A closure for the compressibility of the source terms in Lighthill's acoustic analogy

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Abstract

The compressible nature of the source terms in Lighthill's acoustic analogy can be closed. For weakly compressible flows, in the absence of thermoacoustic effects, the compressibility of the source field is known in terms of solenoidal modes of the vortical flow field. In such flows, the square of the fluctuating Mach number is small and this fact, coupled with the singular nature of the acoustic problem, and the fact that the phase speed of the acoustic sources is the advective speed, is used to formally close the compressible portion of the fluctuating Reynolds stresses. The closure resolves, as expressed in Crow's 1970 paper, the inconsistent incompressible approximation to Lighthill's source term. It is shown that the incompressible approximation to Lighthill's source term, accurate to order of the square of the Mach number, predicts an acoustic field accurate to order Mach number.

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1. Introduction

Many investigations in aeroacoustics begin with Lighthill's (1952, 1954) acoustic analogy: it is derived, without approximation, from the equations of motion and constitutes an exact solution for the acoustic field. The source term in Lighthill's equation while exact, requires in any application of the acoustic analogy, approximation. Any lack of agreement between Lighthill's theory and experiment is then due to approximations made when the source term is modeled. Lighthill (1952, 1954) showed that an estimate of the source term neglecting its compressible nature, and where the density was taken to be the local mean density, constituted a useful approximation. Also underlying Lighthill's analogy is the hypothesis that the flow field, which is the source of the acoustic field, is distinct from an ambient constant property stationary medium enclosing the flow field; in the ambient medium Lighthill's source terms are assumed to be zero. Several issues, limitations, and ambiguities regarding the utility of such assumptions have been investigated by Crow (1970).

In this article the compressible nature of the source terms in Lighthill's acoustic analogy are closed. That this might be possible, for compact sources, is suggested by the singular nature of the aeroacoustic problem: in the near field source region, the wave operator, rescaled on the length and time scales of the source flow, becomes elliptic, Crighton (1975), Kambe (1986). The compressible aspects of the acoustic source are then given by a series of Poisson equations whose source terms are given by the solenoidal of the inner source field. A form of Lighthill's analogy accounting for the compressibility of the vortical source field is obtained and the incompressible approximation to the source terms in Lighthill's acoustic analogy can be understood and, if necessary, avoided.

Problem statement

Lighthill's acoustic analogy is written, Lighthill (1952), as

$$\rho_{,tt} - c_\infty^2 \rho_{,jj} = T_{ij,ij} = [\, \rho u_i u_j + (p - c_\infty^2 \rho)\, \delta_{ij} \,]_{,ij} \,. \tag{1}$$

It is convenient, for the present problem, to work in terms of pressure: Lighthill's analogy can then be written as

$$c_\infty^{-2} p_{,tt} - p_{,jj} = (\rho u_i u_j)_{,ij} + c_\infty^{-2} (p - c_\infty^2 \rho)_{,tt} \tag{2}$$

The quantity $(p - c_\infty^2 \rho)_{,tt}$ is responsible for thermoacoustic effects, Crighton *et al.* (1992); it will be neglected for the class of flows that are the subject of this article. The present analysis is understood to be limited to the class of flows relevant to subsonic aeroacoustics: high Reynolds number (turbulent), weakly compressible, compact flows with a relatively constant mean density. In such a flow it can be shown, using the perturbation series derived below, that the quantity $(p - c_\infty^2 \rho)$ makes contributions that are of $\mathcal{O}(M_t^2)$ with respect to the highest order terms included in the present analysis. Here $M_t = u_c/c$ is the fluctuating Mach number for which u_c is a characteristic fluctuating velocity; its square is a measure of the strength of the effects of compressibility.

In many analyses using Lighthill's analogy an incompressible approximation of the source terms is used: $\rho u_i u_j \simeq \rho_\infty v_i v_j$ where v_i represents the solenoidal velocity field. Lighthill's acoustic analogy is then written as

$$c_\infty^{-2} p_{,tt} - p_{,jj} = \rho_\infty (v_j v_i)_{,ij} \tag{3}$$

and the difference between the compressible and incompressible term, $[\rho u_i u_j - \rho_\infty v_i v_j]_{,ij}$, is neglected. This article addresses the *compressibility correction* term,

$$T^c_{ij,ij} = [\rho u_i u_j - \rho_\infty v_i v_j]_{,ij} \,, \tag{4}$$

which can be viewed as encompassing, in the absence of thermoacoustic effects, the compressible nature of Lighthill's source term. The present focus will be on understanding the 1) scalings of T^c_{ij}, 2) identifying its acoustic nature, and 3) its contribution to the radiated acoustic field. The main result of this article, however, is a closure for the compressible correction term, as a result the full Lighthill source term $(\rho u_i u_j)_{,ij}$ is then known. As a consequence, the incompressible approximation to the source term in Lighthill's acoustic analogy can be analytically and numerically better assessed. It is seen that in many cases of practical interest the incompressible approximation is sufficient. When it is not sufficient the contributions due to compressibility can now be obtained.

Outline

In the next section, §2, some background material and a clarification of diverse issues are provided. In §3 single-time low Mach number expansions of the governing equations are conducted. The consequences and shortcomings of this procedure and its reliance on an unjustifiable imposition of a Helmholtz decomposition are indicated. In §4 a two-time low Mach number expansions is used; this procedure distinguishes acoustic from compressible modes by phase speed – as opposed to the Helmholtz decomposition used by Crow (1970) and in §3. The analysis highlights important kinematic features regarding the coupling between the solenoidal and the compressible strain field which allows a simple and straightforward closure for the compressible portions of the source term. In §5 the nature and scaling of the compressible source terms are investigated.

2. Preliminaries

Some background information useful for motivating the analysis and understanding the procedure is now given. As the singular nature of the problem is in fact what allows a closure for the compressible correction, it is described first.

