ASTEROIDAL TRIPLE-FREE GRAPHS*

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Abstract. An independent set of three vertices such that each pair is joined by a path that avoids the neighborhood of the third is called an *asteroidal triple*. A graph is asteroidal triple-free (AT-free, for short) if it contains no asteroidal triples. The motivation for this investigation was provided, in part, by the fact that the asteroidal triple-free graphs provide a common generalization of interval, permutation, trapezoid, and cocomparability graphs. The main contribution of this work is to investigate and reveal fundamental structural properties of AT-free graphs. Specifically, we show that every connected AT-free graph contains a dominating pair, that is, a pair of vertices such that every path joining them is a dominating set in the graph. We then provide characterizations of AT-free graphs in terms of dominating pairs and minimal triangulations. Subsequently, we state and prove a decomposition theorem for AT-free graphs. An assortment of other properties of AT-free graphs is also provided. These properties generalize known structural properties of interval, permutation, trapezoid, and cocomparability graphs.

Key words. asteroidal triples, asteroidal triple-free graphs, interval graphs, permutation graphs, trapezoid graphs, cocomparability graphs, dominating pairs, graph decompositions, structural graph theory

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1. Introduction. The original motivation for this work was provided by the linear structure that is apparent in various families of graphs including interval graphs, permutation graphs, trapezoid graphs, and cocomparability graphs. Somewhat surprisingly, the linearity of interval, permutation, trapezoid, and cocomparability graphs is described in terms of different and seemingly ad-hoc properties of each of these classes of graphs. For example, in the case of interval graphs, the linearity property is traditionally expressed in terms of a linear order on the set of maximal cliques [3, 4]. For permutation graphs, the linear behavior is explained in terms of the underlying partial order of dimension two [1]. For cocomparability graphs, the linear behavior is expressed in terms of the well-known linear structure of comparability graphs [17], and so on. Our intention is to provide a unifying look at these classes in the hope of identifying the "agent" responsible for their linear behavior.

Before proceeding, it is perhaps appropriate to recall a few definitions. A graph is an *interval graph* if its vertices can be put in a one-to-one correspondence with a set of intervals on the real line in such a way that two vertices are adjacent if and only if the corresponding intervals overlap. A graph is a *comparability graph* if the edges may be given a transitive orientation. A *cocomparability graph* is the complement of a comparability graph. A graph that is at the same time a comparability and a cocomparability graph is said to be a *permutation graph* [13].

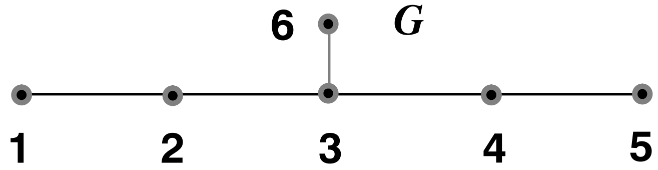

FIG. 1.1. *A graph G*

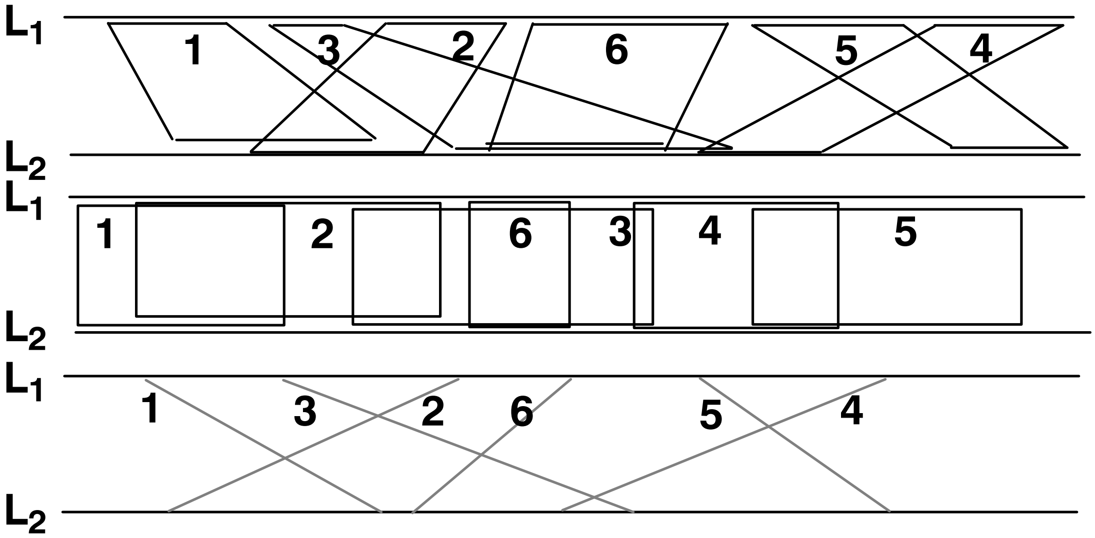

FIG. 1.2. *Trapezoid, interval, and permutation models of the graph in Figure 1.1*

A *trapezoid representation* R consists of two parallel lines (denoted L_1 and L_2) and some trapezoids with two endpoints lying on L_1 and the other two lying on L_2. A graph G is a *trapezoid graph* if it is the intersection graph of such a representation. Specifically, the vertices of G are in one-to-one correspondence with the trapezoids in R and two vertices in G are adjacent if and only if their corresponding trapezoids intersect. If the trapezoids degenerate with the endpoints on L_1 (respectively L_2) coinciding (i.e. the trapezoids become lines) then the intersection graph is a permutation graph. Similarly, if the intervals on L_1 are the mirror image of the intervals on L_2, then the intersection graph is an interval graph. The reader is referred to Figure 1.2 for an illustration of these notions for the graph presented in Figure 1.1. It is shown in [6] that permutation graphs and interval graphs are strictly contained in trapezoid graphs. Furthermore, trapezoid graphs are strictly contained in cocomparability graphs [5]. Cocomparability graphs, and thus trapezoid, permutation, and interval graphs are *perfect* in the sense of Berge [15], i.e. for every induced subgraph the chromatic number equals the clique number.

The trapezoid representation that provides the common thread with interval and permutation graphs also indicates that, in some sense, the graphs can only "grow" linearly. For example, referring to the graph in Figure 1.1 which is at the same time an interval, trapezoid, permutation, and cocomparability graph, we can add a new vertex adjacent to one of the vertices 1, 2, 3, 4, or 5 without destroying membership in any of these families; however, when looking at various intersection models of G featured in Figure 1.2, it seems as though we cannot add a new vertex adjacent to 6 without destroying membership in each family.

More than three decades ago Lekkerkerker and Boland [18] set out to identify the property that prevented a *chordal graph*, namely a graph in which every cycle of length at least four has a chord, from "growing" in three directions at once. For this purpose, they defined an *asteroidal triple* to be an independent set of three vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third. For an illustration, the reader is referred to Figure 1.3 featuring various instances of asteroidal triples.

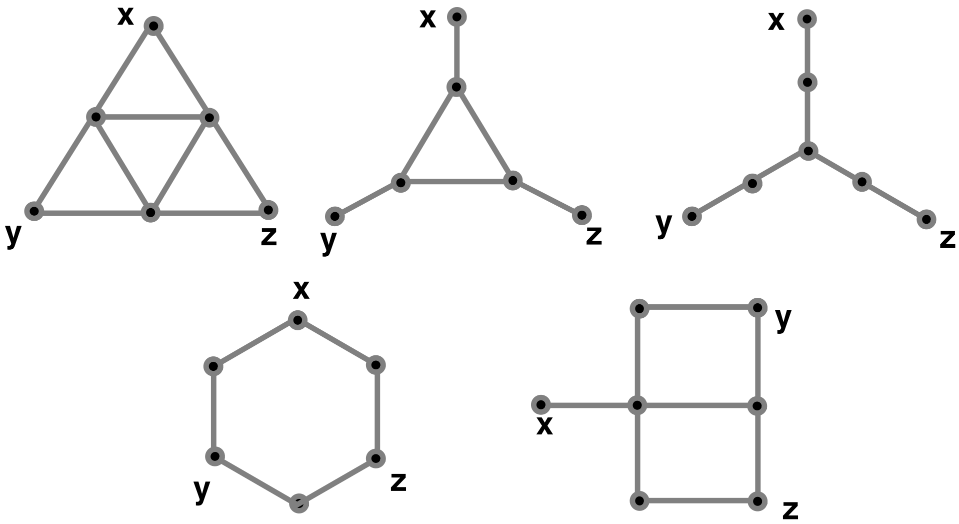

FIG. 1.3. *Various examples of asteroidal triples*

Lekkerkerker and Boland [18] demonstrated the importance of asteroidal triples in the following theorem.

