

# Arc Length Based Grid Distribution For Surface and Volume Grids

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## Abstract

Techniques are presented for distributing grid points on parametric surfaces and in volumes according to a specified distribution of arc length. Interpolation techniques are introduced which permit a given distribution of grid points on the edges of a three-dimensional grid block to be propagated through the surface and volume grids. Examples demonstrate how these methods can be used to improve the quality of grids generated by transfinite interpolation.

## Introduction

For many applications, the construction of grids on parametric surfaces is a necessary step in grid generation. If the surface is defined by a spline or some other type of parametric equation, the grid is usually constructed by generating a grid in parametric space and then evaluating the surface equations at the grid points in parametric space to obtain grid points on the surface. This method causes the grid to depend on the parametric definition of the surface. If the parametric representation of the surface is not smooth, as may be the case when several surfaces are patched together to form a single surface, then the surface grid will not be smooth. There may also be large differences between the distribution of grid points in parametric space and the distribution of points on the surface.

This report will examine the application of arc length distributions in redistributing grid points on surfaces and in three-dimensional volumes. Various redistribution schemes have been developed by Soni [1] and Yu, *et al.* [2]. The objective here is to redistribute the points based on the interpolation of relative arc length from the boundary curves or surfaces. Using this technique relative spacing off of the boundary can be prescribed. The commonly used algebraic interpolation methods do not maintain a uniform spacing along boundary surfaces. In constructing multiblock structured grids, this can lead to discontinuities in grid

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spacing along block boundaries. Of course, in the three-dimensional case there are no parametric equations, and the grid point locations are calculated directly from the arc length distribution. The interpolated relative arc lengths can also be modified to yield a prescribed absolute spacing at the boundaries. The distribution of points depends on arc length and not on the parametric representation of the surface. A more precise distribution can be achieved by repeatedly applying the redistribution scheme in an iterative manner. Convergence has been shown to be fast in most cases and does not appear to depend on the grid size.

### Surface Grid Distribution

A parametric surface is given by the equation

$$p = p(u, v), \quad \text{where } p = (x, y, z).$$

The redistribution scheme for surface grids assumes an initial grid which is associated with a corresponding grid in parametric space. The grid points in parametric space are given as

$$(u_{i,j}, v_{i,j}), \quad i = 1, \dots, I, \quad j = 1, \dots, J.$$

The grid points on the surface are denoted as

$$p_{i,j} = p(u_{i,j}, v_{i,j}), \quad i = 1, \dots, I, \quad j = 1, \dots, J.$$

For this initial grid, relative arc length distributions in each coordinate direction are given as

$$s_{i,j} = \frac{\sum_{k=2}^{k=i} \|p_{k,j} - p_{k-1,j}\|}{\sum_{k=2}^{k=I} \|p_{k,j} - p_{k-1,j}\|} \quad (1)$$

$$t_{i,j} = \frac{\sum_{k=2}^{k=j} \|p_{i,k} - p_{i,k-1}\|}{\sum_{k=2}^{k=J} \|p_{i,k} - p_{i,k-1}\|}. \quad (2)$$

The grid point distribution along the boundary of the surface can be prescribed simply by marching along the boundary curves. However, the problem of distributing points on the interior is not as simple. For one thing, the most common grid generation scheme, usually referred to as transfinite interpolation, relies on interpolation of the parametric variables from their boundary values. As a result, the distance between points in the interior does not relate directly to the distances between corresponding points on the boundary. The parameterization of the surface may also effect the distribution of points in the interior, especially when there are large changes in the derivatives of the cartesian variables with respect to the parametric variables.