The singular nature of the aeroacoustical problem

In the present closure for the effects of compressibility on the source terms in Lighhill's acoustic analogy the singular nature of the aeroacoustical problem is used to advantage. The singular nature of the acoustics problem manifests itself in the nonuniformity, Lauvstad (1970), of an expansion procedure; important physical processes take place on different length scales, Crighton (1975), Kambe (1986), Kambe, Minota and Takaoka (1993). The ratio of the two length scales goes to zero as the small parameter, M_t, goes to zero. When the problem is scaled using the length and time scales of the vortical flow field, $[\ell, \ell/u_c]$, the Lighthill equation, $c_\infty^{-2} p,_{tt} - p,_{jj} = (\rho u_i u_j),_{ij}$ devolves to a Poisson equation for the pressure and one obtains a set of equations that are used to describe an incompressible flow: $\rho_\infty (v_i,_t + v_k v_i,_k) + p,_i = 0$, $v_i,_i = 0$ and

$$\nabla^2 p = -\rho_\infty (v_i v_j),_{ij} \,. \tag{5}$$

The problem is elliptic in nature. On the outer or acoustic scales, $[\ell/M_t, \ell/u_c]$, one obtains the sourceless wave equation

$$c_\infty^{-2} p,_{tt} - p,_{jj} = 0. \tag{6}$$

On the outer scales of the problem the hyperbolic nature of the problem manifests itself. The nature of the two problems is clearly different; Crighton (1975) provides a pithy view. On the inner scales the Laplacian is more singular than the double time derivative. This fact is exploited to obtain closure for $T^c_{ij},_{ij}$; the compressible portions of the source terms will then be related to a series of Poisson (elliptic) equations whose source term is given by the solenoidal field and is therefore known. The essential point is that the compressible nature of the source term is known without solving the acoustic radiation problem. In the way that the leading order contribution to the sound field is obtained from the solenoidal modes, $\rho_\infty v_i v_j$, so also is the compressibility correction, T^c_{ij}.

Observations by Crow

The neglect of the compressible portions of the source term, $T^c_{ij},_{ij}$, is sometimes justified on the grounds that T^c_{ij} is of $\mathcal{O}(M_t^2)$ with respect to $(v_i v_j)$. This implicitly (and unjustifiably) assumes that the compressible correction, $T^c_{ij},_{ij}$, is quadrupole as is $(v_i v_j),_{ij}$. Crow (1970) has examined this idea.

Crow (1970) investigated the structure of the Lighthill equation as a perturbation series in the fluctuating Mach number.

$$M_t^2\, p_{,tt} - p_{,jj} = T^0_{ij,ij} + T^c_{ij,ij} + ... = \rho_\infty (v_i v_j)_{,ij} + M_t^2\, T^1_{ij,ij} + ... \quad . \tag{7}$$

The source term $T^1_{ij,ij}$ represents contributions to the source term from interactions between incompressible and, in Crow's (1970) nomenclature, acoustic modes of the flow. Crow (1970) makes the point that to treat the sound problem by retaining $\mathcal{O}(M_t^2)$ in the wave operator while neglecting in T_{ij} a compressible term of $\mathcal{O}(M_t^2)$, $T^1_{ij,ij}$ is inconsistent [2]. Despite his demonstration of this inconsistency, he concludes that the incompressible approximation to the fluctuating quadrupole is adequate for compact flows with small fluctuating Mach number. This appears to be due to the nonuniformity of the problem; $p_{,tt}$ is not important in regions of the flow where $T^1_{ij,ij}$ is important, and, in regions of the acoustic field where $p_{,tt}$ is important, $T^1_{ij,ij}$ is not important – because of this behavior the effects of the truncation relative to the the term $p_{,tt}$ is mitigated. Crow (1970) makes an additional point: a simple source in T^1_{ij} will contribute terms of the same order as those in the incompressible quadrupole $(v_i v_j)_{ij}$. The major issue then appears to be whether the contribution to the acoustic field due to $M_t^2\, T^1_{ij,ij}$ can be neglected relative to $T^0_{ij,ij}$; this will not true if $T^1_{ij,ij}$ contains a simple source. This important question motivates, in part, this analysis; its resolution requires a delineation of the compressible source terms which is only possible once a closure for these terms is obtained.

The governing equations

The following equations are used to describe the flow:

$$\rho^*_{,t} + u^*_k \rho^*_{,k} = -\rho^* u^*_{p,p} \tag{8}$$

$$\rho^* u^*_{i,t} + \rho^* u^*_k u^*_{i,k} + p^*_{,i} = 0 \tag{9}$$

$$p^*/p_\infty = (\rho^*/\rho_\infty)^\gamma . \tag{10}$$

An asterisk denotes the fact that the superscripted variable is dimensional. The last equation comes from the assumption of isentropy: $\frac{D}{Dt}\, s = 0 \;\Rightarrow\; \rho \frac{D}{Dt}\, h = \frac{D}{Dt}\, p$ from which the equation follows given the ideal gas law. The enthalpy, h, is given by $h(\gamma - 1) = c^2$ where $c^2 = \gamma p^*/\rho^*$ and $\frac{D}{Dt} = \frac{\partial}{\partial t} + u^*_k \frac{\partial}{\partial x_k}$. The momentum and continuity equations can be combined to give $\rho^*_{,tt} - p^*_{,jj} = (\rho^* u^*_i u^*_j)_{,ij}$, which becomes a wave equation for ρ or p if the gas law is used to eliminate one in favor of the other. For clarity of exposition, the viscous terms are not carried; they can be shown, for a high Reynolds number flows, to be of higher order. Perturbing about a reference state, (p_∞, ρ_∞), the nondimensional forms of the pressure and density are taken as $p^* = p_\infty(1 + p)$ and $\rho^* = \rho_\infty(1 + \rho)$.

[2] A similar inconsistency in Ribner's (1962) dilatational acoustic theory has been indicated in Ristorcelli (1997).

