A Stable Penalty Method for the Compressible Navier-Stokes Equations.
I. Open Boundary Conditions.

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Abstract

The purpose of this paper is to present asymptotically stable open boundary conditions for the numerical approximation of the compressible Navier-Stokes equations in three spatial dimensions. The treatment uses the conservation form of the Navier-Stokes equations and utilizes linearization and localization at the boundaries based on these variables.

The proposed boundary conditions are applied through a penalty procedure, thus ensuring correct behavior of the scheme as the Reynolds number tends to infinity. The versality of this method is demonstrated for the problem of a compressible flow past a circular cylinder.

1. Introduction. In the present paper, we discuss boundary conditions for dissipative, wave dominated problems, exemplified by Burgers equation and the three-dimensional, compressible Navier-Stokes equations given in conservation form. The emphasis is on deriving open boundary conditions ensuring the continuous problem to be well-posed and on devising semi-discrete schemes for imposing these conditions, which can be proven asymptotically stable. The boundary conditions and the semi-discrete scheme are valid even in the limit of infinite Reynolds number.

When addressing exterior, wave-dominated, dissipative problems, one is often forced to introduce an artificial boundary for computational reasons. This introduces the well known problem of specifying appropriate boundary conditions at the artificial open boundary. For purely hyperbolic problems, it is well known that enforcing these boundary conditions through the characteristic variables leads to a stable approximation. However, for dissipative wave problems the procedure is considerably more complicated.

Naturally, we must require the boundary conditions to lead to a well-posed continuous problem. For wave problems of dissipative type, the problem must, in order to be compatible with weak boundary layers, remain well-posed even in the limit where the dissipation vanishes and the problem becomes purely hyperbolic. In addition to this, we wish the discrete approximation of the problem to be asymptotically stable, and that the boundary conditions are easily implemented.

For general non-linear problems the issues of well-posedness and asymptotic stability are very complicated, and for most problems relatively little is known. However, as discussed by Kreiss and Lorenz [1], we may, for a large class of operators, simplify the problem significantly if the solutions are smooth. It was shown that in this case it is sufficient to consider the questions of well-posedness and asymptotic stability for the linearized, constant coefficient version of the full problem.

The energy method is applied to the linearized, constant coefficient version of the continuous problem in order to obtain energy inequalities which bound the temporal growth of the solutions to the initial-boundary value problem. This technique allows for handling such complex problems as the Navier-Stokes equations and is in general applicable to symmetrizable problems [2].

The usual way to enforce the boundary conditions in the numerical scheme, once their proper form for the continuous problem is known, is to solve the equation in the interior of the computational domain, and then enforce the boundary conditions at the boundary points. However, this approach does not take into account the fact that the equation should be obeyed arbitrarily close to the open boundary. To circumvent this problem, Funaro and Gottlieb [3, 4], and Carpenter et al. [5] developed the penalty

method which enforces the boundary conditions, as well as taking into account the equation at the boundary. They showed asymptotic stability for the scheme applied to scalar hyperbolic equations and systems of hyperbolic equations. Don and Gottlieb [6] recently showed how this idea can help in applying the Legendre collocation method on Chebyshev grids.

The proofs presented in this paper are all done for semi-discrete schemes. The relation between the stability of the semi-discrete and the fully discrete scheme was recently discussed by Kreiss and Wu[7].

The issue of well posed boundary conditions for the compressible Navier-Stokes equations was previously considered by Gustafsson and Sundström [8], Oliger and Sundström [9], and Nordström [10]. They all used the energy method to derive boundary conditions for the linearized, constant coefficient Navier-Stokes equations in the primitive variable formulation. Dutt [11] introduced an entropy function, which allowed him to derive boundary conditions for the non-linear problem, ensuring that the solution remains bounded in an entropy norm.

The remaining part of this paper is organized as follows. In Sec. 2 we review some well known results on Legendre polynomials and collocation methods. Section 3 discusses Burgers equation, and boundary conditions ensuring well-posedness of the problem are derived. We continue by proposing an asymptotically stable penalty method through which the boundary conditions are enforced. This scheme ensures the correct behavior even in the limit where the problem becomes hyperbolic, and may in general be applied to any non-linear scalar equation. The penalty method for scalar hyperbolic, parabolic, and linear advection-diffusion equations is briefly discussed, and the proposed scheme is evaluated by numerical tests. The importance of properly choosing the penalty parameter is addressed in Sec. 4, where we discuss the effect of the penalty method on the CFL condition when using explicit Runge-Kutta methods for time-stepping linear problems. The results from the linear analysis are s! h! own to carry over to the non-line

2. Legendre Polynomials and Collocation Methods. The schemes which we analyze in the present paper are all based on Legendre collocation methods. This choice is merely dictated by a wish to obtain analytical results, and the methods extend trivially to other collocation methods and even to finite difference/finite element methods.

The Legendre polynomial of order N is defined as

$$P_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N} (x^2 - 1)^N \ ,$$

where $|x| \leq 1$. We will in the following only consider collocation methods, where the

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collocation points are given as the Legendre-Gauss-Lobatto points, being defined as the roots of the polynomial $(1 - x^2)P'_N(x)$. There is no known explicit formula for these roots.

Associated with the Gauss-Lobatto points is the quadrature formula, stating that if $f(x)$ is a polynomial of degree $2N - 1$, then

$$(1) \qquad \sum_{k=0}^{N} f(x_k)\omega_k = \int_{-1}^{1} f(\xi)\,d\xi \ ,$$

where x_k are the Legendre-Gauss-Lobatto collocation points, and the Gauss-Lobatto weights, ω_k, are given as

$$(2) \qquad \omega_k = -\frac{2}{N+1}\frac{1}{P_N(x_k)\,P_{N-1}(x_k)} \quad , \ 1 \le k \le N-1$$
$$\omega_0 = \omega_N = \frac{2}{N(N+1)} \ .$$

For further details on the properties of the Legendre polynomials, we refer to [12].

In a Legendre collocation method, the function, $f(x)$, is approximated by a grid function, $f_k = f(x_k)$, where the grid points are the Gauss-Lobatto collocation points. Thus, we construct a global Legendre interpolant, I_N, to obtain the approximation to the function;

$$(I_N f)(x) = \sum_{i=0}^{N} f_k\,h_k(x) \ ,$$

where the interpolating Legendre-Lagrange polynomials are given as

$$h_k(x) = -\frac{(1-x^2)P'_N(x)}{N(N+1)\,(x-x_k)P_N(x_k)} \ .$$

We note that by construction

$$(I_N f)(x_k) = f_k \ .$$

To seek equations for an approximate solution, $(I_N f)(x)$, to a partial differential equation, we need to obtain values for the spatial derivatives at the collocation points. This is done by approximating the differential operator by a matrix operator, with the matrix entries given as

$$\mathcal{D}_{kl} = h'_l(x_k) \ .$$

For the explicit expression of the entries, we refer to [13, 14].

3. Burgers Equation. In this section, we consider Burgers equation

$$(3) \qquad \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = \varepsilon \frac{\partial^2 U}{\partial x^2} \qquad |x| \leq 1 \quad t > 0 \ ,$$

where $\varepsilon \geq 0$. The initial condition is given as

$$U(x,0) = f(x) \ ,$$

with boundary conditions of the form

$$(4) \qquad \alpha U(-1,t) - \beta \varepsilon \frac{\partial U}{\partial x}\Big|_{x=-1} = 0 \ .$$

$$(5) \qquad \gamma U(1,t) + \delta \varepsilon \frac{\partial U}{\partial x}\Big|_{x=1} = 0 \ ,$$

When addressing the issue it is, as discussed in the introduction, sufficient to consider the linearized, constant coefficient version of Burgers equation

$$(6) \qquad \frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = \varepsilon \frac{\partial^2 U}{\partial x^2} \qquad |x| \leq 1 \quad t > 0 \ .$$

Here $\lambda = U_0$ is the uniform solution around which we have linearized. Equation (6) is also known as the linear advection-diffusion equation.

The four real constants, α, β, γ, and δ, in the boundary conditions, Eq.(4)-(5), may not be chosen arbitrarily, since the resulting problem should be well-posed. Bounds yielding a sufficient condition for well-posedness are given in the following Lemma.

LEMMA 3.1. *Equation (6), with boundary conditions given by Eq.(4)-(5) is well-posed if one of the following conditions holds*

>   *(i):* $\beta = 0$, $\delta = 0$.
>   *(ii):* $\beta \neq 0$, $\delta = 0$ *and* $(\varepsilon - \lambda) + 2\alpha/\beta \geq 0$.
>   *(iii):* $\beta = 0$, $\delta \neq 0$ *and* $(\varepsilon + \lambda) + 2\gamma/\delta \geq 0$.
>   *(iv):* $\beta \neq 0$, $\delta \neq 0$ *and* $2(\varepsilon - \lambda)\gamma/\delta + 2(\varepsilon + \lambda)\alpha/\beta + 4(\alpha\gamma)/(\beta\delta) \geq \lambda^2$.

Proof. Construct the energy integral as

$$\frac{1}{2} \frac{d}{dt} \|U\|^2 = -\lambda (U, U_x) + \varepsilon (U, U_{xx}) = \frac{1}{2} \Big[-\lambda U^2 + 2\varepsilon U \, U_x \Big]_{-1}^{1} - \varepsilon \|U_x\|^2 \ .$$

Here we have introduced

$$(U,V) = \int_{-1}^{1} U \, V \, dx \ , \quad (U,U) = \|U\|^2 \ .$$

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Following the similar analysis done in [15], we use the following estimate

$$-\varepsilon\|U_x\|^2 \leq -\frac{\varepsilon}{2}[U(1) - U(-1)]^2 \quad.$$

Applying this, the condition for well-posedness becomes

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 \leq \frac{1}{2}\left[-\lambda U^2 + 2\varepsilon U\,U_x\right]_{-1}^{1} - \frac{\varepsilon}{2}[U(1) - U(-1)]^2 \leq 0 \quad.$$

Condition (i) implies that $U(-1) = U(1) = 0$ such that

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 \leq 0 \quad.$$

For condition (ii) we obtain $U(1) = 0$ and thus

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 \leq -\frac{1}{2}\left(\varepsilon - \lambda + 2\frac{\alpha}{\beta}\right)U^2(-1) \leq 0 \quad,$$

yielding the condition

$$\varepsilon - \lambda + 2\frac{\alpha}{\beta} \geq 0 \quad.$$

Likewise, for condition (iii) we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 \leq -\frac{1}{2}\left(\varepsilon + \lambda + 2\frac{\gamma}{\delta}\right)U^2(1) \leq 0 \quad,$$

showing that this choice yields well-posedness. For condition (iv) we obtain the following condition

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 \leq -\frac{1}{2}\left(\varepsilon - \lambda + 2\frac{\alpha}{\beta}\right)U^2(-1) + \varepsilon U(-1)U(1) - \frac{1}{2}\left(\varepsilon + \lambda + 2\frac{\gamma}{\delta}\right)U^2(1) \leq 0 \quad.$$

This is obeyed if

$$\varepsilon^2 - \left(\varepsilon - \lambda + 2\frac{\alpha}{\beta}\right)\left(\varepsilon + \lambda + 2\frac{\gamma}{\delta}\right) \leq 0 \quad,$$

implying

$$2(\varepsilon - \lambda)\gamma/\delta + 2(\varepsilon + \lambda)\alpha/\beta + 4(\alpha\gamma)/(\beta\delta) \geq \lambda^2 \quad. \qquad \square$$

3.1. The Semi-Discrete Scheme. Equation (3) will be solved using a Legendre collocation method where the collocation points are the Legendre-Gauss-Lobatto points. This involves finding an Nth degree polynomial, $u(x,t)$, satisfying

$$(7) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2} \quad \text{at } x = x_k\,, \quad k \in [1\dots N-1]\,,$$

in the interior. The boundary points are given by boundary conditions of Robin type

$$\alpha u(x_0, t) - \beta\varepsilon\frac{\partial u}{\partial x}\bigg|_{x_0} = g_1(t) \ ,$$

$$\gamma u(x_N, t) + \delta\varepsilon\frac{\partial u}{\partial x}\bigg|_{x_N} = g_2(t) \ ,$$

where $g_1(t)$ and $g_2(t)$ are the boundary conditions. The traditional method of imposing the boundary conditions is to solve Eq.(7) in the interior and enforce the boundary conditions at the boundary points only. However, this approach does not take into account the fact that the equation must be obeyed arbitrarily close to the boundary. In addition to this, it has proven difficult to implement Robin boundary conditions consistently for non-linear problems. To overcome these problems, we follow the line of thought initiated by Funaro and Gottlieb [3, 4] and propose a penalty method for Burgers equation at the Legendre-Gauss-Lobatto collocation points, $x = x_k$, $k \in [0, \ldots, N]$, as

$$
\begin{aligned}
(8) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \quad & \varepsilon\frac{\partial^2 u}{\partial x^2} \\
& - \ \tau_1 Q^-(x)\left[\alpha u(x_0, t) - \beta\varepsilon\frac{\partial u}{\partial x}\bigg|_{x_0} - g_1(t)\right] \\
& - \ \tau_2 Q^+(x)\left[\gamma u(x_N, t) + \delta\varepsilon\frac{\partial u}{\partial x}\bigg|_{x_N} - g_2(t)\right] \quad .
\end{aligned}
$$

where

$$(9) \qquad Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N'(-1)} \ , \quad Q^+(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)} \quad .$$

These two functions have the property of being zero at all Legendre-Gauss-Lobatto collocation points, except at the two endpoints of the domain. Although Q^- and Q^+ here are defined as delta-functions at the boundary, we may equally well chose other definitions. As shown by Don and Gottlieb [6], this approach may also be applied for implementing Legendre methods on Chebyshev grids.