THEOREM 1.1. [18] *A graph is an interval graph if and only if it is chordal and asteroidal triple-free.*

Thus, it appears that the condition of being asteroidal triple-free (AT-free, for short) prohibits a chordal graph from growing in three directions at once. The top three graphs in Figure 1.3 are examples of chordal graphs that are not interval graphs.

Later, Golumbic et al. [16] showed that cocomparability graphs (and, thus, permutation and trapezoid graphs) are also AT-free. Subsequently, it was shown that the perfect AT-free graphs strictly contain the cocomparability graphs [5]. Since C_5 is AT-free, the AT-free graphs need not be perfect. However, an easy argument shows that the celebrated Strong Perfect Graph Conjecture is true for asteroidal triple-free graphs [19].

More than two decades ago, Gallai [14] in his monumental work on comparability graphs obtained the first characterization of AT-minimal graphs (i.e. graphs that contain an asteroidal triple and are minimal with this property) in terms of fifteen families of subgraphs. Actually, Gallai's list is not complete. Since he was only interested in graphs with no induced C_5, all the AT-minimal graphs containing a C_5 are missing from [14]. For a full list of AT-minimal graphs the interested reader is referred to [7]. After Gallai's paper, little work has been done on AT-free graphs.

The main contribution of this work is to provide a number of structural results concerning asteroidal triple-free graphs. To anticipate, our main results[1] are:

1. We show that every connected AT-free graph has a dominating pair, that is, a pair of vertices such that every path joining them is a dominating set;
2. We provide properties of dominating pairs in AT-free graphs related to the concepts of connected domination and diameter;
3. We provide a characterization of AT-free graphs in terms of dominating pairs;
4. We provide a characterization of AT-free graphs in terms of minimal triangulations;
5. We provide a decomposition theorem for AT-free graphs.

The remainder of this work is organized as follows. Section 2 provides background material along with definitions of technical terms used throughout the paper. In §3 we study the existence of dominating pairs in connected AT-free graphs. In §4 we discuss properties of dominating pairs in the context of connected domination and show that some dominating pair achieves the diameter of the graph. In §5 we offer two characterizations of AT-free graphs. Specifically, we provide characterizations of AT-free graphs in terms of dominating pairs and in terms of minimal triangulations. In §6 we show that an AT-free graph may be extended to another AT-free graph by attaching, to each vertex in an appropriate dominating pair, a new vertex of degree one. This result leads to a decomposition theorem for AT-free graphs, whereby an AT-free graph is reduced to a single vertex by a sequence of contractions. In §7 we show that in AT-free graphs of diameter greater than three, the sets of vertices that can be in dominating pairs are restricted to two disjoint sets, thus strengthening the intuition about the linear structure of this class of graphs. Finally, §8 offers concluding remarks and poses some open problems.

2. Preliminaries. All graphs in this paper are finite with no loops or multiple edges. We use standard graph-theoretic terminology compatible with [2] to which we refer the reader for basic definitions.

As usual, we shall write $G = (V, E)$ to denote a graph G with vertex-set V and edge-set E. The *complement* of a graph G is the graph $\overline{G}$ having the same vertex-set as G; distinct vertices u and v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G. For a vertex x in G, $N_G(x)$ denotes the set of all the vertices adjacent to x in G. The *degree* of vertex x in the graph G, denoted by $d_G(x)$,

[1] for undefined terms the reader is referred to §2.

is the cardinality of $N_G(x)$. A vertex x will be said to be *pendant* if its degree is one. We let $N'_G(x)$ stand for the set of all the vertices adjacent to x in the complement $\overline{G}$ of G. The notation will be shortened to $N(x)$, $d(x)$, and $N'(x)$, respectively, whenever the context permits. If H is a subset of the vertex-set V of G, then G_H will denote the subgraph of G induced by H. Occasionally, if no confusion is possible, we shall use H as a shorthand for G_H.

A *path* is a sequence $v_0, v_1, \ldots, v_p$ of distinct vertices of G with $v_{i-1}v_i \in E$ for all i $(1 \leq i \leq p)$. A *chord* in a path $v_0, v_1, \ldots, v_p$ is an edge v_iv_j with i and j differing by more than one. A *cycle* of length $p+1$ is a sequence $v_0, v_1, \ldots, v_p$ of distinct vertices of G such that $v_{i-1}v_i \in E$ for all i $(1 \leq i \leq p)$ and $v_pv_0 \in E$. We let P_n and C_n denote the chordless path and cycle with n vertices, respectively. Unless stated otherwise, all paths in this work will be assumed to be chordless.

A set S of vertices of graph G is said to be *dominating* if every vertex outside S is adjacent to some vertex in S. Among dominating sets S that induce connected subgraphs of G, one is often interested in those that have minimum cardinality. In the remainder of this paper such a dominating set will be referred to as a *mccds.* A mccds that induces a path will be referred to as a *path-mccds.*

A path joining vertices x and y is termed an x, y-path. A vertex u *misses* a path π if u is adjacent to no vertex on π; otherwise, u *intercepts* π. In a connected graph, a pair (u, v) of vertices is termed a *dominating pair* if all u, v-paths are dominating. For vertices u and v of graph G, we let $D(u, v)$ denote the set of vertices that intercept all u, v-paths. In this terminology, (u, v) is a dominating pair whenever $D(u, v) = V$. For vertices u, v, and x of graph G, we say that u and v are *unrelated with respect to* x if $u \notin D(v, x)$ and $v \notin D(u, x)$.

Given a connected graph $G = (V, E)$, the distance $d_G(u, v)$ (or $d(u, v)$, for short) between vertices u and v is the length of a shortest path in G joining u and v. The *diameter* of G is defined as

$$\operatorname{diam}(G) = \max_{u,v \in V} \ d_G(u, v).$$

Two vertices u and v such that $d(u, v) = \operatorname{diam}(G)$ are said to *achieve* the diameter.

3. Dominating Pairs in AT-free Graphs. The main purpose of this section is to prove a fundamental domination-related property of AT-free graphs. To state this property, recall that a pair of vertices (x, y) is a dominating pair in a graph G if all x, y-paths in G are dominating sets. As it turns out, connected AT-free graphs always contain dominating pairs. Although it is straightforward to see that connected interval, permutation, trapezoid, and cocomparability graphs all contain dominating pairs, it is somewhat surprising that, up to now, this property had not been noticed for these classes of graphs.

Throughout this section, we assume a connected AT-free graph $G = (V, E)$ along with an arbitrary vertex x of G. We are now in a position to state the main result of this section.

THEOREM 3.1. *Every connected asteroidal triple-free graph contains a dominating pair.*

The conclusion of Theorem 3.1 is implied by the following stronger result.

THEOREM 3.2. *Let x be an arbitrary vertex of a connected asteroidal triple-free graph G. Either (x, x) is a dominating pair or else for a suitable choice of vertices y and z in $N'(x)$, (y, x) or (y, z) is a dominating pair.*

Our proof of Theorem 3.2 relies on a number of intermediate results about connected AT-free graphs that we present next.

CLAIM 3.3. *Let u, v, and w be arbitrary vertices of G. If $u \in D(v, x)$, $w \in D(u, x)$ and u and w are not adjacent, then $w \in D(v, x)$.*

Proof. Suppose that w misses some v, x-path π: $v = v_0, v_1, \ldots, v_k = x$. Let j be the largest subscript for which u is adjacent to vertex v_j of π: since $u \in D(v, x)$, such a subscript must exist. But now, w misses the u, x-path, $u, v_j, v_{j+1}, \ldots, v_k = x$ contradicting that $w \in D(u, x)$. ∎

In the remainder of this section, we shall use "unrelated" as a shorthand for "unrelated with respect to x". The reader is referred to Figure 3.1 for an illustration. The paths confirming that

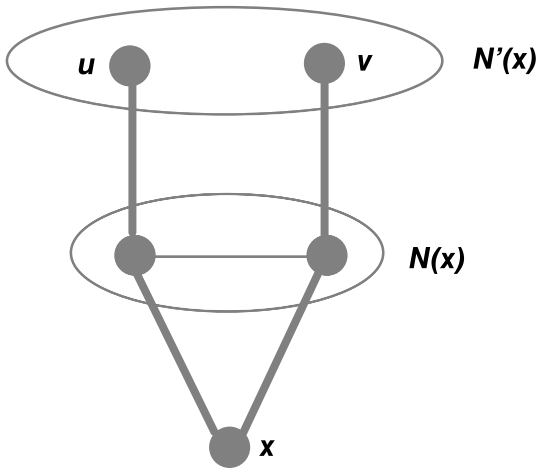

FIG. 3.1. *Illustrating unrelated vertices*

vertices u and v are unrelated are drawn in heavy lines. We further assume that F is an arbitrary connected component of $N'(x)$.