A redistribution scheme will now be described. Suppose that the boundary points on the surface have been determined by some one-dimensional scheme for

distributing points along a parametric curve with a prescribed distribution of arc length. The relative arc lengths along the boundary are given by the boundary values of the distributions  $s$  and  $t$  defined in (1) and (2). Namely, the relative arc lengths along  $j = 1$  and  $j = J$  are  $s_{i,1}$  and  $s_{i,J}$ , respectively. Similarly, the relative arc lengths along  $i = 1$  and  $i = I$  are  $t_{1,j}$  and  $t_{I,j}$ , respectively. Now define a desired distribution of arc length over the surface by interpolating the given arc length distributions on the boundary. The new distributions  $\sigma$  and  $\tau$  are defined as

$$\sigma_{i,j} = \frac{j-1}{J-1}s_{i,J} + \frac{J-j}{J-1}s_{i,1} \quad (3)$$

$$\tau_{i,j} = \frac{i-1}{I-1}t_{I,j} + \frac{I-i}{I-1}t_{1,i} \quad (4)$$

where  $i = 1, \dots, I$ ,  $j = 1, \dots, J$ . Now consider  $u$  and  $v$  as functions of arc length. With the given values at  $(s_{i,j}, t_{i,j})$ ,  $i = 1, \dots, I$ ,  $j = 1, \dots, J$ , interpolation can be used to find values at  $(\sigma_{i,j}, \tau_{i,j})$ ,  $i = 1, \dots, I$ ,  $j = 1, \dots, J$ . The new values of the parametric variables  $u$  and  $v$ , corresponding to the desired arc length distributions in (3) and (4), define the new surface grid. Due to interpolation error, the arc length distributions of the new grid may not actually match the desired distributions  $\sigma_{i,j}$  and  $\tau_{i,j}$ . This procedure can then be incorporated into an iterative scheme, and repeated until the desired distribution is achieved to within a certain tolerance. The accuracy of the method will depend on the interpolation procedure used to generate the new parametric values. All of the examples of surface grids have used two-dimensional bilinear interpolation. Other schemes can be used, and these will be discussed later when considering volume grids.

There are cases where redistribution in only one direction is desired. In such cases, a simple one-dimensional redistribution scheme can be employed. However, the present multidimensional scheme can also be used to generate one-dimensional redistribution of grid points, for example, by setting  $\sigma_{i,j} = s_{i,j}$  instead of (3) to produce redistribution along constant  $i$  grid lines.

Now suppose absolute spacing off of the boundary of the surface is to be imposed. For example, one may wish to have a constant distance between a boundary curve and the adjacent grid line in the interior. It is assumed that the desired distribution of points along the boundary of the surface has been obtained using a method for generating points along parametric curves in space. The boundary grid points are to remain fixed with the spacings along the boundary blended into the interior in such a way that the spacing off the boundary is to be some specified (possibly nonconstant) value. For a fixed value of  $j$ ,  $1 < j < J$ , the desired absolute spacing at the ends are denoted by  $a_1$  and  $a_{I-1}$ . The interior spacings are to be determined. The interpolated relative spacings are  $b_i = \sigma_{i+1,j} - \sigma_{i,j}$ . Now define a perturbation of the relative spacings as  $e_i$ ,  $i = 1, \dots, I-1$ , which is to satisfy the equations

$$e_1 = a_1 - b_1 \quad (5)$$

$$e_{I-1} = a_{I-1} - b_{I-1} \quad (6)$$

$$\sum_{i=1}^{I-1} e_i = 0. \quad (7)$$

If such a set of values can be determined, then a new distribution  $\bar{\sigma}_{i,j}$  can be defined by setting  $a_i = b_i + e_i$  and

$$\bar{\sigma}_{i,j} = \sum_{k=1}^{i-1} a_k. \quad (8)$$

Now  $\bar{\sigma}_{1,j} = 0$  and  $\bar{\sigma}_{I,j} = \sigma_{I,j} = 1$ . The problem of finding a perturbation  $e$  that yields a monotone distribution remains an unsolved problem for a general surface with arbitrary boundary spacing. However, many practical grid generation problems have been solved using a simple quadratic

$$e_i = A(i-1)^2 + B(i-1) + C \quad (9)$$

where the coefficients are calculated from equations (5) - (7) by solving the resulting linear system of three equations in the unknowns  $A$ ,  $B$ , and  $C$  in (9). If this procedure is used for each  $j$ ,  $1 < j < J$ , and the interpolated relative distribution  $\sigma$  in (3) is replaced by  $\bar{\sigma}$  in (8), then the desired absolute spacing is achieved at the boundaries  $j = 1$  and  $j = J$ . In a similar manner, absolute spacing at the boundaries  $i = 1$  and  $i = I$  can be imposed by replacing the interpolated distribution  $\tau$  in (4) by a perturbed distribution  $\bar{\tau}$ .