We note here that the penalty method as given by Eq.(8) combines the boundary conditions and the governing equation into one equation. When using the penalty method, the boundary conditions are not exactly obeyed at the boundary. However, the method remains spectrally accurate, as we will soon illustrate. One may also observe that the scheme is equivalent to the traditional approach for τ_1, τ_2 approaching infinity.

In order to obtain the energy inequality, we consider only homogeneous boundary conditions. As discussed previously in [1], this is no restriction, since we may always

introduce a variable transform such the boundary conditions become homogeneous. In the following Lemma we state the bounds on τ_1 and τ_2 ensuring that the linearized, constant coefficient version of Eq.(8) is asymptotically stable.

LEMMA 3.2. *Assume* $u(x,t)$ *exists and let* $\tau_{a,b}^-$ *and* $\tau_{a,b}^+$ *be defined as*

$$\tau_{a,b}^- = \frac{1}{\omega\varepsilon b}[\varepsilon + 2\kappa - 2\sqrt{\kappa^2 + \varepsilon\kappa - 1/2\varepsilon\omega|\lambda|}] \ ,$$
$$\tau_{a,b}^+ = \frac{1}{\omega\varepsilon b}[\varepsilon + 2\kappa + 2\sqrt{\kappa^2 + \varepsilon\kappa - 1/2\varepsilon\omega|\lambda|}] \ ,$$

where $\kappa = \omega a/b$ *and*

$$\omega = \frac{2}{N(N+1)} \ ,$$

is the Legendre weight at the end-points.
   Then if

$$\tau_{\alpha,\beta}^- \leq \tau_1 \leq \tau_{\alpha,\beta}^+ \ ,$$
$$\tau_{\gamma,\delta}^- \leq \tau_2 \leq \tau_{\gamma,\delta}^+ \ ,$$

then the linearized, constant coefficient version of Eq.(8) is asymptotically stable and the solution is bounded as

$$\frac{1}{2}\frac{d}{dt}\|u\|_N^2 \leq -\varepsilon\sum_{k=1}^{N-1}\left(\frac{\partial u}{\partial x}(x_k)\right)^2\omega_k \ .$$

Proof. We start be defining the discrete, weighted scalar product as

$$(u,v)_N = \sum_{k=0}^{N} u(x_k)\,v(x_k)\,\omega_k \ , (u,u)_N = \|u\|_N^2 \ .$$

and note that since we are using a Legendre collocation method, we have, through Eq.(1), the identity

$$(u,v_x)_N = (U,V_x) \ .$$

This makes it straightforward to apply partial differentiation. Following the results stated previously, it is sufficient to obtain the energy estimate for the linearized, constant coefficient version of Eq.(8);

$$\frac{1}{2}\frac{d}{dt}\|u\|_N^2 = -\frac{\lambda}{2}[u\,u_x]_{-1}^1 + \varepsilon[u\,u_x]_{-1}^1 - \varepsilon\|u_x\|_N^2$$
$$-\tau_1\omega u(-1)[\alpha u(-1) - \beta\varepsilon u_x(-1)] - \tau_2\omega u(1)[\gamma u(1) + \delta\varepsilon u_x(1)] \ .$$

Here the subscripts designate differentiation and ω is the Legendre weight at the endpoints (Eq.(2)). Using the quadrature rule allows for rewriting

$$\|u_x\|_N^2 = u_x^2(-1)\omega + u_x^2(1)\omega + \sum_{k=1}^{N-1} u_x^2(x_k)\omega_k \ .$$

Contrary to the approach followed by Funaro and Gottlieb [3, 4], we recast the problem of stability into an algebraic eigenvalue problem. For the present problem, this may seem an additional complication. However, we find that for more complicated problems, this approach greatly simplifies the proofs.

Isolating the terms contributing to stability at each boundary, we obtain two conditions for asymptotic stability;

$$\mathbf{u}_-^T \mathcal{H}^- \mathbf{u}_- \le 0 \ , \quad \mathbf{u}_+^T \mathcal{H}^+ \mathbf{u}_+ \le 0 \ ,$$

where $\mathbf{u}_- = [u(-1), u_x(-1)]^T$, $\mathbf{u}_+ = [u(1), u_x(1)]^T$ and

$$\mathcal{H}^- = \frac{1}{2} \begin{bmatrix} \lambda - 2\alpha\omega\tau_1 & -\varepsilon(1-\beta\omega\tau_1) \\ -\varepsilon(1-\beta\omega\tau_1) & -2\varepsilon\omega \end{bmatrix} \ ,$$

$$\mathcal{H}^+ = \frac{1}{2} \begin{bmatrix} -\lambda - 2\gamma\omega\tau_2 & \varepsilon(1-\delta\omega\tau_2) \\ \varepsilon(1-\delta\omega\tau_2) & -2\varepsilon\omega \end{bmatrix} \ .$$

Since both matrices are symmetric, the problem is reduced to ensuring that $\mathcal{H}^-$ and $\mathcal{H}^+$ are negative, semi-definite. The eigenvalues of the two matrices are found to be

$$\rho_{1,2}(\mathcal{H}^-) = \frac{1}{8}\left(-\zeta^- \pm \sqrt{(\zeta^-)^2 + 16\varepsilon(\beta^2\omega^2\varepsilon\tau_1^2 - 2\omega(\beta\varepsilon + 2\alpha\omega)\tau_1 + 2\omega\lambda + \varepsilon)}\right)$$

$$\rho_{1,2}(\mathcal{H}^+) = \frac{1}{8}\left(-\zeta^+ \pm \sqrt{(\zeta^+)^2 + 16\varepsilon(\delta^2\omega^2\varepsilon\tau_2^2 - 2\omega(\delta\varepsilon + 2\gamma\omega)\tau_2 - 2\omega\lambda + \varepsilon)}\right) \ ,$$

where $\zeta^- = -2\lambda + 4\omega\varepsilon + 4\alpha\omega\tau_1$ and $\zeta^+ = 2\lambda + 4\omega\varepsilon + 4\gamma\omega\tau_2$. It is evident that negative semi-definiteness is ensured if

$$\beta^2\omega^2\varepsilon\tau_1^2 - 2\omega(\beta\varepsilon + 2\alpha\omega)\tau_1 + 2\omega\lambda + \varepsilon \le 0$$

$$\delta^2\omega^2\varepsilon\tau_2^2 - 2\omega(\delta\varepsilon + 2\gamma\omega)\tau_2 - 2\omega\lambda + \varepsilon \le 0 \ .$$

The roots of the two polynomials are

$$\tau_1^\pm = \frac{1}{\omega\varepsilon\beta}\left(\varepsilon + 2\kappa_- \pm 2\sqrt{\kappa_-^2 + \varepsilon\kappa_- - 1/2\varepsilon\omega\lambda}\right) \ ,$$

$$\tau_2^\pm = \frac{1}{\omega\varepsilon\delta}\left(\varepsilon + 2\kappa_+ \pm 2\sqrt{\kappa_+^2 + \varepsilon\kappa_+ + 1/2\varepsilon\omega\lambda}\right) \ ,$$

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where $\kappa_- = \omega\alpha/\beta$ and $\kappa_+ = \omega\gamma/\delta$. We introduce

$$\tau_{a,b}^- = \frac{1}{\omega\varepsilon b}[\varepsilon + 2\kappa - 2\sqrt{\kappa^2 + \varepsilon\kappa - 1/2\varepsilon\omega|\lambda|}] ,$$

$$\tau_{a,b}^+ = \frac{1}{\omega\varepsilon b}[\varepsilon + 2\kappa + 2\sqrt{\kappa^2 + \varepsilon\kappa - 1/2\varepsilon\omega|\lambda|}] ,$$

where $\kappa = \omega a/b$. Since

$$\tau_{a,b}^- \geq \frac{|\lambda|}{2a\omega} + \frac{1}{4}\varepsilon\frac{1}{\omega^2} ,$$

for $\varepsilon \ll 1$, this ensures $\zeta^- > 0$ and $\zeta^+ > 0$.

Hence, stability is ensured for

$$\tau_{\alpha,\beta}^- \leq \tau_1 \leq \tau_{\alpha,\beta}^+ ,$$

$$\tau_{\gamma,\delta}^- \leq \tau_2 \leq \tau_{\gamma,\delta}^+ ,$$

with the solution satisfying

$$\frac{1}{2}\frac{d}{dt}\|u\|_N^2 \leq -\varepsilon \sum_{k=1}^{N-1} u_x^2(x_k)\omega_k . \qquad \square$$

3.1.1. Remarks on the Penalty Method for Linear Equations. The results stated in Lemma 3.2 may be used to derive the appropriate penalty parameter for a large class of linear equations. We consider the general linear advection-diffusion equation, Eq.(6), with the Robin boundary conditions given in Eq.(4)-(5). Solving this problem by a penalty method, equivalent to that given by Eq.(8), requires bounds on the penalty parameters in order to ensure stability of the scheme.

In the following, we will give these bounds for reference and will return to the numerical validation of these results in Sec. 4. Some of these results may be found in [3, 4, 6], but are here given in a more general framework. Note that $\omega \sim \mathcal{O}(N^2)$.

(i) Hyperbolic Equations $(\varepsilon = 0)$.

1. $\lambda > 0$. Well-posedness is ensured by choosing $\alpha > 0$ and $\beta = \gamma = \delta = 0$. Thus, for this case we will only need bounds on τ_1.

$$\tau_{\alpha,0}^- = \frac{\lambda}{2\omega\alpha} , \quad \tau_{\alpha,0}^+ = \infty .$$

The scheme for the hyperbolic case is stable for

$$\infty \geq \tau_1 \geq \frac{\lambda}{2\omega\alpha} .$$

2. $\lambda < 0$. Well-posedness is ensured by choosing $\gamma > 0$ and $\alpha = \beta = \delta = 0$. Thus, for this case we will only need bounds on τ_2.

$$\tau_{\gamma,0}^- = \frac{|\lambda|}{2\omega\gamma} \quad , \quad \tau_{\gamma,0}^+ = \infty \quad .$$

The scheme for the hyperbolic case is stable for

$$\infty \geq \tau_2 \geq \frac{|\lambda|}{2\omega\gamma} \quad .$$

(ii) Parabolic Equations $(\lambda = 0,\ \varepsilon > 0)$. Necessary and sufficient conditions for well-posedness may be obtained by choosing the four parameters, α, β, γ, and δ, properly as stated in Lemma 3.1 [15]. We only state the results for the bounds of τ_1, since the results for τ_2 are equivalent.

1. Dirichlet boundary condition, $(\alpha > 0,\ \beta = 0)$.

$$\tau_{\alpha,0}^- = \frac{\varepsilon}{4\alpha}\frac{1}{\omega^2} \quad , \quad \tau_{\alpha,0}^+ = \infty \quad .$$

Stability is ensured for

$$\infty \geq \tau_1 \geq \frac{\varepsilon}{4\alpha}\frac{1}{\omega^2} \quad .$$

2. Neumann boundary condition, $(\alpha = 0,\ \beta > 0)$.

$$\tau_{0,\beta}^- = \frac{1}{\beta\omega} \quad , \quad \tau_{0,\beta}^+ = \frac{1}{\beta\omega} \quad .$$

Stability is ensured for

$$\tau_1 = \frac{1}{\beta\omega} \quad .$$

3. Robin boundary condition, $(\alpha > 0,\ \beta > 0)$.

$$\tau_{\alpha,\beta}^- = \frac{1}{\omega\varepsilon\beta}[\varepsilon + 2\kappa - 2\sqrt{\kappa^2 + \varepsilon\kappa}] \quad ,$$

$$\tau_{\alpha,\beta}^+ = \frac{1}{\omega\varepsilon\beta}[\varepsilon + 2\kappa + 2\sqrt{\kappa^2 + \varepsilon\kappa}] \quad ,$$

where $\kappa = \omega\alpha/\beta$. Stability is ensured for

$$\tau_{\alpha,\beta}^+ \geq \tau_1 \geq \tau_{\alpha,\beta}^- \quad .$$

(iii) Advection-Diffusion Equations $(\lambda \neq 0,\ \varepsilon \geq 0)$. Again we must ensure well-posedness by proper choice of the four parameters, as given in Lemma 3.1. We only state the results for the bounds of τ_1 as the results for τ_2 are equivalent.

1. Dirichlet boundary condition ($\alpha > 0$, $\beta = 0$).

$$\tau^-_{\alpha,0} = \frac{|\lambda|}{2\alpha}\frac{1}{\omega} + \frac{\varepsilon}{4\alpha}\frac{1}{\omega^2} \quad , \quad \tau^+_{\alpha,0} = \infty \quad .$$

Stability is ensured for

$$\infty \geq \tau_1 \geq \frac{|\lambda|}{2\alpha}\frac{1}{\omega} + \frac{\varepsilon}{4\alpha}\frac{1}{\omega^2} \quad .$$

2. Neumann boundary condition ($\alpha = 0$, $\beta > 0$).

$$\tau^-_{0,\beta} = \frac{1}{\beta\omega} - \sqrt{\frac{2|\lambda|\omega}{\beta^2}} \quad , \quad \tau^+_{0,\beta} = \frac{1}{\beta\omega} + \sqrt{\frac{2|\lambda|\omega}{\beta^2}} \quad .$$

Stability is ensured for

$$\frac{1}{\beta\omega} + \sqrt{\frac{2|\lambda|\omega}{\beta^2}} \geq \tau_1 \geq \frac{1}{\beta\omega} - \sqrt{\frac{2|\lambda|\omega}{\beta^2}} \quad .$$

3. Robin boundary conditions, $\alpha > 0$, $\beta > 0$. Results are given in Lemma 3.2.

3.2. Numerical Tests. As we aim at solving the full non-linear Burgers equation, and not the linearized, constant coefficient version, we need to validate the results obtained from the linear analysis. We have solved Burgers equation using the scheme given by Eq.(8) and employing a standard Legendre collocation method [13, 16].