CLAIM 3.4. *F contains no unrelated vertices.*

Proof. If u and v are unrelated vertices in F, then the connectedness of F implies that $\{u, v, x\}$ is an asteroidal triple. □

CLAIM 3.5. *If u and v are vertices in F and if $v \notin D(u, x)$, then $D(u, x) \subset D(v, x)$.*

Proof. From Claim 3.4 it follows that $u \in D(v, x)$. Let w be an arbitrary vertex in $D(u, x) \backslash D(v, x)$. Clearly $w \notin N(x)$. If w and u are not adjacent, then Claim 3.3 guarantees that $w \in D(v, x)$; if w and u are adjacent, then clearly $w \in F$. If w misses some v, x-path then, in particular, v and w are not adjacent. Thus, with π standing for some u, x-path missed by v, $\pi \cup \{w\}$ contains a w, x-path missed by v. But now, v and w are unrelated, contradicting Claim 3.4. Consequently, $w \in D(v, x)$ and $D(u, x) \subseteq D(v, x)$; the inclusion is strict since $v \notin D(u, x)$. □

A vertex y in F is called *special* if $D(u, x) \subseteq D(y, x)$ for all vertices u in F. The following statement provides a characterization of special vertices.

CLAIM 3.6. *A vertex y in F is special if and only if $F \subseteq D(y, x)$.*

Proof. First, if the vertex y is special then, for every vertex v in F, $D(v, x) \subseteq D(y, x)$. In particular, $v \in D(v, x)$ implying that $F \subseteq D(y, x)$.

Conversely, suppose that $F \subseteq D(y, x)$. Let u be an arbitrary vertex in F and let w be an arbitrary vertex in $D(u, x)$. If w belongs to F then, since $F \subseteq D(y, x)$, $w \in D(y, x)$; if w does not belong to F, then u and w are not adjacent and Claim 3.3 guarantees that $w \in D(y, x)$, confirming that $D(u, x) \subseteq D(y, x)$. Since u is arbitrary, the claim follows. □

CLAIM 3.7. *F contains a special vertex.*

Proof. Choose a vertex y in F with $D(y, x) \subset D(t, x)$ for no vertex t in F. If y is not special then, by Claim 3.6, we find a vertex v in F with $v \notin D(y, x)$. By Claim 3.5, $D(y, x) \subset D(v, x)$ contradicting our choice of y. □

CLAIM 3.8. *Let v be an arbitrary vertex in $N'(x) \setminus F$. Either $v \in D(w, x)$ for all vertices w in F, or $v \notin D(w, x)$ for all vertices w in F.*

Proof. Suppose not; for a suitable choice of vertices w and w' in F, we have $v \in D(w, x)$ and $v \notin D(w', x)$. Let π stand for a w', x-path missed by v, and let π' stand for a w, w'-path entirely within F. But now $\pi \cup \pi'$ contains a w,x-path missed by v, contrary to our assumption. ∎

CLAIM 3.9. *Let v be a vertex in $N'(x) \setminus F$. If $F \not\subset D(v,x)$ then, for a special vertex u^* in F, $u^* \notin D(v,x)$.*

Proof. Write $U = \{u \in F \mid u \notin D(v,x)\}$. Since $F \not\subset D(v,x)$, U is nonempty. Choose a vertex u^* in U such that $D(u^*,x) \subset D(u,x)$ for no vertex u in U. If u^* is not special then, by Claim 3.6, there exists some vertex w in $F \backslash D(u^*,x)$. In particular, u^* and w are not adjacent. By Claim 3.5, $D(u^*,x) \subset D(w,x)$; by our choice of u^*, w must belong to $F \setminus U$. This, however, implies that $w \in D(v,x)$. Since $w \notin D(u^*,x)$, Claim 3.4 implies that $u^* \in D(w,x)$. Since u^* and w are not adjacent, Claim 3.3 guarantees that $u^* \in D(v,x)$, which is the desired contradiction. ∎

Call a vertex u of $N'(x)$ *strong* if $N'(x) \subset D(u,x)$. It is easy to verify that if u is a strong vertex, then (u,x) is a dominating pair in G. From now on, we shall tacitly assume that $N'(x)$ contains no strong vertices. A pair (y,z) of vertices in distinct components of $N'(x)$ is an *admissible pair* if $D(y,x) \cup D(z,x) \subset D(t,x) \cup D(t',x)$ for no vertices t, t' in distinct components of $N'(x)$.

Notice that if $N'(x)$ is connected, Claim 3.7 implies that $N'(x)$ contains a special vertex which, by virtue of Claim 3.6, is strong. We shall, therefore, assume that $N'(x)$ is disconnected. Now, the absence of strong vertices in $N'(x)$ guarantees the existence of admissible pairs. As it turns out, admissible pairs play a crucial role in our arguments. We now study some of their properties.

CLAIM 3.10. *Let Y and Z be two distinct components of $N'(x)$ and let vertices y in Y and z in Z be an admissible pair. Then, $Y \not\subset D(z,x)$ and $Z \not\subset D(y,x)$.*

Proof. Assume $Z \subset D(y,x)$; then, in particular, $z \in D(y,x)$. To see that $D(z,x) \subseteq D(y,x)$, note that for an arbitrary vertex w in $D(z,x)$, $w \in D(y,x)$ whenever $w \in Z$ and that, by virtue of Claim 3.3, $w \in D(y,x)$ whenever $w \notin Z$.

Since y is not strong, we find a vertex y' in $N'(x) \backslash D(y,x)$. But now, either (z,y') or (y,y') contradicts our choice of (y,z). To see this, note that if y' belongs to Y then, by Claim 3.5, $D(y,x) \subset D(y',x)$ and so

$$D(y,x) \cup D(z,x) = D(y,x) \subset D(y',x) \subseteq D(y',x) \cup D(z,x).$$

If y' does not belong to Y, then $D(y,x) \cup D(z,x) = D(y,x) \subseteq D(y',x) \cup D(y,x)$. Since y' does not belong to $D(y,x)$, the inclusion is strict. The fact that $Y \not\subset D(z,x)$ follows by a similar argument. ∎

CLAIM 3.11. *If (y,z) is an admissible pair, then $N'(x) \subset D(y,x) \cup D(z,x)$.*

Proof. We assume, without loss of generality, that vertices y and z belong to distinct connected components Y and Z of $N'(x)$, respectively. If the claim is false, we find a vertex w in $N'(x) \setminus (D(y,x) \cup D(z,x))$. Clearly, $w \notin D(y,x)$ and $w \notin D(z,x)$.

Since G is AT-free, it is easy to verify that

(3.1) no distinct vertices t, t', t'' in $N'(x)$ are pairwise unrelated with respect to x.

We claim that

(3.2) w does not belong to $Y \cup Z$.

If the vertex w belongs to Y then, by Claim 3.5, $D(y,x) \subset D(w,x)$ and since $w \notin D(z,x)$,

$$D(y,x) \cup D(z,x) \subset D(z,x) \cup D(w,x),$$

contradicting that (y,z) is an admissible pair. The proof of the fact that $w \notin Z$ is similar and, thus, omitted.

Further, we claim that for a suitable choice of vertices u and v in $N'(x)$

$$u \in D(y,x)\backslash(D(z,x) \cup D(w,x)) \text{ and } v \in D(z,x)\backslash(D(y,x) \cup D(w,x)). \tag{3.3}$$

To justify (3.3), observe that by (3.2), y, z and w belong to distinct components of $N'(x)$. Since (y,z) is an admissible pair,

$$D(y,x) \cup D(z,x) \not\subset D(z,x) \cup D(w,x),$$

and, therefore, the required vertex u exists. A similar argument asserts the existence of vertex v.

Next, we claim that

$$y \in D(z,x) \cup D(w,x) \text{ and } z \in D(y,x) \cup D(w,x). \tag{3.4}$$

To see this, note that if $y \notin D(z,x) \cup D(w,x)$, then our choice of w guarantees that y and w are unrelated. Therefore, it must be that $z \in D(y,x) \cup D(w,x)$, for otherwise y, z, and w would be pairwise unrelated, contradicting (3.1). Consider the vertex v specified in (3.3); since $z \in D(y,x) \cup D(w,x)$ and $v \in D(z,x)\backslash(D(y,x) \cup D(w,x))$, Claim 3.3 implies that z and v are adjacent. But now $\{y,v,w\}$ is an asteroidal triple; this follows since y and w are unrelated, and both v,w and v,y are unrelated by (3.3) and Claim 3.8. Along similar lines, one can prove that $z \in D(y,x) \cup D(w,x)$. Thus, (3.4) must hold.

Further, we claim that

$$u \in Y \text{ and } v \in Z. \tag{3.5}$$

By (3.4), $y \in D(z,x) \cup D(w,x)$; by (3.3), $u \in D(y,x)\backslash(D(z,x) \cup D(w,x))$. It follows that u and y are adjacent, for otherwise we contradict Claim 3.3. The fact that $v \in Z$ is proved similarly.