The equations, where $[\rho, p]$ are now dimensionless, become

$$
\begin{aligned}
\rho,_t \ + u_k^* \rho,_k &= -(1+\rho) u_p^*,_p && (11)\\
(1+\rho)[u_i^*,_t \ + u_k^* u_i^*,_k] \ + \ \frac{c_\infty^2}{\gamma} \, p,_i &= 0 && (12)\\
p - \gamma\rho &= \frac{1}{2}\gamma(\gamma-1)\rho^2 && (13)\\
\rho,_{tt} \ - \frac{c_\infty^2}{\gamma} \, p,_{jj} &= [(1+\rho) u_i^* u_j^*],_{ij} \, . && (14)
\end{aligned}
$$

The velocities are still dimensional. In these equations $c_\infty^2 = \gamma p_\infty / \rho_\infty$. These equations will be used as the starting point. The following definitions for the strain and the rotation will also be needed: $s_{ij} = \frac{1}{2}[v_i,_j + v_j,_i]$ and $r_{ij} = \frac{1}{2}[v_i,_j - v_j,_i]$. The superscript "$c$" as in s_{ij}^c and r_{ij}^c will denote the strain and the rotation associated with the compressible portion of the velocity field.

3. A single time perturbation analysis

A small fluctuating Mach number expansion of the compressible Navier-Stokes equations is now conducted. The analysis is understood to be addressing high Reynolds number, weakly compressible, turbulent flows – the class of flows relevant to subsonic aeroacoustics for which isentropy implies $dp = c^2 d\rho$. Similarites to the Crow (1970) analysis are discussed. Shortcomings are highlighted.

A single time perturbation analysis

Regular expansions of the form $p = \epsilon^2 \ [\ p_1 + \epsilon^2 p_2 + ...]$, $\rho = \epsilon^2 \ [\ \rho_1 + \epsilon^2 \rho_2 + ...]$, $u_i/u_c = v_i + w_i = v_i + \epsilon^2 \ [\ w_{1i} + \epsilon^2 w_{2i} + ...]$, are inserted into the governing equations. To the lowest order a set of equations that are the same as that used to describe an incompressible flow,

$$
\begin{aligned}
v_i,_t \ + v_p v_i,_p \ + p_1,_i &= 0 && (15)\\
v_i,_i &= 0 && (16)\\
\nabla^2 p_1 &= -(v_i v_j),_{ij} && (17)\\
p_1 &= \gamma \rho_1 && (18)
\end{aligned}
$$

are obtained. The velocity has been made nondimensional with u_c and $\epsilon^2 = \gamma M_t^2 = \gamma u_c^2 / c_\infty^2$ where $c_\infty^2 = \gamma p_\infty / \rho_\infty$. The independent variables are still dimensional — length and time scales have not been chosen. The fluctuating pressure, p_1 is obtained from a Poisson equation whose source term is known in terms of the solenoidal modes of the velocity field; the leading order density fluctuation is linearly related to the leading order pressure. The next order equations are

$$
\rho_1,_t \ + v_p \rho_1,_p \ = \ -w_{1k},_k \ = \ -d_1 \qquad (19)
$$

$$
\begin{aligned}
w_{1i,t} + v_p w_{1i,p} + w_{1p} v_{i,p} + p_{2,i} &= -\rho_1 (v_{i,t} + v_p v_{i,p}) & (20) \\
p_2 - \gamma \rho_2 &= \frac{1}{2} \gamma (\gamma - 1) \rho_1^2 & (21) \\
-p_{2,jj} &= (w_{1i} v_j + w_{1j} v_i + \rho_1 v_i v_j)_{,ij} - \frac{1}{\gamma} p_{1,tt} . & (22)
\end{aligned}
$$

The compressible portion of the source flow, $[w_{1i}, \rho_1, p_2]$, does not involve a wave equation. In the near field region the compressible pressure is felt effectively instantaneously. Since $\gamma\rho_1 = p_1$, the next order continuity equation yields a *diagnostic* relationship for the dilatation, $d_1 = w_{1k,k}$,

$$
-\gamma d_1 = p_{1,t} + v_k p_{1,k} = \frac{D}{Dt} p_1 . \tag{23}
$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}$. The dilatation is, to leading order, not an acoustic quantity – it does not propagate with the acoustic phase speed. The dilatation, $d_1 = w_{1i,i}$ is related to the pressure that satisfies $\nabla^2 p_1 = -(v_i v_j)_{,ij}$. It can be shown, Crow(1970), that the solenoidal pressure falls off from the source region as $p_1 \sim x^{-3}$. Using the continuity equation indicates that $w_{1i,i} = d_1 \sim \frac{D}{Dt} p_1 \sim x^{-3}$ and $w_{1i} \sim x^{-2}$ which differs from the acoustic scaling, $w_{1i} \sim x^{-1}$.

Lighthill's acoustic analogy is reconstituted, in dimensional form, as

$$
c_\infty^{-2} p_{,tt} - \nabla^2 p = \rho_\infty [v_i v_j]_{,ij} + \rho_\infty [w_i v_j + w_j v_i + \rho v_i v_j]_{,ij} . \tag{24}
$$

The identifications $w_i = \gamma M_t^2 u_c w_{1i}$ and $\rho = \gamma M_t^2 \rho_1$ have been made. This is what Crow (1970) has called a self-consistent form of Lighthill's acoustic analogy — $\mathcal{O}(M_t^2)$ terms on both sides of the equation are retained.

Imposition of a Helmholtz decomposition

In order to close the source terms expressions for ρ_1 and w_{1i} are required. This is readily accomplished as the leading order problem $[v_i, p_1]$ is considered known.

In the perturbation anslysis above the small parameter was used to order various dynamical aspects of the physics. The problem, following Crow(1970), is now kinematically partitioned: the Helmholtz decomposition allows, for any suitably compact vector field, a unique decomposition into the curl of vector, v_i, and the gradient of a scalar, w_i: $u_i = v_i + w_i = v_i + \phi_{,i}$. The decomposition implies, of course, $w_{1i} = \phi_{1,i}$ and $\nabla^2 \phi_1 = d_1$ and the usual Biot-Savart relation $\nabla^2 v_i = \varepsilon_{ijk} \omega_{k,j}$. Here $\omega_i = \varepsilon_{ijk} v_{j,k}$ is the vorticity. The compressible modes are therefore determined by the inversion of the following Poisson equation for the potential

$$
-\gamma \nabla^2 \phi_1 = \frac{D}{Dt} p_1 , \tag{25}
$$

using the methods of Green's functions. The homogeneous solution is the trivial solution. The velocity field is given by

$$-\gamma\nabla^2 w_{1i} \;=\; (\frac{D}{Dt}\;\; p_1)_{,i} \qquad\qquad \Rightarrow \qquad\qquad w_{1i} \;=\; -\;\frac{1}{\gamma}\;\nabla^{-2}(\frac{D}{Dt}\;\; p_1)_{,i}\,. \tag{26}$$