Burgers equation, Eq.(3), has a rightward traveling wave solution (see e.g. [1]) of the form

$$(10) \qquad U(x,t) = -a\tanh\left(a\frac{x - ct}{2\varepsilon}\right) + c \quad , x \in [-\infty, \infty] \ , t \geq 0 \ ,$$

where the free-stream values

$$\lim_{x\to-\infty} U(x,t) = b_{-\infty} \quad , \quad \lim_{x\to\infty} U(x,t) = b_{\infty} \quad ,$$

are associated with the wave-speed, c, and the constant, $a \geq 0$, as

$$c = \frac{b_{-\infty} + b_{\infty}}{2} \quad , \quad a = \frac{b_{-\infty} - b_{\infty}}{2} \quad .$$

Following the results in Lemma 3.1 (condition (iv): $\alpha = \lambda$, $\beta = 1$, $\gamma = 0$, $\delta = 1$), we expect the non-linear problem to be well-posed for boundary conditions of the type

$$\lambda U(-1,t) - \varepsilon\frac{\partial U(-1,t)}{\partial x} = g_1(t) \quad , \quad \varepsilon\frac{\partial U(1,t)}{\partial x} = g_2(t) \quad ,$$

where $\lambda \geq 0$ is the value around which we have linearized. In the present study, we have used the free-stream value at the inflow, i.e. $\lambda = b_{-\infty}$.

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Since we know an exact solution, the boundary conditions may be given exactly at all times using Eq.(10). As initial condition we use

$$U(x,0) = -a \tanh\left(a\frac{x}{2\varepsilon}\right) + c \ .$$

The solution is time-stepped using a classical 4th-order Runge-Kutta method, where the boundary conditions are imposed at the intermediate time-levels.

Using the values of the penalty parameters given in Lemma 3.2 results in a stable scheme. However, the CFL-number, relating the maximum allowable time step to the spatial resolution as

$$\Delta t_{\max} \leq \frac{CFL}{|U|\Delta x_{\min}^{-1} + \varepsilon \Delta x_{\min}^{-2}} \ ,$$

will have to be very small in order to ensure stability. Here $|U|$ signifies the maximum absolute value of U. Thus, with the theoretical value of the penalty parameter, the proposed method compares unfavorably with the traditional method, due to severe time step restrictions. Fortunately, the limits of the penalty parameters, in between which asymptotic stability is ensured, are obtained as a result of a conservative energy estimate and hence are not very accurate.

We have used the values of penalty parameter (see Lemma 3.2) as;

$$\tau_1 = \frac{\tau_{\lambda,1}^-}{4} \quad , \quad \tau_2 = \frac{\tau_{0,1}^-}{4} \ .$$

These values are found to lead to a stable scheme, provided $\varepsilon N^2 \gg 1$. In Eq.(3), ε plays the role of an inverse Reynolds number. The constraint, $\varepsilon N^2 \gg 1$, simply states that increasing the Reynolds number requires increased spatial resolution, which is a natural restriction. For advection dominated problems, stability is obtained by increasing the penalty parameters towards the values stated in Lemma 3.2.

With these values of the penalty parameters, we have been able to perform the simulations with a CFL number of 4, which is equivalent to what is usually allowed when using the traditional method. Thus, by fine-tuning the penalty parameters we were able to avoid any effect of the penalty method on the CFL-condition. The following section contains a study of the effect of the penalty method on the CFL-condition and guidelines for fine-tuning the penalty parameter for practical applications.

In Fig. 1 we show the temporal evolution of the traveling wave solution when using the proposed scheme as given by Eq.(8). The simulation is done with $N = 64$ and $\varepsilon = 0.1$. We observe no spurious reflections from the open boundary and the kink is seen to travel undisturbed out of the domain. Table 1 shows the error at $T = 1.00$, where the kink has propagated half way through the boundary. It is evident that
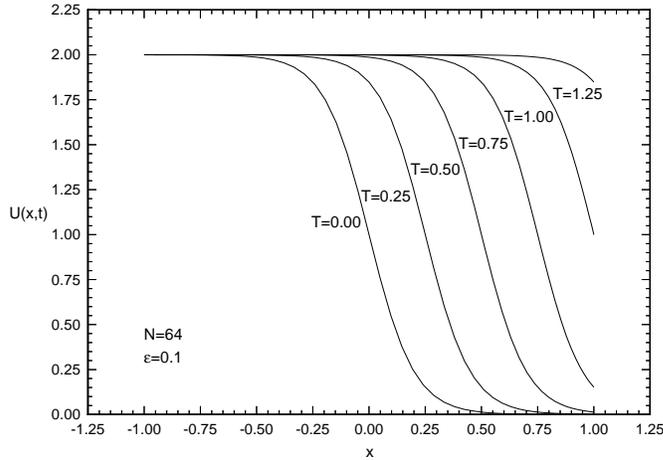
FIG. 1. *Traveling wave solution of Burgers equation.*

TABLE 1

Error in the spectral simulation of Burgers equation using the penalty method. The maximum error (L_∞) occurs at the boundary.

N	L_2	L_∞
16	1.07E-02	3.26E-02
32	7.64E-05	3.50E-04
64	3.36E-09	2.21E-08
128	1.56E-11	7.62E-11

the proposed scheme maintains the spectral accuracy. The time-step is so small that time-stepping errors may be neglected.

4. CFL-Restrictions for the Penalty Method. As discussed briefly in the previous Section, choosing too large a penalty parameter results in severe CFL-restrictions. For this reason, it is vital to understand how the penalty method alters the eigenvalue spectrum of the operators and consequently changes the CFL-restriction.

In the present section we will study these effects for the linear advection and diffusion operators for Legendre collocation methods. For completeness, we will also give the results for Chebyshev collocation methods, which are widely used when solving non-linear problems. The analysis will consider both 3rd- and 4th-order Runge Kutta methods, which are often employed when addressing problems of the type considered here. At the end of the section we will compare the results from our linear analysis with simulations of the non-linear Burgers equation.

Consider now the semi-discrete linear, constant coefficient problem

$$\begin{array}{lll} (\mathbf{q})_t = \mathcal{L}_N \mathbf{q} & \mathbf{x}_k \in \Omega \;\;, & t \geq 0 \\ \mathbf{q} = 0 & \mathbf{x}_k \in \Omega \;\;, & t = 0 \quad, \\ \mathcal{B}_N \mathbf{q} = 0 & \mathbf{x}_k \in \Gamma \;\;, & t \geq 0 \end{array}$$

(11)

where $\mathbf{q} = (\mathbf{q}(\mathbf{x}_0), \dots, \mathbf{q}(\mathbf{x}_N))^T$, $k \in [0, \dots, N]$, $\mathcal{L}_N$ is the discrete approximation of the operator for the interior and $\mathcal{B}_N$ determines the appropriate discrete boundary conditions. We assume that the semi-discrete approximation is a consistent approximation of the continuous problem. A time-differencing scheme, where the boundary conditions are enforced exactly at the boundary points, may then be expressed as

$$\mathbf{q}^{n+1} = K_N(\Delta t, \; \mathcal{L}_N) \mathbf{q}^n$$
$$\mathcal{B}_N \mathbf{q}^{n+1} = 0 \;\;.$$

Here $\mathbf{q}^n$ signifies the solution vector at time-step n. Thus, for strong stability we must require

$$|K_N(\Delta t, \; \mathcal{L}_N)| < 1 \;\;.$$

However, employing the penalty method changes the time-stepping scheme as

$$\mathbf{q}^{n+1} = K_N(\Delta t, \mathcal{L}_N - \tau \mathcal{B}_N) \mathbf{q}^n$$

and strong stability is ensured if

$$|K_N(\Delta t, \mathcal{L}_N - \tau \mathcal{B}_N)| < 1 \;\;,$$

explaining why the CFL-condition depends strongly on the correct choice of the penalty-parameter.

In the following analysis we consider explicit Runge-Kutta time stepping methods, which, for time independent operators, may be expressed as

$$K_N^p(\Delta t, \mathcal{L}_N) = \sum_{i=0}^{p} \frac{1}{i!} (\Delta t \mathcal{L}_N)^i \;\;,$$

where p is the order of the scheme. We have for simplicity assumed that the boundary conditions are included in the operators. Assuming $\mathcal{L}_N = \mathcal{S}_N \Lambda_N \mathcal{S}_N^{-1}$, where $|S_N|$ and $|S_N^{-1}|$ are bounded and independent of N, strong stability of the Runge-Kutta schemes is obtained if

$$|K_N^p(\Delta t, \; \mathcal{L}_N)| = \mathcal{S}_N \left| \sum_{i=0}^{p} \frac{1}{i!} (\Delta t \Lambda_N)^i \right| \mathcal{S}_N^{-1} = \left| \sum_{i=0}^{p} \frac{1}{i!} (\Delta t \Lambda_N)^i \right| < 1 \;\;.$$

TABLE 2

Scaling constants for the advection operator. The proper boundary conditions are of Dirichlet type (D).

Advection Operator	C_L		C_C	
$\lambda \neq 0$ $\varepsilon = 0$	3rd RK	4th RK	3rd RK	4th RK
D Exact BC	21	35	27	32
Penalty BC	10	17	10	11

Hence, the problem is reduced to finding the eigenvalue spectrum of the operator $\mathcal{L}_N$ and choose Δt accordingly.

In the present study we consider the linear advection-diffusion operator;

$$\mathcal{L}_N = \lambda \frac{\partial}{\partial x} + \varepsilon \frac{\partial^2}{\partial x^2} \ ,$$

with the Robin boundary condition operators

$$\mathcal{B}_N^- = \alpha - \varepsilon \beta \frac{\partial}{\partial x} \quad , \quad \mathcal{B}_N^+ = \gamma + \varepsilon \delta \frac{\partial}{\partial x} \ .$$

The boundary conditions for the exact method are enforced through the operator as described in [16].

In order to compare time-step restrictions as found for the two different approaches, we now define the two CFL-like constants, C_L and C_C, as

$$\Delta t_L \leq \frac{C_L}{\lambda N(N+1) + \varepsilon N^2(N+1)^2} \quad , \quad \Delta t_C \leq \frac{C_C}{\lambda N^2 + \varepsilon N^4} \ ,$$

where the subscripts refer to Legendre(L) and Chebyshev(C) operators, respectively. These constants are determined by solving the eigenvalue problem and calculating the maximum Δt which ensures stability and supplies an upper bound on the time-step.

Table 2 and 3 shows the calculated values of C_L and C_C for the advection and the diffusion operator. The results are the same for the full advection-diffusion operator as for the diffusion operator, provided $\varepsilon N^2 \gg 1$, and is therefore omitted.

It is clear from Table 2 that using the penalty method for enforcing boundary conditions on purely advective problems results in a significant reduction of the maximum allowable time-step. However, more importantly, Table 3 shows that for problems where the diffusion operator dominates the eigenvalue spectrum, the penalty method allows for increasing the time-step with as much as 50 %. The effect is most pronounced when using a 4th-order Runge-Kutta method for time-stepping a Chebyshev collocation scheme.

In order to explain the results in Table 2 and 3, we compare in Fig. 2 the spectrum of the Legendre collocation advection (Fig. 2a) and diffusion (Fig. 2b) operators when

TABLE 3

Scaling constants for the diffusion operator. Results are given for possible combinations of Dirichlet (D), Neumann (N) and Robin (R) boundary conditions.

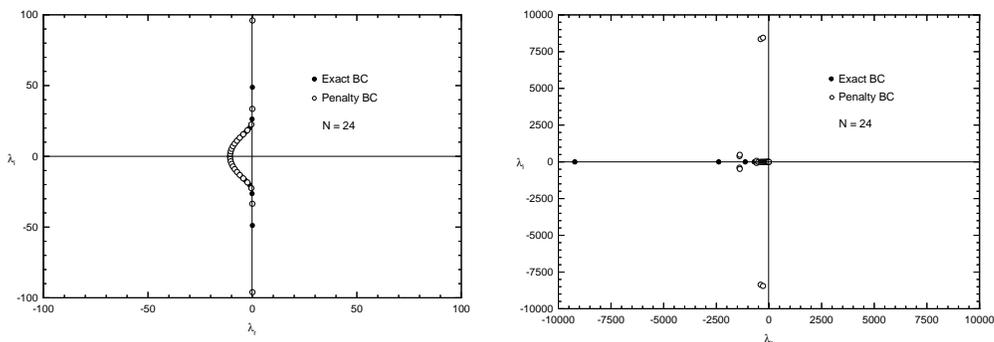| Diffusion Operator		C_L		C_C	
$\lambda = 0 \ \varepsilon > 0$		3rd RK	4th RK	3rd RK	4th RK
D-D/D-N/D-R	Exact BC	99	109	53	58
	Penalty BC	81	123	56	84
N-R	Exact BC	99	109	53	58
	Penalty BC	130	135	91	96
R-R	Exact BC	99	109	53	58
	Penalty BC	130	141	93	97

FIG. 2. Eigenvalue spectrum $(\lambda = \lambda_r + i\lambda_i)$ for the Legendre advection operator (2a) and the Legendre diffusion operator (2b) as obtained by using exact boundary conditions (•) and the penalty method (◦).

enforcing Dirichlet boundary conditions through the exact method and the penalty method.