To complete the proof of Claim 3.11, we first observe that (3.5), (3.3), and Claim 3.8 combined, guarantee that $u \notin D(v,x)$ and $v \notin D(u,x)$, and so u and v are unrelated. Similarly, by (3.5), (3.3), and Claim 3.8, the vertices u and w are unrelated, as are v and w. But now, the vertices u, v, and w are pairwise unrelated, contradicting (3.1). With this, the proof of Claim 3.11 is complete. ☐

We are now in a position to give the proof of Theorem 3.2.

Proof. (Theorem 3.2) If $N'(x)$ is empty, then (x,x) is a dominating pair. If $N'(x)$ is nonempty but contains a strong vertex y, then clearly (x,y) is a dominating pair. Otherwise, let (y,z) be an admissible pair in $N'(x)$. We assume, without loss of generality, that y and z belong to distinct connected components Y and Z of $N'(x)$, respectively. By Claim 3.10, Claim 3.9, and Claim 3.8 we find special vertices y^* in Y and z^* in Z such that $y^* \notin D(z^*,x)$ and $z^* \notin D(y^*,x)$. Put differently, y^* and z^* are unrelated. Furthermore, since y^* and z^* are special, we have $D(y,x) \cup D(z,x) \subseteq D(y^*,x) \cup D(z^*,x)$, implying that (y^*,z^*) is also an admissible pair.

We claim that

$$(y^*,z^*) \text{ is a dominating pair in } G.$$

By Claim 3.11, any vertex v that misses some y^*,z^*-path must be in $N(x)$. (Observe that v and x are distinct, since every y^*,z^*-path contains at least one vertex in $N(x)$.) Since y^* and z^* are unrelated, y^* misses some z^*,x-path π and z^* misses some y^*,x-path π'. But now, we have reached a contradiction; $\{y^*,z^*,v\}$ is an asteroidal triple. To see this, note that, by assumption, v misses some y^*,z^*-path; in addition y^* misses the z^*,v-path $\pi \cup \{v\}$ and z^* misses the y^*,v-path $\pi' \cup \{v\}$. ☐

It is perhaps interesting to note that Claim 3.4 suggests the following characterization of AT-free graphs. The proof is immediate and left to the reader.

THEOREM 3.12. *A graph G is AT-free if and only if for every vertex x of G, no component F of $N'(x)$ contains unrelated vertices.*

4. Distance Properties of Dominating Pairs. The purpose of this section is to examine various distance-related properties featured by dominating pairs in connected AT-free graphs. Specifically, we study the maximum distance between vertices of a dominating pair, as well as the relationship between dominating pairs and minimum cardinality connected dominating sets. In particular, we show that in every connected AT-free graph, some dominating pair achieves the diameter (Theorem 4.3) and some dominating pair forms the endpoints of a path-mccds (Theorem 4.6). To begin, we state a property of connected AT-free graphs that will be used throughout this section.

CLAIM 4.1. *A connected asteroidal triple-free graph G is a clique if and only if it contains no non-adjacent dominating pair.*

Proof. The "only if" part is trivial. To prove the "if" part, note that if G is not a clique then, for some vertex x of G, $N'(x)$ is nonempty. By Theorem 3.2, there exist vertices $y, z \in N'(x)$ such that either (x, y) is a dominating pair (with x and y non-adjacent) or, failing this, (y, z) is a dominating pair. In the latter case, the vertices y and z belong to distinct connected components of $N'(x)$ and, consequently, must be non-adjacent. □

In the remainder of this section we assume a connected AT-free graph G which is not a clique. Claim 4.1 guarantees that we can find a non-adjacent dominating pair (x, y_0) in G. Let F be the connected component of $N'(x)$ containing y_0, and let Y be the set of vertices y in F for which (x, y) is a dominating pair in G. A vertex a in $F \setminus Y$ is called an *attractor* if $Y \subset D(a, x)$.

CLAIM 4.2. *F contains no attractors.*

Proof. If the statement is false then the set A of attractors in $F \setminus Y$ is nonempty. Let a^* be a vertex in A for which $D(a^*, x) \subset D(a, x)$, for no vertex a in A. We claim that (a^*, x) is a dominating pair in G. If the statement is false, we find a vertex t that misses some a^*, x-path π. However,

(i) $t \notin A$ by our choice of a^* and Claim 3.5, combined;
(ii) $t \notin Y$ because $Y \subset D(a^*, x)$;
(iii) $t \notin N'(x) \setminus F$, for otherwise t would miss a y_0, x-path. Such a path is contained in the concatenation of π with a y_0, a^*-path in F;
(iv) $t \notin F \setminus (A \cup Y)$. Since $Y \subset D(a^*, x)$, t must be adjacent to every vertex in Y, implying that t belongs to A, a contradiction.

□

The next result concerns the maximum distance between vertices in a dominating pair.

THEOREM 4.3. *In every connected asteroidal triple-free graph some dominating pair achieves the diameter.*

Our proof of Theorem 4.3 relies on the following intermediate result.

LEMMA 4.4. *Let G be a connected AT-free graph and let vertices x and a of G be such that $d(x, a) = diam(G)$. If (x, y) is a dominating pair with vertex y in $N'(x)$, then there exists a vertex z such that (x, z) is a dominating pair and $d(x, z) = diam(G)$.*

Proof. Clearly, we may assume that $d(x, a) \geq 2$. Let Y be the set of vertices y in $N'(x)$ such that (x, y) is a dominating pair.

We assume that a does not belong to Y, for otherwise there is nothing to prove. Observe that Y is contained in the component of $N'(x)$ containing a; otherwise, $d(x, y) = 2$ and $d(x, a) = 2$, since every path joining x and y must dominate a.

By virtue of Claim 4.2, a cannot be an attractor; we find a vertex y in Y such that $y \notin D(a, x)$. In particular, a and y are non-adjacent. Consider an arbitrary shortest x, y-path $\pi(x, y)$: $x = u_0,\ u_1,\ \ldots,\ u_k = y$. Since (x, y) is a dominating pair, a must be adjacent to some vertex u_j. Since a and y are non-adjacent, $j < k$. But now, $\mathrm{diam}(G) = d(x, a) \leq d(x, u_j) + 1 \leq d(x, y)$ $\leq \mathrm{diam}(G)$, implying that (x, y) is a dominating pair with $d(x, y) = \mathrm{diam}(G)$. This completes the proof of Lemma 4.4. □

We now give a proof of Theorem 4.3.

Proof. (Theorem 4.3) Let vertices x and a be such that $d(x, a) = \mathrm{diam}(G)$. Let C be the connected component of $N'(x)$ containing a. We may assume that x is in no dominating pair involving a vertex in $N'(x)$, otherwise we are done by Lemma 4.4. By the proof of Theorem 3.2, there exists a dominating pair (y, z) with vertices y and z belonging to distinct components of $N'(x)$. We observe that precisely one of y and z belongs to C; otherwise, $d(y, z) = 2$ and we are done. (To see this, note that if neither of y and z is in C, then a must be adjacent to a neighbor of x; therefore, $\mathrm{diam}(G) = d(a, x) = 2$ and $2 \leq d(y, z) \leq \mathrm{diam}(G)$, implying that (y, z) is a dominating pair of distance $\mathrm{diam}(G)$.) Furthermore, we may assume that $d(y, z) < \mathrm{diam}(G)$; otherwise, (y, z) is the desired dominating pair.

Assume without loss of generality that y belongs to C and that z belongs to some component C' ($\neq C$) of $N'(x)$. If there exists a shortest z, y-path $\pi(z, y)$: $z = u_0, u_1, \ldots, u_k = y$ such that a is adjacent to u_j, for some $j < k$, then $\mathrm{diam}(G) = d(x, a) \leq d(x, u_j) + 1 \leq d(z, u_j) + 1 \leq d(z, y) \leq \mathrm{diam}(G)$, and (y, z) is the required dominating pair. Otherwise, y is the only vertex on π adjacent to a and $\mathrm{diam}(G) = d(x, a) \leq d(a, z) \leq d(y, z) + 1 \leq \mathrm{diam}(G)$. Therefore $d(a, z) = \mathrm{diam}(G)$ and the conclusion follows by Lemma 4.4. ◻

Thus, in a connected AT-free graph, some dominating pair achieves the diameter. We now consider shortest dominating paths and their relation to connected dominating sets. In the remainder of this section we shall find it convenient to make use of a special notation that we now introduce. When referring to a path π, we shall denote by $\pi - y$ the path obtained from π by removing y, one of its endpoints. Similarly, we let $\pi + x$ denote the path obtained from π by the addition of x as a new endpoint.