There are several other simplifications that can be made to this expression using the fact that w_{i1} is a double convolution. This does not suit present intentions. The fluctuating density is, of course, linearly related to the fluctuating pressure $\gamma\rho_1 = p_1$. Thus all terms in $[w_{1i}v_j + w_{1j}v_i + \rho_1 v_i v_j]$ are closed once

$$\nabla^2 p_1 = -\;[v_i v_j]_{,ij} \tag{27}$$

is solved. The Lighthill equation, to order M_t^2, has now been formally closed. This is similar to Crow's (1970) analysis; Crow (1970), however, was interested in different questions and did not pursue the question of a closure for the compressible correction $T^c_{ij},_{ij} = [\rho u_i u_j -\; \rho_\infty v_i v_j]_{,ij} = \rho_\infty[w_i v_j + w_j v_i + \rho v_i v_j]_{,ij}$.

There are some theoretical and practical problems with this result.

- The Helmholtz decomposition has partitioned the flow field such that all the vorticity is in the leading order problem, $[v_i, p_1]$; all the effects of compressibility are in the irrotational mode. The Helmholtz decomposition precludes any representation of a compressible rotational mode.
- Inspection of the evolution equation for w_{1i} shows that w_{1i} is not irrotational.
- Contributions to the source term $w_{1i},_j\, v_j,_i = (s^c_{ij} + r^c_{ij})v_j,_i$ does not account for the interaction term $r^c_{ij}v_j,_i$ since the rotational compressible are not captured in this representation. Here $s^c_{ij} = \frac{1}{2}[w_{1i},_j + w_{1j},_i\,]$ and $r^c_{ij} = \frac{1}{2}[w_{1i},_j - w_{1j},_i\,]$
- Any calculation of the compressible correction requires a double convolution. Any solution for the acoustic field requires a triple convolution.

These shortcomings can be resolved with a multi-time analysis.

4. A two-time perturbation analysis

The ability of a two-time expansion to distinguish, *in the source field*, compressible from acoustic modes is essential to closing for the compressible aspects of the source terms in Lighthill's acoustic analogy. The multi-scale procedure distinguishes acoustic modes, with acoustic phase speed from compressible modes, with advective phase speed. This is a distinction that is not possible with the methodology employed above.

In the flow field there are two velocity scales, the sound speed, c, and a characteristic fluctuating velocity, u_c. In the flow field one length scale is recognized: ℓ. Two time scales can be identified: a fast time that scales with the eddy crossing time, ℓ/c_∞, and a slow time that scales with the eddy turnover time, ℓ/u_c. The ratio of these two time scales is the turbulent Mach number and is the small parameter in a two-timing procedure. The problem is recognized as being driven by the vortical flow which evolves on the advective (slow) time scale: if there were no flow evolving on the slow time there would be no compressible or acoustic field. The problem can be viewed as a forced linear oscillator with forcing coming from the slow vortical modes with advective scales $[u_c, \ell]$. While simple in concept diverse subtelties associated with the aeroacoustical problem can be more readily seen if the two-time procedure is first applied to the forced linear oscillator.

The forced linear oscillator

Consider the forced linear oscillator driven with frequency ω much slower than the natural frequency of the oscillator, ω_0, $\omega/\omega_0 << 1$:

$$\ddot{u} + \omega_0^2\, u = \omega_0^2 cos\, \omega t. \tag{28}$$

The exact solution is written as a sum of homogeneous and particular solutions:

$$u(t;\omega_0,\omega) = u_H(t;\omega_0) + u_P(t;\omega_0,\omega) = A_1 e^{+i\omega_0 t} + A_2 e^{-i\omega_0 t} + \frac{\omega_0^2}{\omega_0^2 - \omega^2}\, cos\, \omega t. \tag{29}$$

Note that the homogeneous solution has no relation whatsoever to the forcing, $u_H(t;\omega_0) \neq f(\omega)$. This property of the homogeneous solution will be used to eliminate its consideration as a solution relevant to the acoustic source generated by the flow field. If the initial conditions are such that the eigensolutions are not stimulated, $A_1 = A_2 = 0$, the solution is the particular solution, $u(t) = \frac{\omega_0^2 cos\omega t}{\omega_0^2 - \omega^2} = cos(\omega t) + \mathcal{O}(\frac{\omega^2}{\omega_0^2})$.

The forced linear oscillator is now treated by the method of multiple-scales. The role a small parameter will be played by $\epsilon = \omega/\omega_0 << 1$. Time can be rescaled, $t' = \omega_0 t$ to produce, after dropping the primes,

$$\ddot{u} + u = \cos\, \epsilon t. \tag{30}$$

Following the usual multi-scale procedures the original time variable, $[\,t\,]$, is replaced by two independent time variables, $[t_0, t_1]$; t_0 is the fast time scale and t_1 is the slow time scale. The multi-time-scale *ansatz* for the dependent variable is $u(x,t) = u(x,t_0,t_1)$, where $t_0 = t$ and $t_1 = \epsilon t$.

The time derivative of $u(x,t)$ is then written, using the chain rule, $\frac{d}{dt}\,u(t;\epsilon) = \frac{\partial t_0}{\partial t}\frac{\partial}{\partial t_0}\,u(t_0,t_1;\epsilon) + \frac{\partial t_1}{\partial t}\frac{\partial}{\partial t_1}\,u(t_0,t_1;\epsilon) = \frac{\partial}{\partial t_0}\,u(t_0,t_1,\epsilon) + \epsilon\frac{\partial}{\partial t_1}\,u(t_0,t_1,\epsilon)$. Expanding $u(t;\epsilon) = u_0(t_0,t_1) + \epsilon u_1(t_0,t_1) + \epsilon^2 u_2(t_0,t_1) + ...$ the problem for the forced oscillator takes the form

$$\frac{\partial^2}{\partial t_0^2}u_0 + u_0 = cos\,t_1 \tag{31}$$

$$\frac{\partial^2}{\partial t_0^2}u_1 + u_1 = -2\frac{\partial^2}{\partial t_0 \partial t_1}u_0. \tag{32}$$