For the advection operator (Fig. 2a) we observe that the effect of the penalty method is to introduce an extreme complex conjugate eigenvalue-pair, which dominates the spectrum and eventually determines the maximum allowable time-step. This results in the decreased CFL-number as observed in Table 2.

The effect on the diffusion operator is more complicated and depends strongly on the value of the penalty parameter. As proved by Gottlieb and Lustman [15], the diffusion operator with exact Robin boundary conditions has a real, negative and distinct eigenvalue spectrum. This property is preserved if a sufficiently large value of τ is used in the penalty method. However, by decreasing the penalty parameter the two most extreme eigenvalues split into two pairs of complex conjugate eigenvalues, which move towards the imaginary axis, as τ is decreased. In Fig. 2b we show the eigenvalue spectrum for the optimal choice of τ. The important observation to make

is that moduli of these new eigenvalues are smaller than the original extreme negative real eigenvalue. Additionally, since the dominating eigenvalue now is complex, it clearly becomes advantageous to use the 4th-order Runge-Kutta method due to the increased extension of the stability region along ! the imaginary axis as compared to

The validity of this conclusion is, however, strongly dependent on the proper choice of the penalty parameter. The values derived in the previous section do indeed ensure asymptotic stability, but with a significant reduction in the maximum allowable CFL-number as a result. Fortunately, as mentioned previously, the limits of the penalty parameters are based on a conservative energy estimate and are therefore not very accurate. In the following we give the penalty parameters used to obtain the results given in Table 2 and 3. These values result in a stable scheme as long as the problem is purely advective or $\varepsilon N^2 \gg 1$, and allows in most cases for a significant increase in the time-step.

(i) Legendre Collocation Methods

1. Dirichlet Boundary Conditions.

$$\tau = \frac{|\lambda|}{4} N(N+1) + \frac{\varepsilon}{64} N^2 (N+1)^2 \ .$$

2. Neumann Boundary Conditions.

$$\tau = \frac{N(N+1)}{8} \ .$$

3. Robin Boundary Conditions.

$$\tau = \frac{\tau^-_{\alpha,\beta}}{4} \ .$$

(ii) Chebyshev Collocation Methods

1. Dirichlet Boundary Conditions.

$$\tau = \frac{|\lambda|}{2} N^2 + \frac{\varepsilon}{50} N^4 \ .$$

2. Neumann Boundary Conditions.

$$\tau = \frac{N^2}{8} \ .$$

3. Robin Boundary Conditions.

$$\tau = \frac{\tau^-_{\alpha,\beta}}{4} \quad \text{with} \quad \kappa = \frac{\alpha N^2}{\beta} \ .$$

TABLE 4

Maximum allowable CFL-number obtained from direct numerical simulation of Burgers equation.

	Legendre		Chebyshev	
	3rd RK	4th RK	3rd RK	4th RK
Exact BC	3.50	4.00	4.25	4.50
Penalty BC	3.00	4.50	4.75	6.75

Note, that the only difference between the parameter values quoted here and those found is Lemma 3.2, is a factor of 1/4 on those terms related to the diffusion operator. This reduction is found to lead to optimal time-step restrictions.

We would like to stress the importance of choosing the appropriate value of the penalty parameter. It is our experience, that this is best done by deriving the theoretical value of this parameter through an analysis similar to that done in Sec. 3.1. This leads to a parameter which scales correctly with the resolution and other significant parameters. If the time-step restriction is dominated by a viscous time-scale, it is very likely that the theoretical estimate leads to severe time-step restrictions. However, the theoretical value may often be decreased considerably, and good results may be obtained after only a few tests. As we have seen for Burgers equation, decreasing the penalty parameter four times leads to acceptable CFL-restrictions. We are not aware of any systematic way of determining the optimal factor by which the theoretical value should be decreased, but it may usually be determined by trial and error through a few tests.

To conclude our study we have solved Burgers (Eq.(3)) with initial condition

$$(12) \qquad\qquad U(x,0) = (1 - x)(1 - x^2) \ ,$$

and homogeneous Dirichlet boundary conditions. A typical temporal evolution is shown in Fig. 3. In Table 4 we show the maximum CFL-number resulting in a stable scheme. These results confirm that the results from the linear analysis carries over to the non-linear problem.

5. The Compressible Navier-Stokes Equations. In the present section, we obtain energy estimates for the solution to the three-dimensional compressible Navier-Stokes equations given in conservation form. Additionally, we derive open boundary conditions taking into account the full stress-tensor, and prove well-posedness for the continuous problem. The derivations follow the approach introduced in [8, 9]. The main difference being that we develop the theory for the conservation form of the Navier-Stokes equations and that we include the off-diagonal terms of the stress-tensor
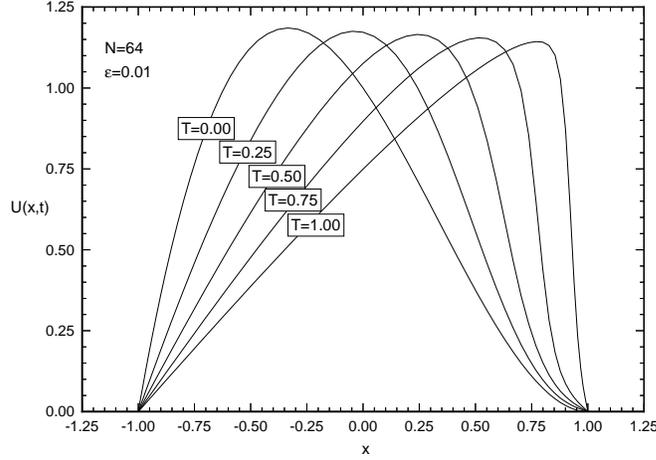
FIG. 3. *Temporal evolution of Burgers equation with initial conditions given by Eq.(12).*

in the full derivations. In the second part of this section, we continue by showing how to apply the boundary conditions and prove asymptotic stability of the semi-discrete scheme.

Consider now the non-dimensionalized, compressible Navier-Stokes equations given in conservation form

$$(13) \qquad \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = \frac{1}{\mathrm{Re}} \left(\frac{\partial \mathbf{F}_\nu}{\partial x} + \frac{\partial \mathbf{G}_\nu}{\partial y} + \frac{\partial \mathbf{H}_\nu}{\partial z} \right) \ ,$$

with $\mathbf{x} \in \Omega = [-1,1]^3$. The state vector, $\mathbf{q}$, and the inviscid flux vectors are given as

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{bmatrix} \ , \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (E+p)u \end{bmatrix} \ , \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ (E+p)v \end{bmatrix} \ , \quad \mathbf{H} = \begin{bmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ (E+p)w \end{bmatrix} \ .$$

Here ρ is the density, u, v, w are the three Cartesian velocity components, E is the total energy and p is the pressure. In the remaining part of the paper we will use (x, y, z) and (x_1, x_2, x_3) interchangeably to denote the spatial coordinates. The total energy

$$E = \rho \left(T + \frac{1}{2} \left(u^2 + v^2 + w^2 \right) \right) \ ,$$

and the pressure is related through the ideal gas law

$$p = (\gamma - 1)\rho T \ ,$$

19

where T is the temperature field and $\gamma = c_p/c_v$ is the ratio between the heat capacities at constant pressure (c_p) and volume (c_v), respectively.

The viscous flux vectors are given as

$$\mathbf{F}_\nu = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{yx} \\ \tau_{zx} \\ \tau_{xx}u + \tau_{yx}v + \tau_{zx}w + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial x} \end{bmatrix} , \quad \mathbf{G}_\nu = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ \tau_{zy} \\ \tau_{xy}u + \tau_{yy}v + \tau_{zy}w + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial y} \end{bmatrix} ,$$

$$\mathbf{H}_\nu = \begin{bmatrix} 0 \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{zz} \\ \tau_{xz}u + \tau_{yz}v + \tau_{zz}w + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial z} \end{bmatrix} .$$

Considering only Newtonian fluids, the stress tensor elements are given as

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad , \quad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad ,$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad , \quad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \quad ,$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \quad , \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad ,$$

where μ is the dynamic viscosity, λ is the bulk viscosity and k is the coefficient of thermal conductivity.

The equations are normalized using the reference values, $u_{\mathrm{ref}} = u_0$, $\rho_{\mathrm{ref}} = \rho_0$, $p_{\mathrm{ref}} = \rho_0 u_0^2$, $T_{\mathrm{ref}} = u_0^2/c_v$ and a reference length L, where (ρ_0, u_0) is some uniform state, e.g. the ambient free-stream conditions of the flow. This gives a Reynolds number as $\mathrm{Re} = \rho_0 u_0 L/\mu_0$ and a Prandtl number as $\mathrm{Pr} = c_p\mu_0/k_0$.

5.1. Well-Posedness and Open Boundary Conditions for the Continuous Problem. Consider the linearized, constant coefficient form of Eq.(13). The viscous fluxes are split as

$$\mathbf{F}_\nu = \mathbf{F}_P + \mathbf{F}_M^y + \mathbf{F}_M^z = \begin{bmatrix} 0 \\ (\lambda + 2\mu)\frac{\partial u}{\partial x} \\ \mu\frac{\partial v}{\partial x} \\ \mu\frac{\partial w}{\partial x} \\ (\lambda + 2\mu)u\frac{\partial u}{\partial x} + \mu v\frac{\partial v}{\partial x} + \mu w\frac{\partial w}{\partial x} + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial x} \end{bmatrix}$$

$$
+ \begin{bmatrix} 0 \\ \lambda\frac{\partial v}{\partial y} \\ \mu\frac{\partial u}{\partial y} \\ 0 \\ \lambda u\frac{\partial v}{\partial y} + \mu v\frac{\partial u}{\partial y} \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda\frac{\partial w}{\partial z} \\ 0 \\ \mu\frac{\partial u}{\partial z} \\ \lambda u\frac{\partial w}{\partial z} + \mu w\frac{\partial u}{\partial z} \end{bmatrix} \quad ,
$$

$$
\mathbf{G}_\nu = \mathbf{G}_P + \mathbf{G}_M^x + \mathbf{G}_M^z = \begin{bmatrix} 0 \\ \mu\frac{\partial u}{\partial y} \\ (\lambda + 2\mu)\frac{\partial v}{\partial y} \\ \mu\frac{\partial w}{\partial y} \\ \mu u\frac{\partial u}{\partial y} + (\lambda + 2\mu)v\frac{\partial v}{\partial y} + \mu w\frac{\partial w}{\partial y} + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial y} \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ \mu\frac{\partial v}{\partial x} \\ \lambda\frac{\partial u}{\partial x} \\ 0 \\ \lambda v\frac{\partial u}{\partial x} + \mu u\frac{\partial v}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda\frac{\partial w}{\partial z} \\ \mu\frac{\partial v}{\partial z} \\ \lambda v\frac{\partial w}{\partial z} + \mu w\frac{\partial v}{\partial z} \end{bmatrix} \quad ,
$$

$$
\mathbf{H}_\nu = \mathbf{H}_P + \mathbf{H}_M^x + \mathbf{H}_M^y = \begin{bmatrix} 0 \\ \mu\frac{\partial u}{\partial z} \\ \mu\frac{\partial v}{\partial z} \\ (\lambda + 2\mu)\frac{\partial w}{\partial z} \\ \mu u\frac{\partial u}{\partial z} + \mu v\frac{\partial v}{\partial z} + (\lambda + 2\mu)w\frac{\partial w}{\partial z} + \frac{\gamma k}{\mathrm{Pr}}\frac{\partial T}{\partial z} \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 \\ \mu\frac{\partial w}{\partial x} \\ 0 \\ \lambda\frac{\partial u}{\partial x} \\ \lambda w\frac{\partial u}{\partial x} + \mu u\frac{\partial w}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mu\frac{\partial w}{\partial y} \\ \lambda\frac{\partial v}{\partial y} \\ \lambda w\frac{\partial v}{\partial y} + \mu v\frac{\partial w}{\partial y} \end{bmatrix} \quad .
$$

Introducing the transformation Jacobians

$$
\mathcal{A}_1 = \frac{\partial \mathbf{F}}{\partial \mathbf{q}} \quad , \quad \mathcal{A}_2 = \frac{\partial \mathbf{G}}{\partial \mathbf{q}} \quad , \quad \mathcal{A}_3 = \frac{\partial \mathbf{H}}{\partial \mathbf{q}} \quad ,
$$

$$
\mathcal{B}_{11} = \frac{\partial \mathbf{F}_P}{\partial \mathbf{q}_x} \quad , \quad \mathcal{B}_{22} = \frac{\partial \mathbf{G}_P}{\partial \mathbf{q}_y} \quad , \quad \mathcal{B}_{33} = \frac{\partial \mathbf{H}_P}{\partial \mathbf{q}_z} \quad ,
$$

$$
\mathcal{B}_{12} = \left(\frac{\partial \mathbf{F}_M^y}{\partial \mathbf{q}_y} + \frac{\partial \mathbf{G}_M^x}{\partial \mathbf{q}_x} \right) \quad , \quad \mathcal{B}_{23} = \left(\frac{\partial \mathbf{G}_M^z}{\partial \mathbf{q}_z} + \frac{\partial \mathbf{H}_M^y}{\partial \mathbf{q}_y} \right) \quad , \quad \mathcal{B}_{13} = \left(\frac{\partial \mathbf{F}_M^z}{\partial \mathbf{q}_z} + \frac{\partial \mathbf{H}_M^x}{\partial \mathbf{q}_x} \right) \quad ,
$$

allows for writing Navier-Stokes equations as

$$\frac{\partial \mathbf{q}}{\partial t} + \sum_{i=1}^{3} \mathcal{A}_i \frac{\partial \mathbf{q}}{\partial x_i} = \frac{1}{\mathrm{Re}} \sum_{i=1}^{3} \sum_{j=i}^{3} \mathcal{B}_{ij} \frac{\partial^2 \mathbf{q}}{\partial x_i\, \partial x_j} \quad .$$

It is well known that Navier-Stokes equations, although not of hyperbolic nature, support waves very similar to those encountered in the hyperbolic Euler equations. For hyperbolic systems, Gottlieb et al. [17] have shown that enforcing the boundary conditions through the characteristic variables of the system results a stable approximation.