THEOREM 4.5. *Every connected asteroidal triple-free graph has a path-mccds.*

Proof. Let G be a connected AT-free graph, let D be an arbitrary mccds and let (x, y) be an arbitrary dominating pair in G. We may assume that $|D| \geq 3$; otherwise there is nothing to prove. We note that

$$\text{if } \{x, y\} \subset D \text{ then } D \text{ induces a path.} \tag{4.1}$$

This follows from the fact that every x, y-path π in D is a connected dominating set, implying that $D = \pi$.

Next, we claim that

$$\text{if } x \in D \text{ or } y \in D \text{ then some mccds induces a path.} \tag{4.2}$$

To justify (4.2) assume, without loss of generality, that $x \in D$. By (4.1), we may assume that $y \notin D$. Let Y consist of all the vertices in D adjacent to y. Since D is connected, we find a path π joining x and a vertex y' in Y, such that all vertices in $\pi - y'$ are in $D \backslash Y$. Either $D = \pi$ or $\pi + y$ is a dominating path of cardinality at most $|D|$. Thus, (4.2) must hold.

By (4.1) and (4.2), combined, we may assume that neither x nor y belongs to D. Let X and Y be the sets of vertices in D adjacent to x and y, respectively. Observe that X and Y must be disjoint, for otherwise with w standing for an arbitrary vertex in $X \cap Y$, $\{x, w, y\}$ induces a dominating path and there is nothing to prove. Connectedness of D guarantees the existence of vertices x' in X, y' in Y, and of an x', y'-path π in D, all of whose internal vertices are in $D \backslash (X \cup Y)$. We claim that

$$|D \backslash \pi| = 1. \tag{4.3}$$

To see that this is the case, observe that if $D = \pi$ then we are done; if $|D \backslash \pi| > 1$, then $\pi + x + y$ is a dominating path of cardinality at most $|D|$. Thus, (4.3) must hold.

By (4.3) we write $\{z\} = D \backslash \pi$. Since the path $\pi + x$ is of cardinality $|D|$, we find a vertex u that misses $\pi + x$. Similarly, since the path $\pi + y$ is of cardinality $|D|$, we find a vertex v that misses

$\pi + y$. The following are easily seen:
• $u \neq v$ and uy, vx are edges; (otherwise, we contradict that (x, y) is a dominating pair)
• u and v are not adjacent; (else, $\{u, x, y'\}$ is an AT in G)
• $u \neq z$, $v \neq z$, and both uz, vz are edges; (otherwise, we contradict that D is a connected dominating set)
• $x'z$, $y'z$ are both edges; (if $x'z$ is not an edge, then $\{u, x', v\}$ is an AT).
We claim that

$$\{u, z, v\} \text{ is a mccds.}$$

To see this, let w be a vertex that misses the path induced by $\{u, z, v\}$. Since D is dominating, w must be adjacent to some vertex on π. But now, it is easy to confirm that $\{u, v, w\}$ is an AT. □

Next, we show that Theorem 4.5 can be strengthened.

THEOREM 4.6. *In every connected asteroidal triple-free graph the endpoints of some path-mccds are a dominating pair.*

Our proof of Theorem 4.6 relies on the following technical result.

LEMMA 4.7. *Let G be a connected asteroidal triple-free graph and let $\pi(x, a)$ be a path-mccds in G with endpoints a and x. If x belongs to a dominating pair involving a vertex in $N'(x)$, then there exists a vertex y in $N'(x)$ such that (x, y) is a dominating pair and each shortest x, y-path is a mccds.*

Proof. Write $\pi(x, a)$: $x = u_0$, u_1, $\ldots$, $u_k = a$. We may assume that $k \geq 2$. Let C be the component of $N'(x)$ containing a. Observe that every vertex that forms a dominating pair with x must belong to C. To clarify this, suppose such a vertex t belongs to a component C' distinct from C. Then, since the path $\pi(x, a)$ is dominating, t is adjacent to u_1 implying that $d(x, t) = 2 \leq k$, and there is nothing to prove.

Let Y be the set of all special vertices in C. It is easy to see that x forms a dominating pair with every vertex in Y. Thus, we may assume that $a \notin Y$. Note that if some vertex in Y is adjacent to u_j with $j < k$ then we are done; otherwise, a is an attractor, contradicting Claim 4.2. This completes the proof of Lemma 4.7. □

Proof. (Theorem 4.6) For convenience, we inherit the notation of Lemma 4.7. We may assume that $\pi(x, a)$ is a path-mccds and that x is in no dominating pair involving a vertex in $N'(x)$, otherwise we are done by Lemma 4.7. By the proof of Theorem 3.2, there exists a dominating pair (y, z) with y and z in distinct components of $N'(x)$. We observe that precisely one of the vertices y and z belongs to C; otherwise, $d(y, z) = 2 \leq k$ and we are done.

Assume without loss of generality that $y \in C$ and that z belongs to a component C' distinct from C. Note that since $\pi(x, a)$ is dominating, z is adjacent to u_1. Thus, y is adjacent to a and to no other vertex on $\pi(x, a)$, for otherwise $\mathrm{d}(y, z) \leq k$.

We claim that at least one of the paths $\pi(x, a) - x + y$ or $\pi(x, a) - x + z$ is dominating. Observe that both of these paths are of length k and each of them is anchored at a vertex belonging to a dominating pair. Therefore, once we establish this claim the conclusion of Theorem 4.6 follows from Lemma 4.7. If neither of these paths is dominating then
• there exists a vertex v missing $\pi(x, a) - x + y$; trivially, both vx and vz are edges;
• there exists a vertex w missing $\pi(x, a) - x + z$; trivially, both wx and wy are edges.

But now, we have reached a contradiction: $\{a, w, z\}$ is an AT, and the proof of the theorem is complete. □

5. Two Characterizations of AT-free Graphs. The goal of this section is to offer two characterizations of AT-free graphs. To motivate our first characterization, notice that Theorems 3.1 and 3.2 do not lead to a necessary and sufficient condition for a graph to be AT-free. For example, vertices achieving the diameter in the C_6 constitute a dominating pair. Furthermore, if we add a universal vertex to an arbitrary graph, we obtain a graph that has a dominating pair consisting of the universal vertex and any other vertex. Clearly, any attempt to provide a characterization of AT-free graphs involving dominating pairs must not only be based on induced subgraphs, it must also restrict the types of dominating pairs. For example, the graph C_6 contains an AT, yet every induced subgraph has a dominating pair.

The first goal is to provide a characterization of AT-free graphs based on dominating pairs. As indicated previously, such a result must restrict the types of dominating pairs. In particular, we impose an adjacency condition on G with dominating pair (x, y), whereby the connected component of $G \setminus \{x\}$ containing y has a dominating pair (x', y) with x' adjacent to x. As illustrated in Figure 5.1, the graph C_6 fails this criterion. Here, (x, y) is a dominating pair in the graph, yet neither (x', y) nor (x'', y) is a dominating pair in the graph obtained by removing vertex x.

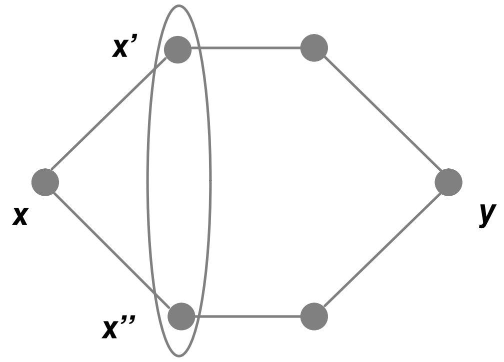

FIG. 5.1. C_6

We begin by stating a simple property of vertices in AT-free graphs which is of independent interest.

CLAIM 5.1. *Let u, v, and y be vertices in a connected asteroidal triple-free graph, such that $v \notin D(u, y)$. If $D(u, y) \not\subset D(v, y)$ then, for some vertex w in $D(u, y)$, v and w are unrelated with respect to y.*

Proof. Let π be a u, y-path missed by v. Let w be an arbitrary vertex in the set $D(u, y) \setminus D(v, y)$. Since w does not belong to $D(v, y)$, w misses some v, y-path. Since w belongs to $D(u, y)$, w intercepts π and, moreover, $\pi \cup \{w\}$ contains a chordless w, y-path missed by v, confirming that v and w are unrelated with respect to y. □

Let $\pi = u_1, u_2, \ldots, u_k$ and $\pi_1 = v_1, v_2, \ldots, v_l$ be two paths. We shall refer to the path $u_1, u_2, \ldots, u_i$ with $i \leq k$ as a *prefix* of π. A vertex w is a *cross* point of π and π_1 if $w = u_i = v_j$ and the four vertices u_{i-1}, v_{j-1}, u_{i+1}, and v_{j+1} are all defined and distinct.