### Volume Grid Distribution

The procedure for generating a volume grid with the arc length distributions inherited from the bounding surface grids is similar to the procedure for generating surface grids. In this case, there are no parametric equations to be considered. Instead, the grid points are computed directly from the arc length parameters.

Suppose an initial grid is given in a three-dimensional volume with grid points

$$p_{i,j,k} = (x_{i,j,k}, y_{i,j,k}, z_{i,j,k}), \quad i = 1, \dots, I, \quad j = 1, \dots, J, \quad k = 1, \dots, K.$$

Relative arc length distributions in each coordinate direction are

$$r_{i,j,k} = \frac{\sum_{m=2}^{m=i} \|p_{m,j,k} - p_{m-1,j,k}\|}{\sum_{m=2}^{m=I} \|p_{m,j,k} - p_{m-1,j,k}\|} \quad (10)$$

$$s_{i,j,k} = \frac{\sum_{m=2}^{m=j} \|p_{i,m,k} - p_{i,m-1,k}\|}{\sum_{m=2}^{m=J} \|p_{i,m,k} - p_{i,m-1,k}\|} \quad (11)$$

$$t_{i,j,k} = \frac{\sum_{m=2}^{m=k} \|p_{i,j,m} - p_{i,j,m-1}\|}{\sum_{m=2}^{m=K} \|p_{i,j,m} - p_{i,j,m-1}\|}. \quad (12)$$

Now it is necessary to define interior arc length distributions by interpolating the arc length distributions on the bounding surfaces. Consider the arc length distribution  $r$  for the grid lines in the  $i$  direction. Its values for  $j = 1$ ,  $j = J$ ,  $k = 1$ , and  $k = K$  are defined from boundary grid point locations. A two-dimensional transfinite interpolation scheme could be used to generate interior values for a fixed  $i$ , however, there is no guarantee that the resulting distribution would be monotone for fixed values of  $j$  and  $k$ . Thus the following monotone interpolation scheme will be described. For each  $j$  and  $k$ , let

$$\begin{aligned} d_1 &= \frac{j-1}{J-1}, & d_2 &= \frac{k-1}{K-1} \\ d_3 &= 1 - d_1, & d_4 &= 1 - d_2 \end{aligned}$$

and define

$$\begin{aligned} \rho_{i,j,k} &= \frac{1}{d_1 d_3 + d_2 d_4} (d_2 d_3 d_4 r_{i,1,k} + d_1 d_3 d_4 r_{i,J,k} \\ &+ d_1 d_2 d_4 r_{i,j,1} + d_1 d_2 d_3 r_{i,j,K}). \end{aligned} \quad (13)$$

A similar interpolation formula is used to interpolate  $s$  and  $t$ . For the distribution  $s$  let

$$\begin{aligned} d_1 &= \frac{k-1}{K-1}, & d_2 &= \frac{i-1}{I-1} \\ d_3 &= 1 - d_1, & d_4 &= 1 - d_2 \end{aligned}$$

and define

$$\begin{aligned} \sigma_{i,j,k} &= \frac{1}{d_1 d_3 + d_2 d_4} (d_2 d_3 d_4 s_{i,j,1} + d_1 d_3 d_4 s_{i,j,K} \\ &+ d_1 d_2 d_4 s_{1,j,k} + d_1 d_2 d_3 s_{I,j,k}). \end{aligned} \quad (14)$$

For  $t$  let

$$\begin{aligned} d_1 &= \frac{i-1}{I-1}, & d_2 &= \frac{j-1}{J-1} \\ d_3 &= 1 - d_1, & d_4 &= 1 - d_2 \end{aligned}$$

and define

$$\begin{aligned} \tau_{i,j,k} &= \frac{1}{d_1 d_3 + d_2 d_4} (d_2 d_3 d_4 t_{1,j,k} + d_1 d_3 d_4 t_{I,j,k} \\ &+ d_1 d_2 d_4 t_{i,1,k} + d_1 d_2 d_3 t_{i,J,k}). \end{aligned} \quad (15)$$

The arc length distributions  $\rho$ ,  $\sigma$ , and  $\tau$  in (13) - (15) are monotone functions of  $i$ ,  $j$ , and  $k$ , respectively, provided the given surface distributions are monotone. It is also clear that the surface distributions are interpolated except along edges where

$d_1d_3 + d_2d_4 = 0$ . Along these edges, it can be shown that the limiting values of the volume distributions are equal to the arc length distribution along the edges.

