. Finally, both terms are available to evaluate $\frac{1}{A}\frac{\partial A}{\partial x_1}$ using Eq. (17).

To obtain the complete $O(\epsilon)$ correction to the growth rate in Eq. (5), one must make a specific choice for $\partial q_0/\partial x_1$, i.e., chose the physical quantity q_0 of interest and the surface normal location, y_1 where the growth rate is to be determined. The term $\partial q_0/\partial x_1$ accounts for the slow change in the shape of the eigenfunction with downstream distance. Two commonly used choices for $q_0(x_1, y_1)$ are (1) u_0 at the y_1 value at which u_0 is at its maximum (referred to as the $u_{\max}$ choice below) and (2) u_0 at a predefined y_1 location. In both cases, $\partial \mathbf{q}_0/\partial x_1$ from Eq. (22) defines the value of $\partial q_0/\partial x_1$ after the selection of variable and surface-normal location has been made.

As shown by Masad and Malik [4], weak surface curvature can be treated as a small perturbation to the problem without curvature. The modifications that are needed to incorporate curvature effects into the equations above are minor and are easily implemented. First, the curvature introduces terms that are proportional to $\frac{1}{r+y}$ where r is the local radius of curvature. For a large r, these additional terms are simply proportional to the small surface curvature $\epsilon\kappa = \frac{1}{r}$. Because the terms are of $O(\epsilon)$, they can be easily added to the matrix $\mathbf{N}$ in Eq. (13). In body-fitted coordinates, the necessary additional terms can be surmised from Lin and Reed [8] and are given as matrix $\mathbf{N}_c$:

$$\mathbf{N}_c = \begin{pmatrix} RV_1\kappa & RU_0\kappa & 0 & 0 \\ -2RU_0\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \end{pmatrix} \tag{23}$$

Secondly, the curvature modifies the x derivative terms so that they are proportional to $\frac{r}{r+y}$. For large r, $\frac{r}{r+y}$ can be approximated by $1 - \frac{y}{r} = 1 - \epsilon\kappa y$. If we keep only those terms that are first order in $\epsilon\kappa$, then the additional $O(\epsilon)$ terms are $\mathbf{M}_c\mathbf{q}_0$, where

$$\mathbf{M}_c = i\alpha_0\kappa y \mathbf{L}_1 \tag{24}$$

Hence, the modified $O(\epsilon)$ equation that must satisfy the solvability condition is

$$\mathbf{L}_{os}\mathbf{q}_1 = -\frac{\partial A}{\partial x_1}\mathbf{M}_0\mathbf{q}_0 - A\left[\mathbf{M}_0\frac{\partial \mathbf{q}_0}{\partial x_1} + \frac{\partial i\alpha_0}{\partial x_1}\mathbf{M}_1\mathbf{q}_0 + (\mathbf{M}_c + \mathbf{N} + \mathbf{N}_c)\,\mathbf{q}_0\right] \tag{25}$$

This equation is solved with the same procedures that are required for Eq. (13).

The numerical solution of these equations involves the discretization of the equations with the use of Chebyshev polynomials. A staggered grid is used for the pressure variable and, hence, the continuity equation. An iterative procedure is used to determine the quasi-parallel eigenvalue and the eigenfunction. Direct solves are used to solve the systems of equations. The computer code used for the calculations is a modified version of SPECLS [9].

3 Verification

Figure 1 shows pointwise comparisons of the nonparallel results with the multiple scales results obtained by El-Hady [2] for incompressible flow at $R = 1000$, for a variety of nondimensional frequencies ($F = 2\pi f^* \nu^* / U^{*2}$, where f^* is the dimensional frequency). The surface-normal axis represents the change in the growth rate ($-\mathrm{Imag}(\epsilon\alpha_1)$) due to the nonparallel effects at the wave number of maximum quasi-parallel growth rate. Both our results and those of El-Hady are based on the streamwise component of velocity at the surface-normal location at which the linearized solution is maximum. The only significant difference between the approaches used involves the order of the system of equations. El-Hady used a first-order system of equations; we use the second-order system. The agreement of the two sets of results strongly suggests that both approaches are correctly implemented.