The leading order solution can be written as a sum of homogeneous and particular solutions,

$$u_0(t_0,t_1) = A_1(t_1)\,e^{+it_0} + A_2(t_1)\,e^{-it_0} + cos\,t_1. \tag{33}$$

The slow time coefficients, $A_i(t_1)$, are determined by the removal of secular terms in the equation for u_1. They are constants for this problem. The solution, when the eigenmodes are not stimulated, $A_1 = A_2 = 0$, is the particular solution: $u_0(t_0,t_1) = u_{0P}(t_1) = cos\,t_1 = cos\,\omega t$ which is, to $\mathcal{O}(\epsilon^2)$, the exact solution given above. The realization that the particular solution $u_0(t_1;\omega)$ is the solution relevant to the forcing and that the homogeneous solution, $u_{0H}(t_0;\omega_0)$, is not associated with the forcing and not relevant to the aeroacoustical problem driven by the fluctuating velocity field.

Multi-scaling the compressible source flow

A multi-scaling of the compressible Navier Stokes equations, *modulo* viscosity, is now conducted. An analogy between the compressible aspects of the source flow and the forced linear oscillator is made. The time scale of the forcing, ω^{-1}, can thought of as an eddy turnover time of the turbulent source field, ℓ/u_c. The slow or advective time scale is ℓ/u_c. The time scale ω_0^{-1}, can be identified with the sound crossing time, ℓ/c_∞. The fast or acoustic time scale is ℓ/u_c. The small parameter is the turbulent Mach number which is the ratio of these two time scales.

Compressible portions of the flow evolving on the fast time scale will be called acoustic modes; compressible portions of the flow evolving on a slow time scale will be called advective compressible modes. The multi-time procedure is crucial to distinguishing these two aspects of the compressible flow. The compressible portions of Lighthill's source terms will be seen to be a function of the advective compressible modes.

Starting with the compressible Navier Stokes equations given in §2 the following *ansatz*, as suggested by earlier results, for the dependent variable is proposed:

$$u_i^* = u_c(v_i + \phi_w w_i) \tag{34}$$

$$p = \phi_0(p_s + \phi_1\,p_C) \tag{35}$$

$$\rho = \phi_0(\rho_s + \phi_1\,\rho_C). \tag{36}$$

The subscript "s" is understood to indicate solenoidal in reference to v_i which has zero divergence and from which both p_s and ρ_s are derived. The subscript "C" will signify the compressible nature of the nondimensional pressure and density fields. Inserting the decompositions into the governing equations produces on the inner length and time scales the problem:

$$v_{i,t} + v_k v_{i,k} + p_{s,i} = 0 \qquad v_{k,k} = 0 \tag{37}$$

$$p_{s,jj} = -(v_i v_j)_{,ij} \qquad \gamma \rho_s = p_s. \tag{38}$$

The leading order problem is described by equations used to model an incompressible flow with characteristic time, length and velocity scales $[\frac{\ell}{u_c}, \ell, u_c]$. The gauge function $\phi_0 = \gamma M_t^2$ is established by balance.

The compressible portion of the problem, $[p_C, \rho_C, w_i]$, is

$$\begin{aligned}
[1+\phi_0(\rho_s+\phi_1\rho_C)][w_{i,t} + v_k w_{i,k} + w_k v_{i,k} + \phi_w w_k w_{i,k}]\phi_w \quad &+ \quad \frac{c_\infty^2}{\gamma}\phi_0\phi_1 p_{C,i} = \\
= -\phi_0(\rho_s \quad &+ \quad \phi_1\rho_C)[v_{i,t} + v_k v_{i,k}] && (39)\\
\phi_0\phi_1[\rho_{C,t} + v_k\rho_{C,k} + \phi_w w_k \rho_{C,k}] + [1+\phi_0(\rho_s+\phi_1\rho_C)]w_{k,k}\,\phi_w \quad &= \quad -\phi_0[\rho_{s,t} + v_k\rho_{s,k}] && (40)\\
\phi_o\phi_1(p_C - \gamma\rho_C) \quad &= \quad \frac{1}{2}\gamma(\gamma-1)\phi_0^2(\rho_s+\phi_1\rho_C)^2 && (41)
\end{aligned}$$

The dependent variables of this problem are all nondimensional: $[v_i, w_i, p_s, p_C, \rho_s, \rho_C]$ are in units of $[u_c, p_\infty, \rho_\infty]$. It is important to note that *the independent variables are dimensional*: length and time scales have not yet been chosen. The scalings for length and time will determine ϕ_1 and ϕ_w. Choosing the ℓ/c and ℓ for the scalings the only consistent balance of the momentum and continuity equations produces $\phi_w = M_t^2$ and $\phi_1 = M_t$.

The compressible independent and dependent variables are now expanded in a series of form:

$$\begin{aligned}
\frac{D}{Dt}\,f(x,t) \quad &= \quad \frac{D}{Dt_0}\,f(x,t_0,t_1) \; + \; \epsilon\frac{D}{Dt_1}\,f(x,t_0,t_1) && (42)\\
w_i(x,t_0,t_1) \quad &= \quad w_{0i}(x,t_0,t_1) + \epsilon\, w_{1i}(x,t_0,t_1) + ... && (43)\\
p_C(x,t_0,t_1) \quad &= \quad p_0(x,t_0,t_1) + \epsilon\, p_1(x,t_0,t_1) + ... && (44)\\
\rho_C(x,t_0,t_1) \quad &= \quad \rho_0(x,t_0,t_1) + \epsilon\, \rho_1(x,t_0,t_1) + ... \quad . && (45)
\end{aligned}$$

Here $\frac{D}{Dt_0} \equiv \frac{\partial}{\partial t_0} + v_k\frac{\partial}{\partial x_k}$ and $\frac{D}{Dt_1} \equiv \frac{\partial}{\partial t_1} + v_k\frac{\partial}{\partial x_k}$. The equations are made nondimensional with the integral length scale, ℓ, and the fast time scale, ℓ/c. Inserting the expansions for the dependent variables and the temporal differential operators produces to leading order:

$$\begin{aligned}
\frac{D}{Dt_0}\;w_{0i} + p_{0,i} \quad &= \quad 0 && (46)\\
\frac{D}{Dt_0}\;p_0 + w_{0k,k} \quad &= \quad -\frac{D}{Dt_1}\;p_s && (47)\\
p_0 - \gamma\rho_0 \quad &= \quad 0. && (48)
\end{aligned}$$