The grid point locations for the interpolated relative arc length distributions  $\rho$ ,  $\sigma$ , and  $\tau$  must now be calculated by interpolating the existing grid with its relative arc length distributions given by  $r$ ,  $s$ , and  $t$ . For most applications, a trilinear interpolation scheme has proven to be satisfactory. However, it does require a Newton iteration scheme to determine the local coordinates of an  $(\rho, \sigma, \tau)$  point in a cell of the grid defined by the points

$$(r_{i,j,k}, s_{i,j,k}, t_{i,j,k}), \quad i = 1, \dots, I, \quad j = 1, \dots, J, \quad k = 1, \dots, K.$$

If the grid is highly skewed or if the aspect ratio is very large, the system of equations which must be solved is ill-conditioned and Newton's method may not converge. Thus an alternate interpolation method has been implemented to take care of these cases. The alternate method is a simple inverse distance scheme used for scattered data interpolation, and commonly referred to as Shepard's method [3]. Here the set of interpolation points for determining the coordinate values at  $(\rho_{i,j,k}, \sigma_{i,j,k}, \tau_{i,j,k})$  is limited by considering only grid points near  $(r_{i,j,k}, s_{i,j,k}, t_{i,j,k})$ . However, the stencil of interpolation points should be large enough to surround  $(\rho_{i,j,k}, \sigma_{i,j,k}, \tau_{i,j,k})$ . Otherwise, the method is the same as that used to interpolate scattered data with the interpolation coefficients computed from the inverse square of the distances from  $(\rho_{i,j,k}, \sigma_{i,j,k}, \tau_{i,j,k})$  to the nearby  $(r, s, t)$  grid points in (10) - (12). There are no systems of equations to solve and the method can be used to interpolate cartesian coordinates with any normalized arc length distribution in the unit cube. On the negative side, the poor precision properties of inverse distance interpolation will effect the smoothness of the grid. It should only be considered when trilinear interpolation fails.

Either of these interpolation methods can be incorporated into an iterative scheme. Other interpolation methods can be used. If redistribution is only required in one direction, then a one-dimensional spline interpolation along grid lines works well. Alternatively, as described above for surfaces, multidimensional redistribution can be limited to one or two directions by using the existing distributions  $r$ ,  $s$ , or  $t$  instead of the interpolated values  $\rho$ ,  $\sigma$ , or  $\tau$ . A succession of one-dimensional interpolations can be used for multi-dimensional redistribution, but that method has not always been successful in practice.

### Examples

Figure 1 demonstrates an application of the redistribution scheme in generating a smooth uniformly distributed grid on the fuselage of an F15 fighter. Note the reduction of the kink in the grid lines near the nose and the improvement in grid spacing on the engine shroud.

Figure 2 contains an application of the three-dimensional redistribution method for a multiblock grid about an aircraft wing. The spacing discontinuity along the vertical grid lines passing through the trailing edge clearly shows the block boundaries in the initial grid. This discontinuity is eliminated after a redistribution of grid points based on interpolated edge arc lengths. Of course, this technique is not designed to deal with skewness and the grid is still highly skewed near the leading edge.

Some computations have been done to investigate the convergence properties when the redistribution scheme is incorporated into an iterative algorithm. Figure 3 contains plots of the residuals for both a coarse 10 by 10 grid and a fine 100 by 100 grid for a parametric surface. This result, and similar results for other surfaces, indicates that the convergence rate is independent of the grid size. This is certainly not true for most iterative schemes. Figure 4 compares a volume grid redistribution using the trilinear interpolation method and the inverse distance interpolation method. In cases where the grid was not excessively skewed and the aspect ratio was not extreme, the trilinear interpolation method converged with the desired distribution of grid points. On the other hand, the inverse distance interpolation method generally converged much slower or diverged as in Figure 4. This is probably caused by the lower precision of the inverse distance interpolation.