For later reference, we now investigate properties of asteroidal triples. Let G be a graph containing an AT. Choose an induced subgraph H of G with the least number of vertices such that some triple $\{x, y, z\}$ is an AT in H. Let $\pi(x, y)$, $\pi(x, z)$, and $\pi(y, z)$ be paths in H demonstrating that $\{x, y, z\}$ is an AT. In the following we write $\pi(x, y) : x = u_1, u_2, \ldots, u_k = y$, $\pi(x, z) : x = v_1, v_2, \ldots, v_l = z$, and $\pi(z, y) : z = w_1, w_2, \ldots, w_t = y$. Clearly, the choice of H guarantees that x, y, and z have degree at most two.

CLAIM 5.2. *No pair of paths among $\pi(x, y)$, $\pi(x, z)$, and $\pi(y, z)$ has a cross point.*

Proof. Suppose that the paths $\pi(x, y)$ and $\pi(x, z)$ have a cross point w, such that $w = u_i = v_j$. Observe that the definition of a cross point and the minimality of H, combined, guarantee that $3 \leq i$

and $3 \leq j$. Since the paths demonstrate that $\{x, y, z\}$ is an AT, $i \leq k-2$ and $j \leq l-2$. But now, in $H' = H \setminus \{v_{j-1}\}$, y misses the x, z-path $u_1, u_2, \ldots, u_i = v_j, v_{j+1}, \ldots, z$ and x misses the y, z-path $y, u_{k-1}, \ldots, u_i = v_j, v_{j+1}, \ldots, z$. Thus, $\{x, y, z\}$ is an AT in H', contradicting the minimality of H. $\square$

CLAIM 5.3. *Let i be the largest subscript for which there exists a subscript j such that $u_i = v_j$ and $u_{i+1} \neq v_{j+1}$. Then, $i = j$ and $u_t = v_t$ for all $1 \leq t \leq i$.*

Proof. Since y are z are distinct and $u_1 = v_1$, the subscript i in the statement of the claim always exists. Since, by Claim 5.2, u_i cannot be a cross point, we must have $u_{i-1} = v_{j-1}$. Let t be the least value for which $u_{i-t} \neq v_{j-t}$. We may assume that such a t exists, for otherwise there is nothing to prove.

Clearly, $u_1 = v_1$ implies that $t \leq \min\{i-2, j-2\}$. Consequently, we can remove vertex v_{j-t} from H, while still ensuring that $\{x, y, z\}$ is an AT in the remaining graph. This contradiction completes the proof of the claim. $\square$

LEMMA 5.4. *There exist unique vertices x', y', z' in H such that*

(i) The unique path between x and x' is a prefix of both $\pi(x, y)$ and $\pi(x, z)$;

(ii) The unique path between y and y' is a prefix of both $\pi(y, x)$ and $\pi(y, z)$;

(iii) The unique path between z and z' is a prefix of both $\pi(z, x)$ and $\pi(z, y)$.

Proof. Claim 5.3 guarantees that one can associate with x a unique vertex x' corresponding to the largest subscript for which $u_i = v_i$. Put differently, the path $x = u_1, u_2, \ldots, u_i = x'$ in H is the common prefix of both $\pi(x, y)$ and $\pi(x, z)$. In a perfectly similar way one can define vertices y' and z'. $\square$

As it turns out, vertices x', y', z' have a number of interesting properties. We present some of them next.

CLAIM 5.5. *The vertices x', y', and z' are either all distinct or else they coincide.*

Proof. Suppose that exactly two of the vertices x', y', z' coincide. Symmetry allows us to assume that $x' = y'$. Write $x' = u_i$ and $y' = w_{t-k+i}$. Since $x'(= y')$ cannot be a cross point of $\pi(x, z)$ and $\pi(z, y)$, we must have $v_{i+1} = w_{t-k+i-1}$. Now an argument similar to that of the proof of Claim 5.3 guarantees that the subpaths of $\pi(x, z)$ and $\pi(z, y)$ between z and x' coincide, a contradiction. $\square$

Claim 5.5 and the minimality of H, combined, imply the following result.

COROLLARY 5.6. *Vertices x', y', and z' coincide if and only if H is isomorphic to the graph in Figure 5.2.*

CLAIM 5.7. *Vertex x' is distinct from x if and only if $d_H(x) = 1$. Furthermore, if x', y' and z' are distinct, and $x' \neq x$ then xx' is an edge.*

Proof. First, observe that if $x' = x$ then, by Claim 5.3, $d_H(x) = 2$. Conversely, if vertices x and x' are distinct, then $\pi(x, y)$ and $\pi(x, z)$ have at least one edge in common, confirming that $d_H(x) = 1$.

To settle the second part of the claim, assume that $x' = u_i$ with $3 \leq i$. Since x', y', z' are distinct, u_{i-1} misses the path $\pi(y, z)$ and, thus, $\{u_{i-1}, y, z\}$ is an AT in $H \setminus \{x\}$. The conclusion follows. $\square$

For reasons that will become clear later, we shall say that a connected graph H with a dominating pair satisfies the *spine property* if for every *non-adjacent* dominating pair (α, β) in H, there exists a neighbor α' of α, such that (α', β) is a dominating pair of the connected component of $H \setminus \{\alpha\}$ containing β. We are now in a position to state the first main result of this section.

THEOREM 5.8. (THE SPINE THEOREM) *A graph G is asteroidal triple-free if and only if every connected induced subgraph H of G satisfies the spine property.*

Proof. To settle the "only if" part, let G be an AT-free graph and let H be any connected induced subgraph of G. We may assume that H is not a clique (complete), since otherwise it has

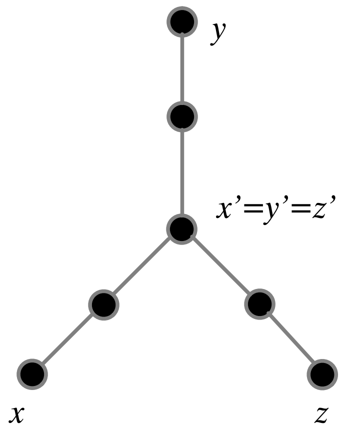

FIG. 5.2. *Illustrating Corollary 5.6*

the spine property. By Claim 4.1, H has a non-adjacent dominating pair (α, β). Let C_β denote the connected component of $H \setminus \{\alpha\}$ that contains β. Let A denote $N(\alpha) \cap C_\beta$. We choose a vertex $\tilde{\alpha}$ in A such that $D(\tilde{\alpha}, \beta) \subset D(t, \beta)$ for no vertex t in A.

We claim that

$$(\tilde{\alpha}, \beta) \text{ is a dominating pair in } C_\beta. \tag{5.1}$$

To see that (5.1) holds, suppose that a vertex t in C_β misses some $\tilde{\alpha}, \beta$-path. Observe that t must belong to A, for otherwise this path extends to an α, β-path in H missed by t, contradicting that (α, β) is a dominating pair. Our choice of $\tilde{\alpha}$ guarantees that $D(\tilde{\alpha}, \beta) \not\subset D(t, \beta)$. By Claim 5.1 we find a vertex w in $D(\tilde{\alpha}, \beta)$ such that t and w are unrelated with respect to β. Note that w belongs to A, otherwise the t, β-path missed by w would extend to an α, β-path missed by w. But now, w and t are in the same component of $N'(\beta)$ and are unrelated with respect to β, contradicting Claim 3.4. This completes the proof of the "only if" part.

To prove the "if" part, let H be an induced subgraph of G with the least number of vertices in which some set $\{x, y, z\}$ is an AT. Further, let $\pi(x, y)$, $\pi(x, z)$, and $\pi(y, z)$ be (chordless) paths in H demonstrating that $\{x, y, z\}$ is an AT.

CLAIM 5.9. *If H has an adjacent dominating pair, it also has a non-adjacent dominating pair.*

Proof. Suppose that (a, b) is an adjacent dominating pair in H and let $A = \{v \mid av \in E, bv \notin E\}$; $B = \{v \mid bv \in E, av \notin E\}$ and $C = \{v \mid av, bv \in E\}$. By the minimality of H, every vertex of $H \backslash \{x, y, z\}$ is on at least one π path. If $x = a$, then y and z are in B and $H \backslash \{b\}$ contains an AT on $\{x, y, z\}$. Thus we may assume that $\{a, b\} \cap \{x, y, z\} = \emptyset$. Furthermore it is easy to see that A and B each contain at least one of $\{x, y, z\}$, otherwise one of a or b can be removed from H without destroying the AT. We now have two cases.