The set of equations are equivalent to those of the forced linear oscillator, $\dot{y_1} - y_2 = 0$ and $\dot{y_2} + \omega_0^2\, y_1 = cos\,(\omega t)$. The analogy can be seen in Fourier space where the complications of the spatial derivative on w_{0i} are removed. (The second order equations are given in the appendix.) The compressible velocity gradients evolve according to:

$$\frac{D}{Dt_0}\; w_{0i},_j \;=\; -\, p_0,_{ij} - v_p,_j\, w_{0i},_p\,, \tag{49}$$

from which the equations for the divergence and the vorticity are readily obtained,

$$\frac{D}{Dt_0}\; w_{0j},_j \;=\; -\, p_0,_{jj} - v_p,_j\, w_{0j},_p\,. \tag{50}$$

$$\frac{D}{Dt_0}\; \omega_{0q} \;=\; -\, \epsilon_{qij} v_p,_j\, w_{0i},_p\,. \tag{51}$$

The associated wave equations are given in the Appendix.

The particular solution and its consequences

Recalling the forced oscillator analog, the general solution for the leading order problem is written as a sum of homogeneous and particular solutions, $w_{0i} = w_{0iH}(x,t_0) + w_{1iP}(x,t_1)$, $p_0 = p_{0H}(x,t_0) + p_{1P}(x,t_1)$. The homogeneous solution is independent of the forcing and is relevant to acoustic fields that are generated by initial or boundary conditions and is not of interest to the acoustic source problem.

The portion of the compressible problem driven by the slow solenoidal modes is of interest; the solution of relevance is the particular solution, $w_{0i} = w_{0iP}(x,t_1), p_0 = p_{0P}(x,t_1)$. As was seen in the forced linear oscillator example, the particular solution, to $\mathcal{O}(\epsilon^2)$, is found by setting $\frac{D}{Dt_0}\;[w_0, p_0] = 0$. Thus:

$$w_{0k},_k \;=\; -\,\frac{D}{Dt_1}\; p_s \tag{52}$$

$$p_0(x,t_1) \;=\; \rho_0(x,t_1) = 0. \tag{53}$$

The two-time expansion indicates slow compressible modes are, at this order, irrotational, and found by

$$w_{0k},_k = \nabla^2\phi(x,t_1) = -\,\frac{D}{Dt_1}\; p_s. \tag{54}$$

Rotational modes can exist at this order; they are associated with the homogeneous solution and are a function of arbitrary initial conditions. The equation for the divergence with, $\frac{D}{Dt_0}\; w_{0j},_j = 0$, indicates that the particular solution has the following property:

$$v_p,_j\, w_{0j},_p = v_p,_j\, \phi,_{jp} = 0. \tag{55}$$

This is a consequence of the two-time expansion; it is, as will be seen, a very useful fact. Kinematically it is understood as indicating that $v_{p,j}\, w_{0j,p} = s_{pj} s^c_{jp} = 0$; the principal axes of the compressible strain rate are orthogonal to the principle axes of the solenoidal strain field. In the compressible portions of the acoustic source quantities such as $(w_{1i} v_j)_{,ij}$ can be simplified using the above fact: $(v_j \phi_{,i})_{,ij} = v_j \nabla^2 \phi_{,j} + v_{i,j}\, \phi_{,ij} = v_j \nabla^2 \phi_{,j} = -\, v_j (\frac{D}{Dt_1}\, p_s)_{,j}$ since $v_{p,j}\, \phi_{,jp} = 0$. The compressible correction term, $T^c_{ij,ij} = \rho_\infty [w_{0i} v_j + w_{0j} v_i + \rho_1 v_i v_j]_{,ij} = \rho_\infty [\phi_{,i}\, v_j + \phi_{,j}\, v_i + \rho_s v_i v_j]_{,ij}$, is then written as

$$T^c_{ij,ij} = \rho_\infty [-\, 2\, (v_j \frac{D}{Dt_1}\; p_s)_{,j} + (\rho_s v_i v_j)_{,ij}\,] \tag{56}$$

and Lighthill's acoustic analogy is reconstituted, in dimensional variables, as

$$c_\infty^{-2} p_{,tt} - \nabla^2 p = \rho_\infty (v_i v_j)_{,ij} + \frac{1}{\gamma}\, c_\infty^{-2} [(p_s v_i v_j)_{,ij} - 2 (v_j \frac{D}{Dt}\, p_s)_{,j}\,]. \tag{57}$$

As the perturbation analysis is finished, p_s, is now dimensional. The compressible nature of the source terms can be obtained from the solenoidal field and the equation is closed: p_s is obtained from its Poisson equation; $-\nabla^2 p_s = \rho_\infty (v_i v_j)_{,ij}$. Several features and advantages of the multi-scale closure are summarized:

- The compressible field is, *to leading order*, irrotational. The multi-scale result justifies the assumption of a Helmholtz decomposition in the previous section (and in Crow's (1970) analysis).

- The principal axes of the compressible strain rate are orthogonal to the principle axes of the solenoidal strain field, since $v_{p,j}\, w_{0j,p} = 0$.

- The compressible correction term can be written in terms of the solenoidal modes of the source flow:

$$T^c_{ij,ij} = \frac{1}{\gamma}\, c_\infty^{-2} [(p_s v_i v_j)_{,ij} - 2 (v_j \frac{D}{Dt}\, p_s)_{,j}\,]. \tag{58}$$

- The solenoidal field, $[v_i, p_s]$, is calculated from the leading order equations — a set of equations that is describes an incompressible flow.

- The compressible correction term does not require a double convolution.