### Conclusions and Recommendations

Arc length distribution schemes that have been widely used for defining grids on curves can be generalized to generate grids on surfaces and in volumes. With this approach to generating a structured grid, the spacings set along block edges can in fact be propagated into the entire region. Further generalizations might include surface curvatures or solution gradients in determining the distributions. Thus, this technique could be used to cluster grid points in regions of high curvature or where solution gradients are large.

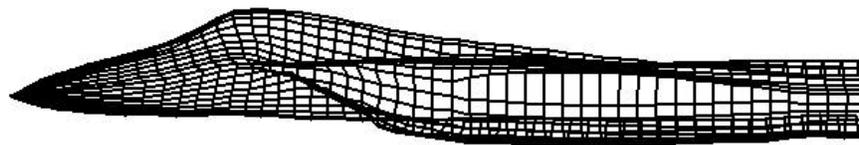
There are still areas where additional work is needed. Imposing absolute spacing at boundary curves and surfaces is not always possible due to the inability to come up with a monotone distribution in some cases. There is also a need for a more accurate and stable interpolation scheme for interpolating on highly skewed distribution grids. More accurate scattered data interpolation methods do exist and should be considered for interpolation on highly skewed grids. However, most of these schemes require derivatives of cartesian variables with respect to arc length.

Although grids with desired distributions have been generated for many different surfaces and regions, the question remains as to whether it is always possible to generate a grid with a given arc length distribution. The answer would seem to be no, especially in the case where the grid is extremely skewed. For a skewed grid, the arc length distributions in two coordinate directions might be different, but

the coordinate lines themselves would be nearly parallel. The distributions along two such lines would be highly dependent and it does not appear that satisfying two separate arc length distributions would be possible.

#### References

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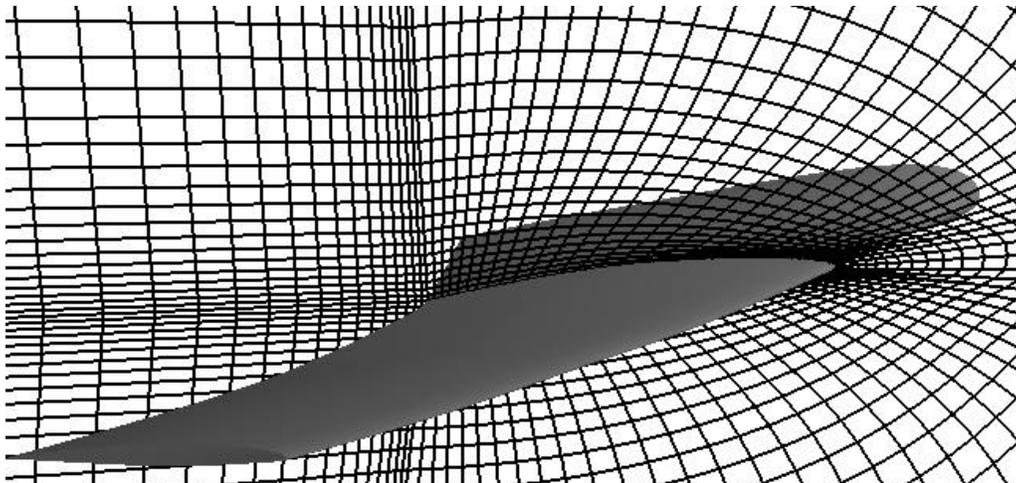