For Navier-Stokes equations, we linearize around a uniform state, $\mathbf{q}_0$, by fixing all the matrices. We transform into characteristic variables by diagonalizing $\mathcal{A}_1$ through a similarity transform $\Lambda = \mathcal{S}^{-1} \mathcal{A}_1 \mathcal{S}$, where Λ is the eigenvalue matrix and $\mathcal{S}$ and $\mathcal{S}^{-1}$ are the matrix of right and left eigenvectors, respectively. These matrices are given in the Appendix. Applying this, the symmetrized, linearized set of equations transforms into

$$(14) \qquad \mathcal{Q}^T \mathcal{Q} \frac{\partial \mathbf{R}}{\partial t} + \sum_{i=1}^{3} \mathcal{A}_i^s \frac{\partial \mathbf{R}}{\partial x_i} = \frac{1}{\mathrm{Re}} \sum_{i=1}^{3} \sum_{j=i}^{3} \mathcal{B}_{ij}^s \frac{\partial^2 \mathbf{R}}{\partial x_i\, \partial x_j} \quad ,$$

where $\mathbf{R} = \mathcal{S}^{-1} \mathbf{q}$ are the characteristic variables. We have introduced a positive definite, symmetrizing diagonal matrix, $\mathcal{Q}^T \mathcal{Q}$, given as

$$\mathcal{Q}^T \mathcal{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2c_0^2}{\gamma-1} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

where $c_0 = \sqrt{\gamma p_0 / \rho_0}$ is the uniform state sound speed. Also we define the symmetrized matrices

$$\mathcal{A}_i^s = \mathcal{Q}^T \mathcal{Q}\, \mathcal{S}^{-1} \mathcal{A}_i \mathcal{S} \quad , \quad \mathcal{B}_{ij}^s = \mathcal{Q}^T \mathcal{Q}\, \mathcal{S}^{-1} \mathcal{B}_{ij} \mathcal{S} \quad .$$

The explicit form of the symmetric matrices are given in the Appendix. The characteristic variables, $\mathbf{R} = [R_1,\, R_2,\, R_3,\, R_4,\, R_5]^T$, are given as

$$\mathbf{R} = \begin{bmatrix} \rho u - u_0 \rho + \frac{\gamma-1}{c_0}\left(E + \frac{1}{2}(u_0^2 + v_0^2 + w_0^2)\rho - u_0 \rho u - v_0 \rho v - w_0 \rho w\right) \\ \rho v - v_0 \rho \\ \rho - \frac{\gamma-1}{c_0^2}\left(E + \frac{1}{2}\rho(u_0^2 + v_0^2 + w_0^2) - u_0 \rho u - v_0 \rho v - w_0 \rho w\right) \\ \rho w - w_0 \rho \\ -(\rho u - u_0 \rho) + \frac{\gamma-1}{c_0}\left(E + \frac{1}{2}\rho(u_0^2 + v_0^2 + w_0^2) - u_0 \rho u - v_0 \rho v - w_0 \rho w\right) \end{bmatrix} \quad .$$

We are now ready to state the following

LEMMA 5.1. *Assume there exists a solution, $\mathbf{q}$, which is periodic or held at a constant value at the y- and z-boundary. If the boundary conditions in the x-direction are given such that*

$$\forall (y,z) \in \Omega_y \times \Omega_z \; : \; -\frac{1}{2} \left[\mathbf{R}^T \mathcal{A}_1^s \mathbf{R} - \frac{2}{\mathrm{Re}} \sum_{j=1}^{3} \mathbf{R}^T \mathcal{B}_{1j}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=-1}^{1} \leq 0 \; ,$$

and the fluid properties are constrained by

$$\mu_0 \geq 0 \; , \quad \lambda_0 \leq 0 \; , \quad \lambda_0 + \mu_0 \geq 0 \; , \quad \frac{\gamma k_0}{\mathrm{Pr}} \geq 0 \; , \quad \gamma \geq 1 \; ,$$

then Eq.(14) is a well-posed problem and the solution is bounded as

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\mathbf{R}\|^2 \leq -\frac{1}{\mathrm{Re}} \int_\Omega \left(\sum_{i=1}^{3}\sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) d\Omega \leq 0 \; .$$

Proof. Construct the energy integral as

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\mathbf{R}\|^2 = \int_\Omega \left(-\sum_{i=1}^{3} \mathbf{R}^T \mathcal{A}_i^s \frac{\partial \mathbf{R}}{\partial x_i} + \frac{1}{\mathrm{Re}} \sum_{i=1}^{3}\sum_{j=i}^{3} \mathbf{R}^T \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) d\Omega$$

$$= \int_{\Omega_y} \int_{\Omega_z} -\frac{1}{2} \left[\mathbf{R}^T \mathcal{A}_1^s \mathbf{R} - \frac{2}{\mathrm{Re}} \sum_{j=1}^{3} \mathbf{R}^T \mathcal{B}_{1j}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=-1}^{1} dydz$$

$$- \frac{1}{\mathrm{Re}} \int_\Omega \left(\sum_{i=1}^{3}\sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) d\Omega \; ,$$

where $\Omega = \Omega_x \times \Omega_y \times \Omega_z$. In deriving this expression, we use partial integration and assume the solution to be periodic or held at a constant value along the y- and z-boundaries, i.e. contributions from these boundaries cancel. This is not a severe restriction, as this assumption is valid for a large variety of situations where open boundary conditions are applied.

It is evident that if we can prove

$$(15) \qquad \frac{1}{\mathrm{Re}} \int_\Omega \left(\sum_{i=1}^{3}\sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) d\Omega \geq 0 \; ,$$

then well-posedness may be ensured by properly constructing the boundary operator at the x-boundary.

Since the matrices, $\mathcal{B}_{ij}^s$, are all symmetric, Eq.(15) may be rewritten in a block-quadratic form as

$$\frac{1}{2\text{Re}} \int_\Omega \tilde{\mathbf{R}}^T \mathcal{H}^s \tilde{\mathbf{R}} \, d\Omega \geq 0 \ ,$$

where we have introduced

$$\tilde{\mathbf{R}} = \left[\frac{\partial \mathbf{R}}{\partial x}, \frac{\partial \mathbf{R}}{\partial y}, \frac{\partial \mathbf{R}}{\partial z} \right]^T \ , \quad \mathcal{H}^s = \begin{bmatrix} 2\mathcal{B}_{11}^s & \mathcal{B}_{12}^s & \mathcal{B}_{13}^s \\ \mathcal{B}_{12}^s & 2\mathcal{B}_{22}^s & \mathcal{B}_{23}^s \\ \mathcal{B}_{13}^s & \mathcal{B}_{23}^s & 2\mathcal{B}_{33}^s \end{bmatrix} \ .$$

We observe that $\mathcal{H}^s$ is a 15×15 symmetric matrix, ensuring that the eigenvalue spectrum, $\rho(\mathcal{H}^s)$, is real. Hence, if $\mathcal{H}^s$ is positive semi-definite, Eq.(15) is obeyed. The eigenvalue spectrum, $\rho(\mathcal{H}^s)$, may be found to be

$$
\begin{aligned}
\rho_1 &= \rho_2 = \rho_3 = 0 \\
\rho_4 &= 2(\mu_0 - \lambda_0) \\
\rho_5 &= \rho_6 = 2(\lambda_0 + 3\mu_0) \\
\rho_7 &= \rho_8 = 3\mu_0 - \sqrt{\mu_0^2 + 2(\mu_0 + \lambda_0)^2} \\
\rho_9 &= \rho_{10} = 3\mu_0 + \sqrt{\mu_0^2 + 2(\mu_0 + \lambda_0)^2} \\
\rho_{11} &= 7\mu_0 + 4\lambda_0 - \sqrt{\mu_0^2 + 4(\mu_0 + \lambda_0)(3\mu_0 + 2\lambda_0)} \\
\rho_{12} &= 7\mu_0 + 4\lambda_0 + \sqrt{\mu_0^2 + 4(\mu_0 + \lambda_0)(3\mu_0 + 2\lambda_0)} \\
\rho_{13} &= \rho_{14} = \rho_{15} = \left(\frac{2c_0^2}{(\gamma - 1)^2} + 1 \right) \frac{2(\gamma - 1)k_0}{\text{Pr}} \ .
\end{aligned}
$$

Here subscript '0' signifies the parameter values in the uniform state around which we have linearized. For most real fluids under non-extreme conditions, it is true that μ_0 is positive, λ_0 is negative and the following relationship is obeyed [18]

$$(16) \qquad \frac{\gamma \mu_0}{\text{Pr}} \geq \lambda_0 + 2\mu_0 \geq \mu_0 \ .$$

A simple investigation of the eigenvalues reveals that $\mathcal{H}^s$ is positive semi-definite under these conditions. Thus, Eq.(15) is true provided

$$\mu_0 \geq 0 \ , \quad \lambda_0 \leq 0 \ , \quad \lambda_0 + \mu_0 \geq 0 \ , \quad \frac{\gamma k_0}{\text{Pr}} \geq 0 \ , \quad \gamma \geq 1 \ .$$

These conditions are only natural as discussed in [19]. In fact, if they are not obeyed, Navier-Stokes equations violates the second law of thermodynamics.

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We now obtain that well-posedness is ensured under the additional condition

$$\forall (y, z) \in \Omega_y \times \Omega_z \quad -\frac{1}{2} \left[\mathbf{R}^T \mathcal{A}_1^s \mathbf{R} - \frac{2}{\mathrm{Re}} \sum_{j=1}^{3} \mathbf{R}^T \mathcal{B}_{1j}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=-1}^{1} \leq 0 \ ,$$

and the solution is bounded as

$$\frac{1}{2} \frac{d}{dt} \| \mathcal{Q}\mathbf{R} \|^2 \leq -\frac{1}{\mathrm{Re}} \int_\Omega \left(\sum_{i=1}^{3} \sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) d\Omega \leq 0 \ ,$$

where $\mathcal{Q}\mathbf{R} = \mathcal{Q} \mathcal{S}^{-1} \mathbf{q}$. $\quad \square$

As stated in Lemma 5.1, appropriate boundary conditions at the x-boundary have to obey

$$-\frac{1}{2} \left[\mathbf{R}^T \mathcal{A}_1^s \mathbf{R} - \frac{2}{\mathrm{Re}} \sum_{j=1}^{3} \mathbf{R}^T \mathcal{B}_{1j}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{-1}^{1} \leq 0 \ .$$

We now define

$$Q^T Q \mathbf{G} = \sum_{j=1}^{3} \mathcal{B}_{1j}^s \frac{\partial \mathbf{R}}{\partial x_j} \ ,$$

where

$$
\begin{aligned}
G_1 &= \frac{k_0 (\gamma - 1)}{2\rho_0 \mathrm{Pr}} \frac{\partial \zeta_1}{\partial x} + \frac{\lambda_0 + 2\mu_0}{2\rho_0} \frac{\partial \zeta_2}{\partial x} + \frac{\lambda_0 + \mu_0}{\rho_0} \left(\frac{\partial R_2}{\partial y} - \frac{\partial R_4}{\partial z} \right) \\
G_2 &= \frac{\mu_0}{\rho_0} \frac{\partial R_2}{\partial x} + \frac{\lambda_0 + \mu_0}{\rho_0} \frac{\partial \zeta_2}{\partial y} \\
G_3 &= -\frac{k_0 (\gamma - 1)}{2\rho_0 c_0 \mathrm{Pr}} \frac{\partial \zeta_1}{\partial x} \\
G_4 &= \frac{\mu_0}{\rho_0} \frac{\partial R_4}{\partial x} + \frac{\lambda_0 + \mu_0}{\rho_0} \frac{\partial \zeta_2}{\partial z} \\
G_5 &= \frac{k_0 (\gamma - 1)}{2\rho_0 \mathrm{Pr}} \frac{\partial \zeta_1}{\partial x} - \frac{\lambda_0 + 2\mu_0}{2\rho_0} \frac{\partial \zeta_2}{\partial x} - \frac{\lambda_0 + \mu_0}{\rho_0} \left(\frac{\partial R_2}{\partial y} - \frac{\partial R_4}{\partial z} \right) \ ,
\end{aligned}
$$

(17)

where we, for simplicity, have introduced

$$\zeta_1 = R_1 + R_5 - \frac{2c_0}{\gamma - 1} R_3 \ , \ \zeta_2 = R_1 - R_5 \ .$$

This allows for rewriting the constraint on the boundary contribution as

$$-\frac{1}{2} Q^T \left[\mathbf{R}^T \Lambda \mathbf{R} - \frac{2}{\mathrm{Re}} \mathbf{R}^T \mathbf{G} \right]_{-1}^{1} Q \leq 0 \ ,$$

where Λ is the diagonal eigenvalue matrix obtained from the similarity transform. We now reformulate this as

(18) $$-\frac{1}{2} \left[\sum_{i=1}^{5} \lambda_i^{-1} \left(\left(|\lambda_i| R_i - \varepsilon \frac{|\lambda_i|}{\lambda_i} G_i \right)^2 - (\varepsilon G_i)^2 \right) \right]_{-1}^{1} \leq 0 \ ,$$

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where λ_i are the wave speeds by which the characteristic variables are advected, as given by the diagonal elements of Λ, and we have introduced $\varepsilon = \mathrm{Re}^{-1}$. This formulation makes it straightforward to devise inflow-outflow boundary conditions, which are maximal dissipative and ensure well-posedness of the complete problem.