Caveat

The analysis presented is relevant to a general fluctuating flow in which there is one characteristic fluctuating velocity scale and whose characteristic Mach number, the turbulent Mach number, is small. For such a flow the slow compressible field is, to leading order, irrotational. The irrotationality of the slow compressible field, for subsonic flows, is a useful approximation as long as the mean strain rate, S, does not approach the eddy crossing time: *ie.* when $S\ell/c \to 1$. When $S\ell/c \sim 1$, the slow compressible modes become rotational and $S\ell/c$ enters the analysis as an additional governing parameter. Thus the present analysis is relevant to applications for which $S\ell/c < 1$; this is the case for many developed turbulent flows for which $u_c \sim S\ell$.

5. Detailing the compressible contribution to the acoustic source

As the analytical work is now complete the presentation returns to the dimensional variables as used in §1. The main result of this work has been the closure for the compressible correction to the Lighthill source terms,

$$T^c_{ij},_{ij} = [\ \rho u_i u_j - \ \rho_\infty v_i v_j],_{ij} \ = \ \frac{1}{\gamma c_\infty^2}[(p_s v_i v_j),_{ij} - 2(v_j \frac{D}{Dt}\ p_s),_j\,].$$

Various properties of this closure for the compressible source terms are now investigated. The far field solution to the wave equation is presented and the nature of the new source terms is defined. The asymptotic scaling of each of the terms is also indicated. It is first noted that the second compressible source term can be rewritten

$$-\ (v_j \frac{D}{Dt}\ p_s),_j = \ (v_j d),_j \tag{59}$$

as the flux of the leading order dilatation using the diagnostic relation, $d = -\ \frac{D}{Dt}\ p_s$, derived above. Note that if the pressure, p_s, were frozen, it would not contribute to the compressible source term.

The nature of the compressible contribution to the source

The contribution of the compressible source terms to the acoustic field is, however, determined by integrating over the volume of the source field. The closed Lighthill analogy reads

$$c_\infty^{-2} p,_{tt} \ - \nabla^2 p = \rho_\infty (v_i v_j),_{ij} + \frac{1}{\gamma}\ c_\infty^{-2}[(p_s v_i v_j),_{ij} - 2(v_j \frac{D}{Dt}\ p_s),_j\,], \tag{60}$$

and the solution, for an unbounded flow, is obtained by convolution,

$$\begin{aligned} p(x,t) \quad &= \quad \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int \rho_\infty [v_i v_j]\ \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \\ &+ \quad \frac{1}{4\pi\gamma}\frac{1}{c_\infty^2}\frac{\partial^2}{\partial x_i \partial x_j}\int [p_s v_i v_j]\ \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \\ &- \quad \frac{2}{4\pi\gamma}\frac{1}{c_\infty^2}\frac{\partial}{\partial x_j}\int [v_j \frac{D}{Dt}\ \ p_s]\ \frac{d\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|} \end{aligned} \tag{61}$$

where the square brackets indicate a quantity evaluated at $[\mathbf{x'}, \mathbf{t_r}]$ where $t_r = t - \frac{|\mathbf{x}-\mathbf{x'}|}{c_\infty}$. is the retarded time.

Following the usual procedures for the far field solution, $\frac{\partial}{\partial x_j} = -\frac{x_j}{x}\frac{1}{c_\infty}\frac{\partial}{\partial t}$, due to symmetry in space and time. The far field solution is,

$$\begin{aligned} p(x,t) &= \frac{1}{4\pi}\frac{1}{c_\infty^2}\frac{1}{x}\frac{x_i x_j}{x^2}\frac{\partial^2}{\partial t^2}\int \rho_\infty [v_i v_j]\, d\mathbf{x}' \\ &+ \frac{2}{4\pi\gamma}\frac{1}{c_\infty^3}\frac{1}{x}\frac{x_j}{x}\frac{\partial}{\partial t}\int [v_j \frac{D}{Dt}\ p_s]\, d\mathbf{x}' \\ &+ \frac{1}{4\pi\gamma}\frac{1}{c_\infty^4}\frac{1}{x}\frac{x_i x_j}{x^2}\frac{\partial^2}{\partial t^2}\int [p_s v_i v_j]\, d\mathbf{x}'. \end{aligned} \tag{62}$$

From a mathematical point of view the three contributions to the acoustic field — ordered according to the expansion parameter — are seen to scale, respectively, as $\mathcal{O}(M_t^2) + \mathcal{O}(M_t^3) + \mathcal{O}(M_t^4)$. It is seen that though T^c_{ij} is $\mathcal{O}(M_t^2)$ with respect to $v_i v_j$, in Lighthill's wave equation, *its contribution to the acoustic field is $\mathcal{O}(M_t)$ with respect to the contribution from $v_i v_j$*. From a more applied viewpoint the contributions to the acoustic pressure scale as

$$p(x,t) \sim \mathcal{O}(u_c^4) + \mathcal{O}(u_c^5) + \mathcal{O}(u_c^6). \tag{63}$$

The first term produces the well known eight power velocity scaling for the sound intensity of a quadrupole source, Lighthill (1952). To leading order $\langle pp \rangle \sim u_c^8$; to next higher order crossterms will produce a u_c^9 scaling.

The second term, involving the flux of the dilatation, can be understood as a lateral quadrupole. Using the momentum equation the flux of the dilatation can be rearranged as

$$v_j \frac{D}{Dt}\ p_s = \frac{D}{Dt}\ (v_j p_s) + \frac{1}{2}(p_s p_s)_{,j}\ , \tag{64}$$

giving a clearer indication of quadrupole nature. The $(p_s p_s)_{,j}$ will make a higher order contribution, at the same order in M_t as the last term. The second term will, nonetheless, be left as a flux of the leading order dilatation. The last source term is a quadrupole.

Asymptotic decay of the compressible portion of the source field

As a consequence of the Biot Savart law, Crow (1970), Howe (1975), Kambe (1986), the boundedness of the first moment of the vorticity indicates that the fall off of the solenoidal velocity associated with a compact vorticity distribution is $v_i \sim x^{-3}$. As a consequence, the incompressible approximation to the source falls off, with distance from the source region, as $v_i v_j \sim x^{-6}$.