(a) original grid

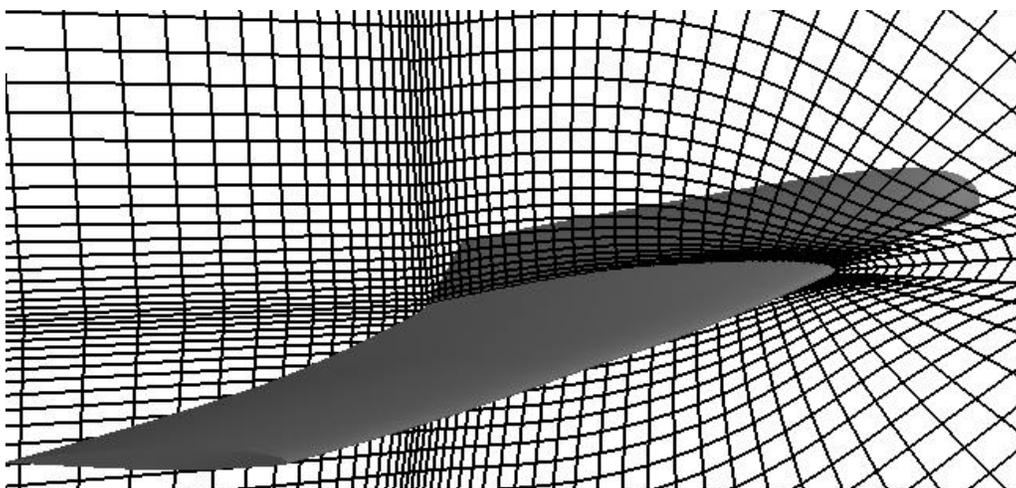


(b) after redistribution

Figure 1. Surface arc length redistribution for F15 fuselage.



(a) original grid



(b) after redistribution

Figure 2. Volume arc length redistribution for multiblock grid about aircraft wing - interior grid surface shown.

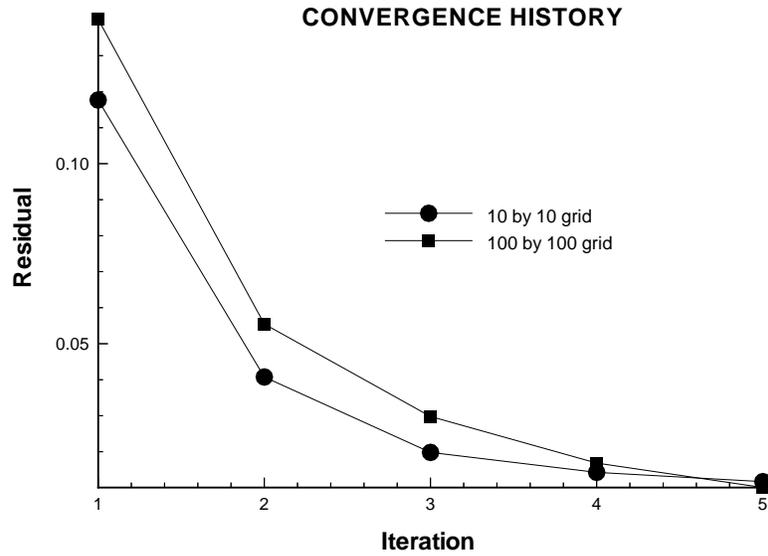


Figure 3. Convergence with coarse and fine surface grids.

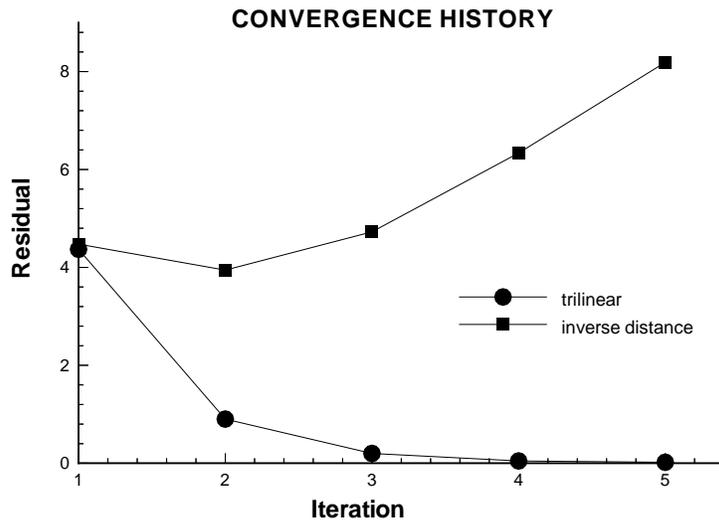


Figure 4. Convergence with trilinear and inverse distance interpolation.