We note in particular that this formulation takes into account the off-diagonal terms of the stress tensor, which is neglected in most previous work [8, 9, 10]. These terms may be of importance if the artificial boundary is introduced into a strongly vortical region of the flow, e.g. a wake flow behind a blunt body.

Inflow Boundary Conditions. At $x = -1$, Eq.(18) becomes

$$\frac{1}{2}\sum_{i=1}^{5}\lambda_i^{-1}\left(\left(|\lambda_i|R_i - \varepsilon\frac{|\lambda_i|}{\lambda_i}G_i\right)^2 - (\varepsilon G_i)^2\right) \leq 0 \ .$$

Subsonic Inflow : $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_5 < 0$

$$\begin{aligned}\lambda_1 R_1 - \varepsilon G_1 &= 0 \\ \lambda_2 R_2 - \varepsilon G_2 &= 0 \\ \lambda_3 R_3 - \varepsilon G_3 &= 0 \\ \lambda_4 R_4 - \varepsilon G_4 &= 0 \\ \varepsilon G_5 &= 0 \end{aligned}$$

(19)

Supersonic Inflow : $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_5 > 0$

$$\begin{aligned}\lambda_1 R_1 - \varepsilon G_1 &= 0 \\ \lambda_2 R_2 - \varepsilon G_2 &= 0 \\ \lambda_3 R_3 - \varepsilon G_3 &= 0 \\ \lambda_4 R_4 - \varepsilon G_4 &= 0 \\ \lambda_5 R_5 - \varepsilon G_5 &= 0 \end{aligned}$$

(20)

Outflow Boundary Conditions. At $x = 1$, Eq.(18) becomes

$$\frac{1}{2}\sum_{i=1}^{5}-\lambda_i^{-1}\left(\left(|\lambda_i|R_i - \varepsilon\frac{|\lambda_i|}{\lambda_i}G_i\right)^2 - (\varepsilon G_i)^2\right) \leq 0 \ .$$

Subsonic Outflow : $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_5 < 0$

$$\begin{aligned}\varepsilon G_2 &= 0 \\ \varepsilon G_3 &= 0 \\ \varepsilon G_4 &= 0 \\ |\lambda_5|R_5 + \varepsilon G_5 &= 0 \end{aligned}$$

(21)

Supersonic Outflow : $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$, $\lambda_4 > 0$, $\lambda_5 > 0$

$$
\begin{array}{ccc}
\varepsilon G_1 = 0 & & \varepsilon G_2 = 0 \\
\varepsilon G_2 = 0 & & \varepsilon G_3 = 0 \\
\varepsilon G_3 = 0 & \text{or} & \varepsilon G_4 = 0 \\
\varepsilon G_4 = 0 & & \varepsilon G_5 = 0
\end{array}
$$

(22)

We note that for both types of outflow boundary conditions, it is only necessary to specify four conditions, since $\varepsilon G_3 = 0 \Rightarrow \varepsilon G_1 = -\varepsilon G_5$. Due to the special structure of $\mathbf{G}$ we also observe that adding an extra condition on εG_1 does not place extra conditions on the solution, since such a condition is redundant. This observation will be used later.

It was shown by Strikwerda [20] that the proper number of boundary conditions for an incomplete, parabolic system, like the compressible Navier-Stokes equations, is 5 in the inflow region and 4 in the outflow region. Our result clearly conforms with that.

We also note that in the limit of infinite Reynolds number, these boundary conditions converge uniformly toward the well known characteristic boundary conditions for the compressible Euler equations [21]. This property is important in order to avoid weak boundary layers of the order $\exp(-x/\varepsilon)$ (see [8]).

5.2. The Semi-Discrete Scheme. Following the line of thought that led to the asymptotically stable scheme for Burgers equation, we propose a Legendre collocation scheme for enforcing open boundary conditions to the compressible Navier-Stokes equations

$$
\begin{aligned}
(23)\quad \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} = \quad & \frac{1}{\mathrm{Re}}\left(\frac{\partial \mathbf{F}_\nu}{\partial x} + \frac{\partial \mathbf{G}_\nu}{\partial y} + \frac{\partial \mathbf{H}_\nu}{\partial z}\right) \\
& -\tau_1 Q^-(x)\mathcal{S}\left(\mathcal{R}^-\left(\mathbf{R} - \mathcal{S}^{-1}\mathbf{g}_1(t)\right) - \frac{1}{\mathrm{Re}}\mathcal{G}^-\mathbf{G}\right) \\
& -\tau_2 Q^+(x)\mathcal{S}\left(\mathcal{R}^+\left(\mathbf{R} - \mathcal{S}^{-1}\mathbf{g}_2(t)\right) + \frac{1}{\mathrm{Re}}\mathcal{G}^+\mathbf{G}\right) \quad .
\end{aligned}
$$

Here $Q^-(x)$ and $Q^+(x)$ are given by Eq.(9) and $\mathcal{S}$ is the right eigenvector matrix as given in the Appendix. The boundary conditions for the state vector are given through the two vectors, $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$, which we for convenience assume to be uniform. The four matrices, $\mathcal{R}^-$, $\mathcal{R}^+$, $\mathcal{G}^-$ and $\mathcal{G}^+$ are chosen such as to construct the appropriate boundary operator as derived in the previous section. Hence, we have for the inflow region

$$
\mathcal{R}^- = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \alpha\lambda_5 \end{bmatrix} \quad , \quad \mathcal{G}^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ,
$$

where $\alpha = 0$ for subsonic conditions and $\alpha = 1$ for supersonic conditions. Likewise we define

$$
\mathcal{R}^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta|\lambda_5| \end{bmatrix} \quad , \quad \mathcal{G}^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ,
$$

where $\beta = 1$ for subsonic conditions and $\beta = 0$ for supersonic outflow conditions.

We have to choose τ_1 and τ_2 such that the semi-discrete scheme is asymptotically stable. The proper choice is stated in the following Lemma.

LEMMA 5.2. *Assume there exists a solution, $\mathbf{q}$, which is periodic or held at a constant value at the y- and z-boundary, and that the fluid properties of the uniform state, $\mathbf{q_0}$, are constrained by*

$$
\mu_0 \geq 0 \quad , \quad \lambda_0 \leq 0 \quad , \quad \lambda_0 + \mu_0 \geq 0 \quad , \quad \frac{\gamma k_0}{\mathrm{Pr}} \geq 0 \quad , \quad \gamma \geq 1 \quad ,
$$

and related as

$$
\frac{\gamma \mu_0}{\mathrm{Pr}} \geq \lambda_0 + 2\mu_0 \geq \mu_0 \quad .
$$

The linearized, constant coefficient version of the scheme given by Eq.(23) is asymptotically stable at the <u>*inflow*</u> *if*

$$
\frac{1}{\omega\kappa}\left(1 + \kappa + \sqrt{1+\kappa}\right) \geq \tau_1 \geq \frac{1}{\omega\kappa}\left(1 + \kappa - \sqrt{1+\kappa}\right) \quad .
$$

Here

$$
\kappa = \frac{\varepsilon}{2\omega}\,\frac{k_0}{\mathrm{Pr}\rho_0 u_0} \quad .
$$

These results are independent on whether the inflow is subsonic or supersonic.

For <u>*supersonic outflow*</u>

$$\frac{1}{\omega}\left(1 + \sqrt{\frac{1}{\kappa}}\right) \geq \tau_2 \geq \frac{1}{\omega}\left(1 - \sqrt{\frac{1}{\kappa}}\right) \quad .$$

For <u>*subsonic outflow*</u>

$$\frac{1}{\omega}\left(1 + \sqrt{\frac{1}{\kappa}}\right) \geq \tau_2 \geq \frac{1}{2\omega} \quad .$$

The solution to Eq.(23) is bounded in the form

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\mathbf{R})\|_N^2 \leq -\frac{1}{\mathrm{Re}}\sum_{k=1}^{N-1}\omega_k \int_{\Omega_y}\int_{\Omega_z}\left[\sum_{i=1}^{3}\sum_{j=i}^{3}\frac{\partial \mathbf{R}^T}{\partial x_i}\mathcal{B}_{ij}^s\frac{\partial \mathbf{R}}{\partial x_j}\right]_{x=x_k} dy\,dz \leq 0 \quad .$$

Proof. Write Eq.(23) is its symmetrized, linearized, constant coefficient version

$$\mathcal{Q}^T\mathcal{Q}\frac{\partial \mathbf{R}}{\partial t} + \sum_{i=1}^{3}\mathcal{A}_i^s\frac{\partial \mathbf{R}}{\partial x_i} = \quad \frac{1}{\mathrm{Re}}\sum_{i=1}^{3}\sum_{j=i}^{3}\mathcal{B}_{ij}^s\frac{\partial^2 \mathbf{R}}{\partial x_i\,\partial x_j}$$

$$-\tau_1 Q^-(x)\left(\mathcal{Q}^T\mathcal{Q}\,\mathcal{R}^-\mathbf{R} - \frac{1}{\mathrm{Re}}\mathcal{Q}^T\mathcal{Q}\,\mathcal{G}^-\mathbf{G}\right)$$

$$-\tau_2 Q^+(x)\left(\mathcal{Q}^T\mathcal{Q}\,\mathcal{R}^+\mathbf{R} + \frac{1}{\mathrm{Re}}\mathcal{Q}^T\mathcal{Q}\,\mathcal{G}^+\mathbf{G}\right) \quad ,$$

where we, without loss of generality, have assumed homogeneous boundary conditions. We construct the energy integral, apply the Gauss-Lobatto quadrature rule and partial integration to obtain

$$(24)\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\mathbf{R}\|_N^2 \quad = \int_{\Omega_y}\int_{\Omega_z} -\frac{1}{2}\left[\mathbf{R}^T\mathcal{A}_1^s\mathbf{R} - 2\varepsilon\sum_{j=1}^{3}\mathbf{R}^T\mathcal{B}_{1j}^s\frac{\partial \mathbf{R}}{\partial x_j}\right]_{x=-1}^{1} dy\,dz$$

$$-\varepsilon\int_{\Omega}\left(\sum_{i=1}^{3}\sum_{j=i}^{3}\frac{\partial \mathbf{R}^T}{\partial x_i}\mathcal{B}_{ij}^s\frac{\partial \mathbf{R}}{\partial x_j}\right) d\Omega$$

$$-\tau_1\omega\int_{\Omega_y}\int_{\Omega_z}\left[\mathbf{R}^T\mathcal{Q}^T\mathcal{Q}\,\mathcal{R}^-\mathbf{R} - \varepsilon\mathbf{R}^T\mathcal{G}^-\sum_{j=1}^{3}\mathcal{B}_{1j}^s\frac{\partial \mathbf{R}}{\partial x_j}\right]_{x=-1} dy\,dz$$

$$-\tau_2\omega\int_{\Omega_y}\int_{\Omega_z}\left[\mathbf{R}^T\mathcal{Q}^T\mathcal{Q}\,\mathcal{R}^+\mathbf{R} + \varepsilon\mathbf{R}^T\mathcal{G}^+\sum_{j=1}^{3}\mathcal{B}_{1j}^s\frac{\partial \mathbf{R}}{\partial x_j}\right]_{x=1} dy\,dz \quad ,$$

where we have used the assumption about periodicity or constant value at the y- and z-boundary. Additionally, we have introduced $\varepsilon = \mathrm{Re}^{-1}$ and ω, which is the Legendre weight at the endpoints and applied the definition

$$\mathcal{Q}^T\mathcal{Q}\,\mathbf{G} = \sum_{j=1}^{3}\mathcal{B}_{1j}^s\frac{\partial \mathbf{R}}{\partial x_j} \quad .$$

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Using the Gauss-Lobatto quadrature rule allows for writing

$$(25) \quad \int_\Omega \left(\sum_{i=1}^{3} \sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right) \quad d\Omega =$$

$$\int_{\Omega_y} \int_{\Omega_z} \left(\sum_{k=1}^{N-1} \omega_k \left[\sum_{i=1}^{3} \sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=x_k} \right.$$

$$+ \omega \left[\sum_{i=1}^{3} \sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=-1}$$

$$\left. + \omega \left[\sum_{i=1}^{3} \sum_{j=i}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ij}^s \frac{\partial \mathbf{R}}{\partial x_j} \right]_{x=1} \right) dy\, dz \geq 0 \ .$$

Here x_k signifies the Legendre-Gauss-Lobatto collocation points. The inequality follows from the analysis done in the proof for Lemma 5.1, and is ensured provided the fluid properties are constrained by

$$\mu_0 \geq 0 \ , \quad \lambda_0 \leq 0 \ , \quad \lambda_0 + \mu_0 \geq 0 \ , \quad \frac{\gamma k_0}{\mathrm{Pr}} \geq 0 \ , \quad \gamma \geq 1 \ .$$

It was shown by Abarbanel and Gottlieb [18] that if a scheme is stable without the contributions from the off-diagonal stress-tensor terms, then it will remain so even if the these terms are included. This is a consequence of the general relation

$$\frac{\gamma \mu_0}{\mathrm{Pr}} \geq \lambda_0 + 2\mu_0 \geq \mu_0 \ ,$$

which roughly gives the relation between the eigenvalues of the normal stress-tensor elements and the off-diagonal elements. Thus, it is sufficient to prove stability in the absence of the off-diagonal contributions.

The penalty parameters, τ_1 and τ_2, has to be chosen such that the boundary term of the energy integral not destroys the stability of the Cauchy-problem. We treat the two boundary contributions separately.