Similarly it has been shown, see Crow (1970), that $p_s \sim x^{-3}$. As a consequence

$$w_{0j,j} \sim \frac{D}{Dt}\ p_s \sim x^{-3} \tag{65}$$

and $w_i \sim M_t^2 x^{-2}$. Using the above facts the source terms in Lighthills acoustic analogy, $v_i v_j + w_i v_j + w_j v_i + p_s v_i v_j$, fall off, from the source region, as:

$$\begin{aligned} v_i v_j &\sim x^{-6} \\ w_i v_j &\sim x^{-5} \\ p_s v_i v_j &\sim x^{-9}. \end{aligned}$$

Thus in the inner source flow there are contributions due to the compressible nature of the fluctuations that are both more and less compact than the incompressible approximation to the source term. The assumption underlying Lighthill's acoustic analogy, that $T_{ij} = 0$ outside the source region, is validated when the compressible nature of the source term is included.

Note that the compressible velocity field scales as x^{-2}; this is not the acoustic scaling assumed in Crow (1970). Crow's (1970) arguments were based on the fact that the compressible component of the velocity scaled as x^{-1} thus $w_i v_j \sim x^{-4}$. As has been shown the compressible correction term, T^c_{ij}, is due to the advective nature of the compressible flow, not its acoustic nature.

6. Summary and conclusions

For a high Reynolds number weakly compressible flow, in the absence of important irreverssible processes, a multi-time scale procedure has been used to distinguish compressible modes with acoustic phase speed from compressible modes with advective phase speed. The advective compressible modes are identified as acoustic source terms. The procedure has produced the following results:

- The assumption underlying Lighthill's acoustic analogy, that $T_{ij} = 0$ outside the source region, is justified when the compressible nature of the source term is included.
- The compressible strain field is orthogonal to the solenoidal strain field: $s_{pj}\phi,_{jp} = 0$.
- A closure for the compressible correction, $T^c_{ij},_{ij} = [\rho u_i u_j - \rho_\infty v_i v_j],_{ij}$, to the incompressible approximation of Lighthill's source term, $\rho_\infty(v_i v_j),_{ij}$, has been obtained.
- Lighthill's acoustic analogy can be written in closed form:
$$c_\infty^{-2} p,_{tt} - \nabla^2 p = \rho_\infty (v_i v_j),_{ij} \;+\; \gamma^{-1} c_\infty^{-2}\,[(p_s v_i v_j),_{ij} - 2(v_j \frac{D}{Dt}\, p_s),_j\,]$$
where $-\nabla^2 p_s = \rho_\infty(v_i v_j),_{ij}$. The incompressible approximation to the Lighthill source term is, of course, $\rho_\infty(v_i v_j),_{ij}$, the dominant term.
- Though T^c_{ij} is of $\mathcal{O}(M_t^2)$ with respect to $v_i v_j$ *its contribution to the acoustic field is of* $\mathcal{O}(M_t)$ *with respect to the contribution from* $v_i v_j$.

- The $\mathcal{O}(M_t^2)$ incompressible approximation to the Lighthill source term, $\rho u_i u_j \approx \rho_\infty v_i v_j$, predicts an acoustic field accurate to $\mathcal{O}(M_t)$.

The results indicate the possibility of investigating the consequences of compressibility analytically. The results also indicate a procedure that extends the possibility of incompressible DNS to investigate the compressible effects on sound generation. This has some utility as incompressible simulation methodologies are better understood, further advanced, and cheaper, allowing larger Reynolds number and longer simulation times.

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Appendix: leading order compressible equations

Additional aspects of the leading order *compressible* problem given in §4 are summarized. The leading order compressible momentum, continuity and thermodynamic equations are

$$\frac{D}{Dt_0}\ w_{0i} + p_{0,i} = 0 \tag{66}$$

$$\frac{D}{Dt_0}\ p_0 + w_{0k,k} = -\frac{D}{Dt_1}\ p_s \tag{67}$$

$$p_0 - \gamma\rho_0 = 0. \tag{68}$$

The equations are nondimensional — the sound speed is unity. The compressible velocity gradients evolve according to:

$$\frac{D}{Dt_0}\ w_{0i,j} = -\,p_{0,ij} - v_{p,j}\,w_{0i,p}\,, \tag{69}$$

from which the equations for the divergence and the vorticity are readily obtained,

$$\frac{D}{Dt_0}\ w_{0j,j} = -\,p_{0,jj} - v_{p,j}\,w_{0j,p}\,. \tag{70}$$

$$\frac{D}{Dt_0}\ \omega_{0q} = -\,\epsilon_{qij}v_{p,j}\,w_{0i,p}\,. \tag{71}$$

On the slow time scale, setting $\frac{D}{Dt_0} = 0$ one finds $p_0 = 0$, $w_{0k,k} = -\frac{D}{Dt_1}\ p_s$ and $v_{p,j}\,w_{0j,p} = 0$. The vorticity results solely from the initial conditions $\frac{D}{Dt_0}\ \omega_{0q} = 0$ on the homogeneous solution. Some wave equations are derivable

$$\frac{D^2}{Dt_0^2}\ p_0 - \nabla^2 p_0 = v_{p,j}\,w_{0j,p} \tag{72}$$

taking $\frac{D}{Dt_0}$ the leading order equations can be combined to give a third order wave equation

$$\frac{D}{Dt_0}\,[\frac{D^2}{Dt_0^2}\ p_0 - \nabla^2 p_0] = v_{p,j}\,\frac{D}{Dt_0}\ w_{0j,p}\,. \tag{73}$$

For the slow portion of the compressible field $\frac{D}{Dt_0}\ w_{0i,j} = 0$, and the following *sourceless* wave equation describes the fast acoustic modes:

$$\frac{D}{Dt_0}\ [\frac{D^2}{Dt_0^2}\ p_0 - \nabla^2 p_0] = 0. \tag{74}$$

In the general case $\frac{D}{Dt_0}\ w_{0i,j} \neq 0$, and a Lilley type wave equation, Goldstein (1976), results:

$$\frac{D}{Dt_0}\ [\frac{D^2}{Dt_0^2}\ p_0 - \nabla^2 p_0] + v_{k,i}\,p_{0,ik} = -v_{k,i}\,v_{p,k}\,w_{0i,p}\,. \tag{75}$$

Had the problem included a mean shear a form of Lilley's equations with the familiar factors of 2 in the refractive term and source terms would have resulted.