Case 1: $x \in A$, $y \in B$, $z \in C$:

Since a and b must be on at least one π path, $\pi(x, z) = x, a, z$ and $\pi(y, z) = y, b, z$. Consider $\pi(x, y) = v_1 (= x), v_2, \ldots, v_k (= y)$. First we note that none of $v_2, \ldots, v_{k-2}$ can be in A since such a vertex together with y and z would form an AT in $H \backslash \{x\}$. Similarly none of $v_3, \ldots, v_{k-1}$ can be in B. Thus all of $v_3, \ldots, v_{k-2}$ (if they exist) must be in C. If v_2 is in C, then $B = \{y\}$ and (a, y) is a non-adjacent dominating pair; if v_{k-1} is in C, then (x, b) is a non-adjacent dominating pair. Thus v_2 is in B, v_{k-1} is in A and all of $v_3, \ldots, v_{k-2}$ are in C. Now if $k > 4$, then $\{v_2, v_{k-1}, z\}$ forms an AT in $H \backslash \{x, y\}$; otherwise (x, b) is a non-adjacent dominating pair.

Case 2: $x \in A$, $y, z \in B$:
Since each of a and b must belong to some π path, we may assume that $a \in \pi(x, y)$ and $\pi(y, z) = y, b, z$. Furthermore we may assume that the degree of x is 2 since otherwise (x, b) would be a non-adjacent dominating pair. We now study $\pi(x, y) = v_1(= x), v_2(= a), \ldots, v_k(= y)$ and note by the fact that $\pi(x, y)$ is chordless that the only vertex of $\pi(x, y)$, other than x that could be in A is v_3. Similarly we let $\pi(x, z) = u_1(= x), u_2, \ldots, u_j(= z)$ and note that no vertex on $\pi(x, z)$ other than x and possibly u_2 may be adjacent to a since otherwise an x, z-path through a contradicts the minimality of H. We distinguish two subcases:
Case 2.1: $v_3 \in A$:
First we show that $k = 4$ (i.e. v_3 is adjacent to y). To see this, note that if (x, b) is not a dominating pair then there exists a chordless x, b-path, P, and a vertex w in A missing P. Furthermore, v_4 must be adjacent to b. If $w = v_3$, then we have an AT on $\{x, v_3, z\}$ in $H\backslash\{y\}$; for the x, z-path consider the induced path on P and the edge bz. If $w \neq v_3$, then w is on $\pi(x, z)$ and we have $\{v_3, y, z\}$ being an AT in $H\backslash\{x\}$; now the v_3, z-path consists of the subpath of $\pi(x, z)$ from z to w together with the edges wa and av_3. Thus $k = 4$.

Now look at $\pi(x, z)$. Since the degree of x is 2, a is not on $\pi(x, z)$. If u_2 is in A, then $j = 3$ (i.e. u_2 is adjacent to z); otherwise $\{u_2, y, z\}$ would be an AT in $H\backslash\{x\}$. Now if u_2v_3 is an edge, then (x, b) is a non-adjacent dominating pair; otherwise (u_2, v_3) is a non-adjacent dominating pair.

Thus we may assume that u_2 is not in A and therefore is adjacent to b. If $j = 3$, then (y, u_2) is a non-adjacent dominating pair. Suppose v_3 is not adjacent to some u_i, $2 < i < j$; then $\{u_i, x, y\}$ forms an AT in $H\backslash\{z\}$. If u_2 is not adjacent to v_3, then $\{x, v_3, z\}$ forms an AT in $H\backslash\{y\}$; otherwise, (x, b) is a non-adjacent dominating pair.
Case 2.2: $v_3 \notin A$:
Thus all of $v_3, \ldots, v_k$ are adjacent to b. Thus (x, b) is a non-adjacent dominating pair since b is adjacent to all vertices of H except x and possibly u_2, which is adjacent to x. $\square$

We now assume that H has a non-adjacent dominating pair (a, b).

Claim 5.10. *Vertices a and b are distinct from x, y, z, x', y', and z'.*

Proof. To begin, we show that a and b are distinct from x, y, and z. Suppose not; we may assume, without loss of generality, that $a = x$. Since (a, b) is a dominating pair, b must belong to $\pi(y, z)$. Consider the x, b-path contained in the concatenation of $\pi(x, y)$ with the $y - b$ portion of $\pi(y, z)$. This path is missed by z, unless vertices b and z are adjacent. A mirror argument shows that b and y are also adjacent.

Since, by assumption, H satisfies the spine property and vertices a and b are non-adjacent, we should be able to find a neighbor b' of b such that (a, b') is a dominating pair in $H \setminus \{b\}$. However, if b' belongs to $\pi(x, y)$, then z misses the corresponding b', a-path; if b' belongs to $\pi(x, z)$, then y misses a b', a-path. The fact that a is distinct from x' follows by an identical argument, whose details are omitted. $\square$

Claim 5.10 has the following interesting corollary.

Claim 5.11. *Each pair of vertices x and x', y and y', and z and z' must coincide.*

Proof. First, observe that the vertices x', y', z' are distinct, for otherwise, by Corollary 5.6, H is isomorphic to the graph in Figure 5.2 which does not satisfy the spine property.

If the statement is false, then we may assume, without loss of generality, that x and x' are distinct. By Claim 5.7, x has degree 1 in H. By Claim 5.10, a (respectively b) is distinct from both x and x', implying that x misses some a, b-path, a contradiction. $\square$

By virtue of Claim 5.11 and Claim 5.7 combined, x, y and z have degree exactly two in H and, moreover, H is biconnected. Without loss of generality, let vertices a and b belong to $\pi(x, y)$ and to $\pi(x, z)$, respectively. Observe that vertices a and y must be adjacent, for otherwise the a, b-path through x is missed by y. Similarly, vertices b and z are also adjacent, else the a, b-path through x

is missed by z. Further, either a or b is adjacent to x, for if not, then the a, b-path through y and z is missed by x. Symmetry allows us to assume, without loss of generality, that a and x are adjacent.

We claim that

$$\text{vertices } b \text{ and } x \text{ are adjacent.} \tag{5.2}$$

Since vertices a and b are not adjacent, and since H is biconnected, the spine property guarantees that we can find a neighbor a' of a, such that (a', b) is a dominating pair of $H \setminus \{a\}$. Clearly, a' cannot be x; if b and x are not adjacent, then a' cannot be y. Therefore, a' must belong to $\pi(y, z)$. But now, x misses the a', b-path containing z, a contradiction. Thus, (5.2) must hold.

To complete the proof of the "if" part, we claim that

$$(b, y) \text{ is a dominating pair.} \tag{5.3}$$

It is clear that once (5.3) is proved, we have reached a contradiction: by Claim 5.10, y cannot be in a dominating pair.

To prove (5.3) consider a vertex c that misses a path π joining b and y. Since (a, b) is a dominating pair, π does not involve a. Trivially, c must belong to $\pi(y, z)$. But now, $\{c, x, y\}$ is an AT in $H \setminus \{a\}$. To see this, note that $\pi + x$ is an x, y-path missed by c; the y, c-path consisting of the portion of $\pi(y, z)$ from y to c is missed by x; finally, $\pi(x, z)$ concatenated with the $c - z$ portion of $\pi(y, z)$ contains a c, x-path missed by y. This completes the proof of Theorem 5.8. □

Let $G = (V, E)$ be a connected AT-free graph and let (x, y) be an arbitrary non-adjacent dominating pair in G. Construct a sequence $x_0, x_1, \ldots, x_k$ of vertices of G and a sequence $G_0, G_1, \ldots, G_k$ of subgraphs of G defined as follows:

i) $G_0 = G$ and $x_0 = x$;
ii) for all i $(0 \le i \le k - 1)$, $x_i y \notin E$ and $x_k y \in E$;
iii) for all i $(1 \le i \le k)$, let G_i stand for the subgraph of G_{i-1} induced by the component of $G_{i-1} \setminus \{x_{i-1}\}$ containing y;
iv) for all i $(1 \le i \le k)$, let x_i be a vertex in G_i adjacent to x_{i-1} and such that (x_i, y) is a dominating pair in G_i.

The existence of the sequence $x_0,\ x_1,\ \ldots,\ x_k$ is guaranteed by the Spine Theorem. The sequence $x_0,\ x_1,\ \ldots,\ x_k,\ y$ will be referred to as a *spine* of G. For an illustration of the Spine Theorem the reader is referred to Figure 5.3. The sequence of graphs featured in Figure 5.3 begins with a graph G with vertex-set $\{a,\ b,\ c,\ d,\ e,\ x,\ y\}$ and with dominating pair (x, y). The sequence continues with the graph $G \setminus \{x\}$ with dominating pair (a, y), and so on. The spine of the graph G is featured in heavy lines.

Note that the existence of a sequence of vertices and a sequence of subgraphs, as defined in i) through iv) above, does not necessarily imply that the graph is AT-free. For example, let (x, y) be the dominating pair $(1, 4)$ of the graph G of Figure 5.4. The vertex sequence $1, 7$ and the subgraph sequence $G, G \setminus \{1\}$ satisfy i) - iv) above; nevertheless, G is not AT-free ($\{2, 4, 6\}$ is an AT). However, the Spine Theorem is not contradicted since the induced subgraph $G \setminus \{7\}$ has a dominating pair (1,4) yet $G \setminus \{1, 7\}$ has no dominating pair consisting of 4 and a neighbor of 1.