Inflow Condition. The contribution of the boundary term at the inflow ($x = -1$) follows from combining Eq.(24) and Eq.(25) and neglecting the off-diagonal contributions to obtain

$$\mathbf{R}^T \left(\frac{1}{2}\mathcal{A}_1^s - \tau_1 \omega \mathcal{Q}^T \mathcal{Q} \mathcal{R}^- \right) \mathbf{R} - \varepsilon \mathbf{R}^T \left(\mathcal{I} - \tau_1 \omega \mathcal{G}^- \right) \mathcal{B}_{11}^s \frac{\partial \mathbf{R}}{\partial x_1} - \varepsilon \omega \sum_{i=1}^{3} \frac{\partial \mathbf{R}^T}{\partial x_i} \mathcal{B}_{ii}^s \frac{\partial \mathbf{R}}{\partial x_i} \leq 0 \ ,$$

where $\mathcal{I}$ is the identity matrix.

First we note that

$$-\varepsilon \omega \frac{\partial \mathbf{R}^T}{\partial x_2} \mathcal{B}_{22}^s \frac{\partial \mathbf{R}}{\partial x_2} - \varepsilon \omega \frac{\partial \mathbf{R}^T}{\partial x_3} \mathcal{B}_{33}^s \frac{\partial \mathbf{R}}{\partial x_3} \leq 0 \ ,$$

since $\mathcal{B}_{22}^s$ and $\mathcal{B}_{33}^s$ are positive semi-definite with an eigenvalue spectrum given as

$$
\begin{array}{ll}
\rho_1(\mathcal{B}_{22}^s) = \rho_1(\mathcal{B}_{33}^s) = 0 & \rho_2(\mathcal{B}_{22}^s) = \rho_2(\mathcal{B}_{33}^s) = \frac{\mu_0}{\rho_0} \\[2mm]
\rho_3(\mathcal{B}_{22}^s) = \rho_3(\mathcal{B}_{33}^s) = 2\frac{\mu_0}{\rho_0} & \rho_4(\mathcal{B}_{22}^s) = \rho_4(\mathcal{B}_{33}^s) = 2\frac{\lambda_0+2\mu_0}{\rho_0} \\[2mm]
\rho_5(\mathcal{B}_{22}^s) = \left(\frac{2c_0^2}{(\gamma-1)^2}+1\right)\frac{(\gamma-1)k_0}{\rho_0 \mathrm{Pr}} & \rho_5(\mathcal{B}_{33}^s) = \left(\frac{2c_0^2}{(\gamma-1)^2}+1\right)\frac{(\gamma-1)k_0}{\rho_0 \mathrm{Pr}}
\end{array} \quad .
$$

Since all matrices are symmetric, the remaining part of the constraint may be expressed in block-quadratic form as

$$
\tilde{\mathbf{R}}^T \mathcal{H}^- \tilde{\mathbf{R}} \leq 0 \ ,
$$

where

$$
\tilde{\mathbf{R}} = \left[\mathbf{R},\ \frac{\partial \mathbf{R}}{\partial x}\right]^T \ , \quad \mathcal{H}^- = \frac{1}{2}\left[\begin{array}{cc} \mathcal{A}_1^s - 2\tau_1\omega\mathcal{Q}^T\mathcal{Q}\mathcal{R}^- & -\varepsilon(1-\tau_1\omega)\mathcal{B}_{11}^s \\[2mm] -\varepsilon(1-\tau_1\omega)\mathcal{B}_{11}^s & -2\varepsilon\omega\mathcal{B}_{11}^s \end{array}\right] \ ,
$$

where we have used $\mathcal{G}^- = \mathcal{I}$. $\mathcal{H}^-$ is a 10×10 symmetric block-matrix. Similar to the approach applied in Sec. 3.2, we have transformed the problem of stability into proving that $\mathcal{H}^-$, for a suitable value of τ_1, is negative semi-definite. The eigenvalue-spectrum, $\rho(\mathcal{H}^-)$, can be found by doing a LU-decomposition. Since $\mathcal{H}^-$ is symmetric, the eigenvalues appear as $\rho_i(\mathcal{H}^-) = U_{ii}$.

We will not give the general form of the eigenvalues here, since they are rather complicated. However, straightforward but very lengthy algebra shows that all eigenvalues are negative if τ_1 is chosen such that

$$
\frac{1}{\omega\kappa}\left(1 + \kappa + \sqrt{1+\kappa}\right) \geq \tau_1 \geq \frac{1}{\omega\kappa}\left(1 + \kappa - \sqrt{1+\kappa}\right) \ ,
$$

where

$$
\kappa = \frac{\varepsilon}{2\omega}\frac{k_0}{\mathrm{Pr}\rho_0 u_0} \ .
$$

This result is independent of whether the inflow is subsonic or supersonic.

Outflow Condition. Neglecting the contribution from the off-diagonal terms yields a criteria for stability at the outflow ($x = 1$)

$$
-\mathbf{R}^T\left(\frac{1}{2}\mathcal{A}_1^s + \tau_2\omega\mathcal{Q}^T\mathcal{Q}\mathcal{R}^+\right)\mathbf{R} + \varepsilon\mathbf{R}^T\left(\mathcal{I} - \tau_2\omega\mathcal{G}^+\right)\mathcal{B}_{11}^s\frac{\partial\mathbf{R}}{\partial x_1} - \varepsilon\omega\sum_{i=1}^{3}\frac{\partial\mathbf{R}^T}{\partial x_i}\mathcal{B}_{ii}^s\frac{\partial\mathbf{R}}{\partial x_i} \leq 0 \ .
$$

Similar to the approach followed in the previous part of the proof, we see that the contributions from $\mathcal{B}_{22}^s$ and $\mathcal{B}_{33}^s$ are always negative and independently ensure stability.

We now rewrite the remaining part of the condition at the outflow in block-quadratic form;

$$
\tilde{\mathbf{R}}^T \mathcal{H}^+ \tilde{\mathbf{R}} \leq 0 \ ,
$$

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where

$$\mathcal{H}^+ = \frac{1}{2}\left[\begin{array}{cc} -\mathcal{A}_1^s - 2\tau_2\omega\mathcal{Q}^T\mathcal{Q}\mathcal{R}^+ & \varepsilon(1-\tau_2\omega)\mathcal{B}_{11}^s \\ \varepsilon(1-\tau_1\omega)\mathcal{B}_{11}^s & -2\varepsilon\omega\mathcal{B}_{11}^s \end{array}\right] .$$

To form $\mathcal{H}^+$ we have assumed $\mathcal{G}^+ = \mathcal{I}$. The additional boundary condition introduced by this replacement is redundant as discussed in Sec. 5.1.2., and, hence, no extra restrictions are put on the system by this approach. The eigenvalue spectrum, $\rho(\mathcal{H}^+)$, may again be found through a LU-decomposition. We state here only the bounds on τ_2 that ensure negative semi-definiteness of $\mathcal{H}^+$ for supersonic outflow

$$\frac{1}{\omega}\left(1 + \sqrt{\frac{1}{\kappa}}\right) \geq \tau_2 \geq \frac{1}{\omega}\left(1 - \sqrt{\frac{1}{\kappa}}\right) .$$

For subsonic outflow the bounds become

$$\frac{1}{\omega}\left(1 + \sqrt{\frac{1}{\kappa}}\right) \geq \tau_2 \geq \frac{1}{2\omega} .$$

Combining Eq.(24) and Eq.(25), we obtain a bound for the growth of the solution

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}\mathbf{R})\|_N^2 \leq -\frac{1}{\mathrm{Re}}\sum_{k=1}^{N-1}\omega_k\int_{\Omega_y}\int_{\Omega_z}\left[\sum_{i=1}^{3}\sum_{j=i}^{3}\frac{\partial\mathbf{R}^T}{\partial x_i}\mathcal{B}_{ij}^s\frac{\partial\mathbf{R}}{\partial x_j}\right]_{x=x_k} dy\,dz \leq 0 . \qquad \square$$

We wish to emphasize that the bounds on τ_1 and τ_2 given in Lemma 5.2 remains valid in the limit when the Reynolds number approaches infinity. This is easily realized by expanding the bounds for $\varepsilon \ll 1$ to obtain

$$\infty > \tau_1 \geq \frac{1}{2\omega} + \varepsilon\frac{1}{8\omega}\kappa ,$$

in the inflow region and

$$\infty > \tau_2 > -\infty \quad , \quad \infty > \tau_2 \geq \frac{1}{2\omega} ,$$

for supersonic and subsonic outflow, respectively. The linearized, constant coefficient version of the Euler equations may be transformed into 5 independent hyperbolic equations for which we should expect the bounds on the penalty parameters to be given by the results in Sec. 3.1.1. We observe that the bounds given above converge uniformly to the expected values in the limit of vanishing viscosity and, thus, the scheme remains stable. The observation that no bounds are necessary on τ_2 for supersonic outflow simply reflects the fact that no boundary conditions are required for the Euler equations at such a boundary.

5.3. Numerical Tests. The proof given in the previous section is only strictly valid for the linearized, constant coefficient version of Navier-Stokes equations. To validate the results and show that it carries over to the full non-linear Navier-Stokes equations, we have implemented the scheme in an existing spectral code (see [22] for details), originally developed for studying two-dimensional compressible flow around an infinitely long circular cylinder.

As spatial approximation scheme was used a standard Fourier-Chebyshev collocation scheme in polar coordinates, (r, θ), with a 3rd-order Runge-Kutta method for time-stepping.

The new scheme is simple to implement in existing codes, as we only need to apply a correction of the flux of the state vector at the boundary. Following the scheme, given by Eq.(23), we need to derive the two vectors $\mathbf{R}$ and $\mathbf{G}$. The characteristic variables are given as

$$
\begin{aligned}
R_1 &= (m_r - \rho u_r) + \frac{p}{c_0} \ , \\
R_2 &= \rho - \frac{p}{c_0^2} \ , \\
R_3 &= m_\theta - \rho u_\theta \ , \\
R_4 &= -(m_r - \rho u_r) + \frac{p}{c_0} \ .
\end{aligned}
$$

where c_0 is the uniform state sound speed.

We have for convenience introduced

$$
u_r = u_0 \hat{k}_1 + v_0 \hat{k}_2 \ , \quad u_\theta = u_0 \hat{k}_2 - v_0 \hat{k}_1 \ ,
$$

which are the radial and azimuthal velocity components, respectively, of the uniform state and

$$
m_r = m_u \hat{k}_1 + m_v \hat{k}_2 \ , \quad m_\theta = m_u \hat{k}_2 - m_v \hat{k}_1 \ ,
$$

are the radial and azimuthal components of the momentum of the flow field. Here $\mathbf{k} = (\hat{k}_1, \hat{k}_2)$ signifies an outward pointing normal-vector at the boundary. The linearized pressure, p, is given as

$$
p = (\gamma - 1) \left[E + \frac{1}{2}\rho(u_0^2 + v_0^2) - u_0 m_u - v_0 m_v \right] \ .
$$

The eigenvalues corresponding to the characteristic functions and determining the direction and propagation velocity of the characteristic waves, are

$$
\lambda_1 = u_r + c_0 \ , \quad \lambda_2 = \lambda_3 = u_r \ , \quad \lambda_4 = u_r - c_0 \ .
$$

33

Following the approach outlined in the previous section, we have likewise derived the viscous correction vector, $\mathbf{G}$, at the outer boundary as

$$G_1 = \frac{(\gamma - 1)k_0}{\text{Pr}} \frac{1}{2\rho_0} \frac{\partial \zeta_1}{\partial r} + \frac{2}{3} \frac{\mu_0}{\rho_0} \frac{\partial \zeta_2}{\partial r} - J \frac{1}{3} \frac{\mu_0}{\rho_0} \frac{\partial R_3}{\partial \theta} \ ,$$

$$G_2 = -\frac{(\gamma - 1)k_0}{\text{Pr}} \frac{1}{2c_0\rho_0} \frac{\partial \zeta_1}{\partial r} \ ,$$

$$G_3 = \frac{\mu_0}{\rho_0} \frac{\partial R_3}{\partial r} + J \frac{1}{6} \frac{\mu_0}{\rho_0} \frac{\partial \zeta_2}{\partial \theta} \ ,$$

$$G_4 = \frac{(\gamma - 1)k_0}{\text{Pr}} \frac{1}{2\rho_0} \frac{\partial \zeta_1}{\partial r} - \frac{2}{3} \frac{\mu_0}{\rho_0} \frac{\partial \zeta_2}{\partial r} + J \frac{1}{3} \frac{\mu_0}{\rho_0} \frac{\partial R_3}{\partial \theta} \ ,$$

where again we have defined

$$\zeta_1 = R_1 + R_4 - \frac{2c_0}{\gamma - 1} R_2 \ , \quad \zeta_2 = R_1 - R_4 \ .$$

Also, we have $J = 1/r$, which is the transformation Jacobian from Cartesian to polar coordinates. We note that no extra calculation of derivatives is needed in order to form the two vectors, since the radial and azimuthal derivatives at the boundary are calculated during evaluation of the interior dynamics when employing a global scheme. Thus, the only additional requirement is to store values of the derivatives of the state vector at the boundary, i.e. the computational requirement for enforcing this new method is negligible.

The boundary conditions are enforced at each intermediate time step of the Runge-Kutta method. Simulations were done with a Reynolds number of 100, a Mach number of 0.4, the diameter (D) of the cylinder being 6.10 cm and the reference temperature was $300°K$. These parameters ensure that the flow field remains subsonic. The resolution was 96 Fourier-modes, 72 Chebyshev modes and the radius (L) of the computational domain was 20 cylinder diameters.