A similar comparison was made to test the implementation of the weak curvature terms. In this case, we consider the incompressible flow over a circular cylinder that is swept at an angle of 60.5° relative to the free stream. The Reynolds number, based on the free-stream speed and the radius of the cylinder, is 2.0×10^6. We consider only steady ($F = 0$) disturbances with a spanwise wave number of 0.8 (normalized with the radius). Weakly nonparallel effects are neglected. The comparisons are made with earlier data from Masad and Malik [4]. Masad and Malik [4] also used a multiple scales analysis to incorporate the curvature effect; however, because their work focused only on the cylinder case, they used a cylindrical coordinate system to derive the perturbation equations. They also used a somewhat less accurate numerical scheme that employed finite differences. However, as shown below, the results that we obtained are in excellent agreement with their results. In Fig. 2(a), the real part of the streamwise wave number α is shown both with and without curvature for various angles θ^* relative to the stagnation point on the cylinder. The effect of the curvature terms is relatively small here, but the results obtained with our multiple scales analysis essentially duplicate those obtained by Masad and Malik [4]. A comparison of the results for the imaginary part of α in Fig. 2(b) show the same excellent agreement.

4 Acknowledgments

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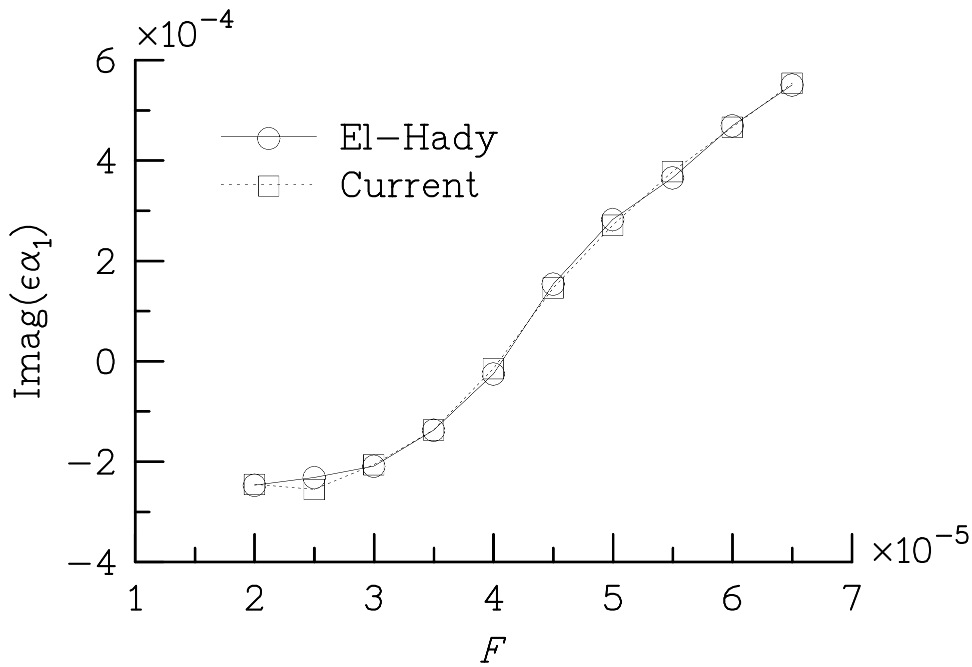
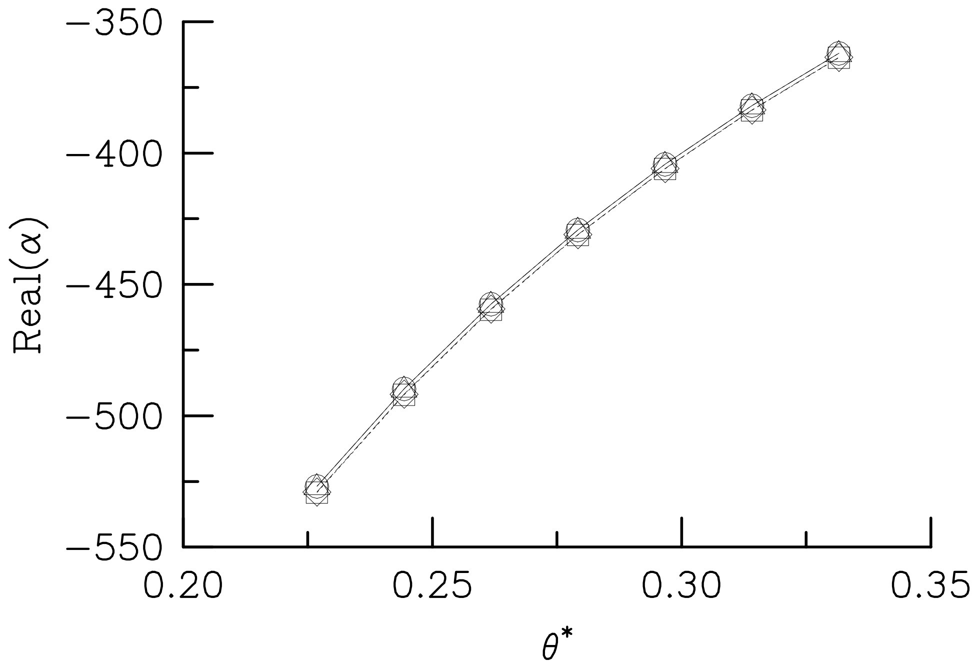
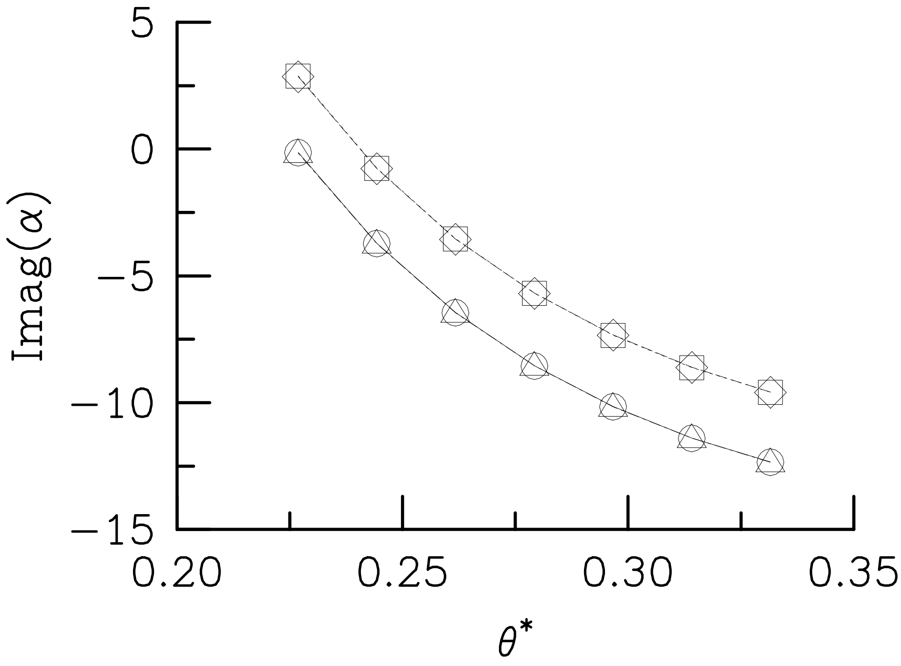


Figure 1. Nonparallel correction to growth rate at $R = 1000$ in a Blasius boundary layer: comparison with El-Hady [2].



(a) Real part of α.



(b) Imaginary part of α.

Figure 2. Multiple scales approach for weak surface curvature: comparison with data from Masad [4]. $R = 2.0 \times 10^6$; flow angle 60°; $F = 0.0$. $\circ$ current quasi-parallel; $\square$ current with curvature; $\triangle$ Masad quasi-parallel; $\diamond$ Masad with curvature.