As penalty parameters we used

$$\tau_1 = \frac{N^2}{4\kappa} \frac{2}{L} (1 + \kappa - \sqrt{1 + \kappa}) \ , \quad \tau_2 = \frac{N^2}{2} \frac{2}{L} \ ,$$

where N is the number of Chebyshev modes, $2/L$ is a factor occurring from the radial mapping of L into $[-1, 1]$ and

$$\kappa = \frac{\varepsilon N^2}{2} \frac{k_0}{\text{Pr}\rho_0 u_0} \ ,$$

This choice appears naturally from the results stated in Lemma 5.2, and the experience gained in Sec. 4, indicating that for dissipative terms we should reduce by a factor of 4 in order to obtain the optimal value of τ. With this choice of penalty

parameters we were able to perform the simulations without any reduction in time-step as compared to the exact method of enforcing the boundary conditions. It should be mentioned, that in the original code only characteristic boundary conditions for the Euler equations were enforced. Comparing with results discussed discussed in Sec. 4, we observe that for 3rd-order Runge-Kutta we should expect the two methods to impose almost equivalent time-step restrictions. This is confirmed by the simulations and shows that the results from the simple linear analysis carries over to the full non-linear Navier-Stokes equations in this case.

In Fig. 4 we show contour-plots of the normalized density and the pressure at T=143.5, corresponding to approximately 23 shedding cycles. The von Karman vortex street is clearly demonstrated, and we observe that the boundary conditions at the outflow boundary affect the flow only slightly. The Strouhal number for the shredding frequency is found to be $St = 0.163$, which is in full accordance with experimental findings [23] and we observe no spurious frequencies or reflections from the artificial boundary back into the flow field (see [22] for a further discussion of this).
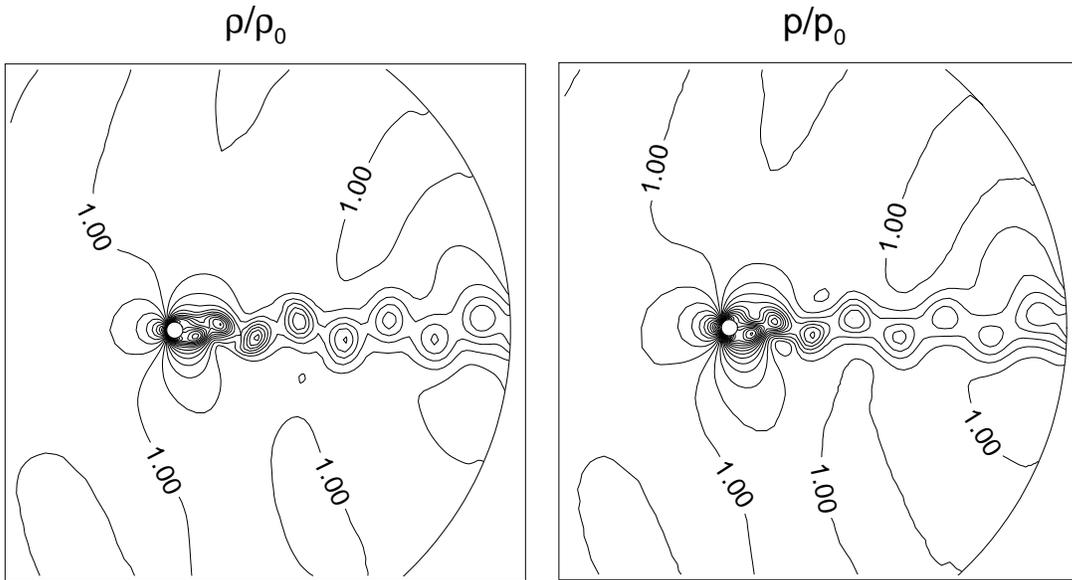
$$\rho/\rho_0 \qquad\qquad p/p_0$$

FIG. 4. *Contour plots of the normalized density, ρ/ρ_0, and the normalized pressure, p/p_0, at the non-dimensional time T=143.5 for a flow at* Re = 100, M = 0.4, D = 6.10 cm *and* $T_0 = 300°K$.

6. Concluding Remarks. The purpose of the present paper has been two-fold. The first goal has been to develop boundary conditions for wave-dominated problems, leading to well-posed total problems. It was argued, that for smooth solutions and the kind of operators we have considered here, it is sufficient to consider the problem of well-posedness for the linearized, constant coefficient version of the non-linear initial-

boundary value problem. Using this allowed for deriving proper boundary conditions to Burgers equation and the three-dimensional, compressible Navier-Stokes equations, and these boundary conditions were shown to ensure well-posedness of the total problem. It should be stressed that the boundary conditions derived for the Navier-Stokes equations takes into account all elements of the stress-tensor, and only very light assumptions were made to derive these. Additionally, they remain valid even in the limit of vanishing viscosity.

Having derived appropriate boundary conditions naturally leads to the question of how to enforce these in a discrete approximation of the problem. This has been the second, and main, contribution of the paper. Recent results [7] on the connection between stability of discrete and semi-discrete approximations, suggest that it is sufficient to consider asymptotic stability for the semi-discrete approximation. We have only considered Legendre collocation methods here. This choice is merely dictated by a wish to obtain analytical results and we have indicated, by numerical tests, that all results carry over to Chebyshev collocation operators. The stability proofs for the semi-discrete approximations to the linearized, constant coefficient versions of Burgers equation and the compressible Navier-Stokes equations are all completed by using the classical energy method. We emphasized that the proposed schemes remain stable even in the limit where the problems become pure! ly hyperbolic.

The proposed penalty method changes the eigenvalue spectra of the discrete approximations of the operators considerably. In order to understand this, we performed a detailed investigation of the effect on the eigenvalue spectra of linear operators. It has been shown that the value of the penalty parameter, which is obtained form the theoretical analysis, often implies that the maximum allowable time-step compares unfavorable with that allowed through more traditional methods. However, we discussed in detail how to remedy this and showed that choosing the penalty parameter properly may allow for increasing the maximum time-step with as much as 50%. Although we are not aware of a systematic way of determining the optimal value of the penalty parameter, we do not see that as any significant disadvantage. Our experience tells that once the theoretical values of the penalty parameters are obtained, only a few tests are needed to obtain the optimal value. Additionally, this only ! has to be done once, and since on

Most of the theoretical results, obtained for linearized, constant coefficient versions of the equations, are confirmed by numerical simulations of the full non-linear equations. It is stressed that the proposed penalty method is very easy to implement in existing codes, which is an attractive feature.

Although all results and numerical simulations in this paper are obtained using

spectral collocation methods, the main conclusions carry over to finite difference/finite element methods. The derivation of the proper boundary operators, be that for Burgers equation or for the compressible, Navier-Stokes equations, is obviously unaffected by the choice of spatial approximation method. The proposed penalty method for enforcing the boundary conditions may be applied in exactly the same manner as discussed here, when using alternative spatial discretization methods. The only difference is the value of the penalty parameter, which will depend strongly on the order of the method. Thus, applying an other method requires one to derive this penalty parameter. This may be done by an approach equivalent to the one utilized here.

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Appendix: Symmetric Matrices for the Navier-Stokes Equations.. Consider the linearized, constant coefficient compressible Navier-Stokes equations in conservation form given as

$$\frac{\partial \mathbf{q}}{\partial t} + \sum_{i=1}^{3} \mathcal{A}_i \frac{\partial \mathbf{q}}{\partial x_i} = \frac{1}{\text{Re}} \sum_{i=1}^{3} \sum_{j=i}^{3} \mathcal{B}_{ij} \frac{\partial^2 \mathbf{q}}{\partial x_i \partial x_j} \quad .$$

The matrix, $\mathcal{A}_1$, diagonalizes under the similarity transform, $\Lambda = \mathcal{S}^{-1} \mathcal{A}_1 \mathcal{S}$, where the right eigenvector matrix, $\mathcal{S}$, and the left eigenvector matrix, $\mathcal{S}^{-1}$, are given as

$$\mathcal{S} = \begin{bmatrix} \alpha & 0 & 1 & 0 & \alpha \\ \alpha(u+c) & 0 & u & 0 & \alpha(u-c) \\ \alpha v & 1 & v & 0 & \alpha v \\ \alpha w & 0 & w & 1 & \alpha w \\ \alpha(H+cu) & v & \frac{1}{2}c^2 M^2 & w & \alpha(H-cu) \end{bmatrix} \quad ,$$

$$\mathcal{S}^{-1} = \begin{bmatrix} \beta\left(\frac{1}{2}(\gamma-1)c^2 M^2 - cu\right) & -\beta((\gamma-1)u-c) & -\beta(\gamma-1)v & -\beta(\gamma-1)w & \beta(\gamma-1) \\ -v & 0 & 1 & 0 & 0 \\ 1 - \frac{1}{2}(\gamma-1)M^2 & \frac{\gamma-1}{c^2}u & \frac{\gamma-1}{c^2}v & \frac{\gamma-1}{c^2}w & -\frac{\gamma-1}{c^2} \\ -w & 0 & 0 & 1 & 0 \\ \beta\left(\frac{1}{2}(\gamma-1)c^2 M^2 + cu\right) & -\beta((\gamma-1)u+c) & -\beta(\gamma-1)v & -\beta(\gamma-1)w & \beta(\gamma-1) \end{bmatrix} \quad .$$

Here

$$\alpha = \frac{1}{2c} \quad , \quad \beta = \frac{1}{c} \quad .$$

Introducing this transformation into the Navier-Stokes equations yields

$$Q^T Q \frac{\partial \mathbf{R}}{\partial t} + \sum_{i=1}^{3} \mathcal{A}_i^s \frac{\partial \mathbf{R}}{\partial x_i} = \frac{1}{\text{Re}} \sum_{i=1}^{3} \sum_{j=i}^{3} \mathcal{B}_{ij}^s \frac{\partial^2 \mathbf{R}}{\partial x_i \partial x_j} \quad ,$$

where $\mathbf{R}$ are the characteristic variables and $\mathcal{Q}^T\mathcal{Q}$ is a positive definite, symmetrizing diagonal matrix.

The symmetrized matrices

$$\mathcal{A}_i^s = \mathcal{Q}^T\mathcal{Q}\,\mathcal{S}^{-1}\mathcal{A}_i\mathcal{S} \ , \quad \mathcal{B}_{ij}^s = \mathcal{Q}^T\mathcal{Q}\,\mathcal{S}^{-1}\mathcal{B}_{ij}\mathcal{S}$$

are given as

$$\mathcal{A}_1^s = \begin{bmatrix} u+c & 0 & 0 & 0 & 0 \\ 0 & 2u & 0 & 0 & 0 \\ 0 & 0 & \frac{2c^2}{\gamma-1}u & 0 & 0 \\ 0 & 0 & 0 & 2u & 0 \\ 0 & 0 & 0 & 0 & u-c \end{bmatrix} , \quad \mathcal{A}_2^s = \begin{bmatrix} v & c & 0 & 0 & 0 \\ c & 2v & 0 & 0 & c \\ 0 & 0 & \frac{2c^2}{\gamma-1}v & 0 & 0 \\ 0 & 0 & 0 & 2v & 0 \\ 0 & c & 0 & 0 & v \end{bmatrix} ,$$

$$\mathcal{A}_3^s = \begin{bmatrix} w & 0 & 0 & c & 0 \\ 0 & 2w & 0 & 0 & 0 \\ 0 & 0 & \frac{2c^2}{\gamma-1}w & 0 & 0 \\ c & 0 & 0 & 2w & c \\ 0 & 0 & 0 & c & w \end{bmatrix} ,$$

$$\mathcal{B}_{11}^s = \frac{1}{2\rho}\begin{bmatrix} (\lambda+2\mu)+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & -(\lambda+2\mu)+\theta \\ 0 & 4\mu & 0 & 0 & 0 \\ -\frac{2c}{\gamma-1}\theta & 0 & \frac{4c^2}{(\gamma-1)^2}\theta & 0 & -\frac{2c}{\gamma-1}\theta \\ 0 & 0 & 0 & 4\mu & 0 \\ -(\lambda+2\mu)+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & (\lambda+2\mu)+\theta \end{bmatrix} ,$$

$$\mathcal{B}_{22}^s = \frac{1}{2\rho}\begin{bmatrix} \mu+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & -\mu+\theta \\ 0 & 4(\lambda+2\mu) & 0 & 0 & 0 \\ -\frac{2c}{\gamma-1}\theta & 0 & \frac{4c^2}{(\gamma-1)^2}\theta & 0 & -\frac{2c}{\gamma-1}\theta \\ 0 & 0 & 0 & 4\mu & 0 \\ -\mu+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & \mu+\theta \end{bmatrix} ,$$

$$\mathcal{B}_{33}^s = \frac{1}{2\rho}\begin{bmatrix} \mu+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & -\mu+\theta \\ 0 & 4\mu & 0 & 0 & 0 \\ -\frac{2c}{\gamma-1}\theta & 0 & \frac{4c^2}{(\gamma-1)^2}\theta & 0 & -\frac{2c}{\gamma-1}\theta \\ 0 & 0 & 0 & 4(\lambda+2\mu) & 0 \\ -\mu+\theta & 0 & -\frac{2c}{\gamma-1}\theta & 0 & \mu+\theta \end{bmatrix} .$$

We have for convenience introduced

$$\theta = \frac{\gamma - 1}{\gamma} \frac{\gamma k}{\Pr} \quad .$$

$$\mathcal{B}_{12}^s = \frac{\lambda + \mu}{\rho} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad \mathcal{B}_{13}^s = \frac{\lambda + \mu}{\rho} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad ,$$

$$\mathcal{B}_{23}^s = \frac{\lambda + \mu}{\rho} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

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