The second goal of this section is to give a characterization of AT-free graphs in terms of minimal triangulations. Let $G = (V, E)$ be an arbitrary graph. A *triangulation* $T(G)$ of G is a set of edges such that the graph $G' = (V, E \cup T(G))$ is chordal. A triangulation $T(G)$ is *minimal* when no proper subset of $T(G)$ is a triangulation of G. Recently, Möhring [20] proved the following result.

THEOREM 5.12. [20] *If G is an asteroidal triple-free graph, then for every minimal triangulation $T(G)$ of G, the graph $G' = (V, E \cup T(G))$ is an interval graph.*

The remainder of this section is devoted to proving the converse of Theorem 5.12. A different proof of the converse was obtained independently by A. Parra [23].

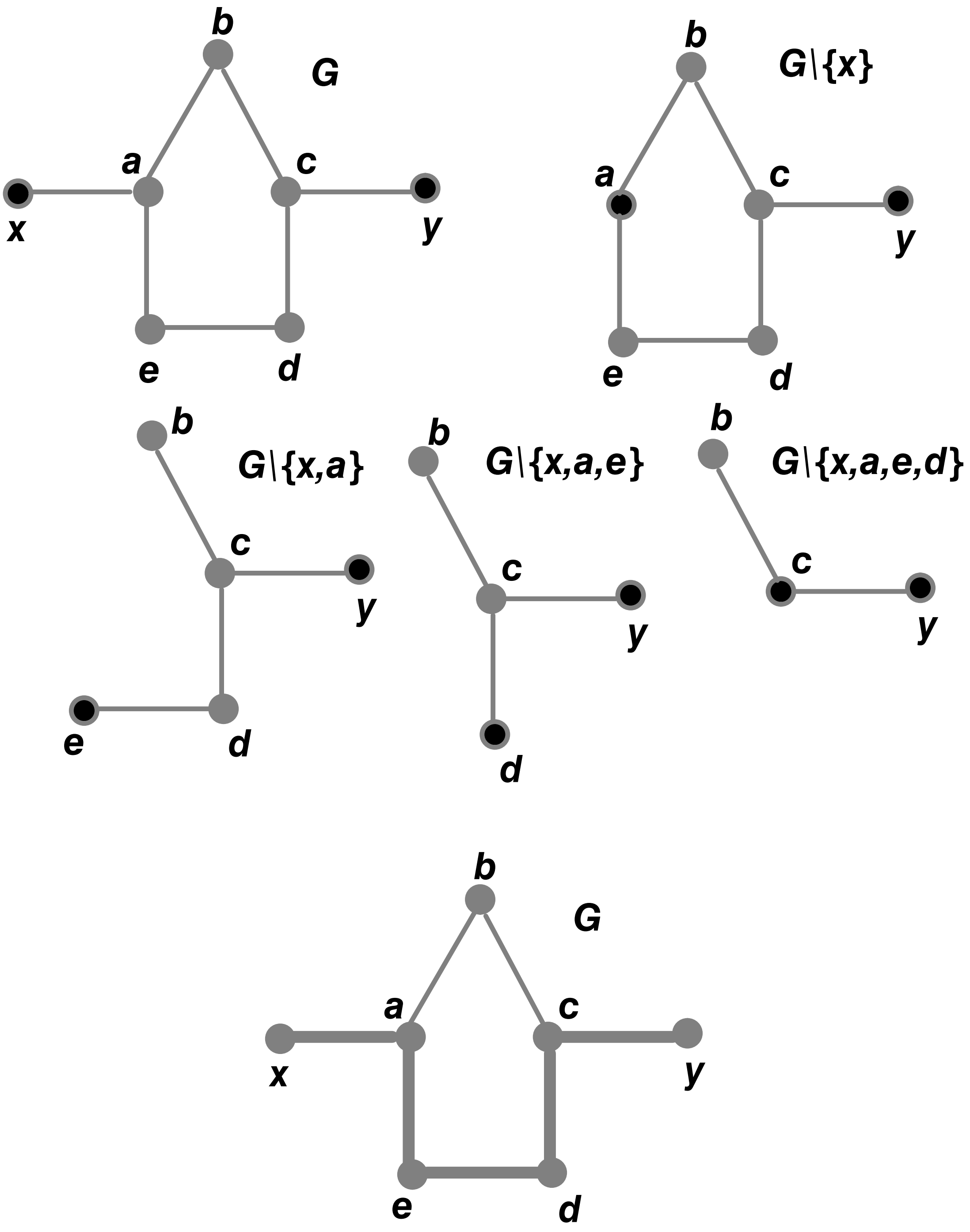

FIG. 5.3. *Illustrating the Spine Theorem*

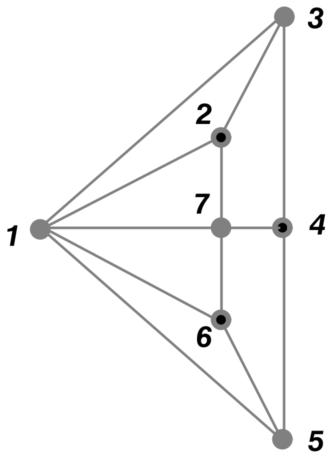

FIG. 5.4. *A graph G*

THEOREM 5.13. *A graph G is asteroidal triple-free if and only if, for every minimal triangulation $T(G)$ of G, the graph $G' = (V, E \cup T(G))$ is an interval graph.*

Our arguments rely, in part, on the following result which is of independent interest.

LEMMA 5.14. *Let G be an arbitrary graph and let $H = (V(H), E(H))$ be an induced subgraph of G. Let $T(H)$ be an arbitrary minimal triangulation of H. There exists a minimal triangulation $T(G)$ of G such that the only edges in $T(G)$ joining vertices in H are those in $T(H)$.*

Proof. If the statement is false, then we select a minimal triangulation $T(G)$ of G that adds as few new edges to H as possible. Since $T(H)$ is a triangulation of H, some edge uv with both u and v in H, present in $T(G)$ but not in $T(H)$, must be the unique chord of a set $\mathcal{C}$ of C_4's, each having (at least) one vertex outside H. Let w and w' be the remaining vertices of such a C_4 with w outside H. The removal from $T(G)$ of the edge uv and the addition of the ww' edge(s), will triangulate all C_4's in $\mathcal{C}$, but may create new cycles, each of which contains at least one vertex (such as w) that is not in H. Each such cycle will be triangulated by adding all possible chords incident with a particular vertex outside H. The addition of these edges may create new cycles that will be triangulated in a similar fashion. Since the graph is finite, we eventually have a triangulation $T'(G)$ that has one fewer H edges than $T(G)$. Any minimal triangulation in $T'(G)$ also has one fewer H edges than $T(G)$, thereby contradicting our choice of $T(G)$. □

Proof. (Theorem 5.13) The "only if" part follows from Theorem 5.12.

To prove the "if" part, let G be a graph containing an AT. Choose an induced subgraph $H = (V(H), E(H))$ of G with the least number of vertices such that some triple $\{x, y, z\}$ is an AT in H. Let $\pi(x, y)$, $\pi(x, z)$, and $\pi(y, z)$ be paths in H demonstrating that $\{x, y, z\}$ is an AT, and write $\pi(x, y) : x = u_1, u_2, \ldots, u_k = y$, $\pi(x, z) : x = v_1, v_2, \ldots, v_l = z$, and $\pi(z, y) : z = w_1, w_2, \ldots, w_t = y$. Clearly, the choice of H guarantees that x, y, and z have degree at most two.

Our plan is to exhibit a minimal triangulation $T(H)$ of H that results in a non-interval graph $H' = (V(H), E(H) \cup T(H))$. For this purpose, let x', y', and z' be the vertices specified in Lemma 5.4 and consider the triangulation $T(H)$ of H returned by the following procedure:

Step 1. If $x' = y' = z'$ then set $T(H) \leftarrow \emptyset$ and return;

Step 2. Let F be the graph obtained from H by removing vertices x, y, z and by adding the edges u_2v_2 (in case $x = x'$), $u_{k-1}w_{t-1}$ (in case $y = y'$), and $v_{l-1}w_2$ (in case $z = z'$). Let $T(F)$ be an arbitrary minimal triangulation of F. Return $T(H) \leftarrow T(F) \cup \{xu_2, xv_2, yu_{k-1}, yw_{t-1}, zv_{l-1}, zw_2\}$ (in case $x \neq x'$ one adds the edge xx' instead of the edges xu_2 and xv_2, etc).

Now Claim 5.5 along with an easy ad-hoc argument shows that $T(H)$ is a minimal triangulation of H and that $\{x, y, z\}$ is still an AT in the graph $H' = (V(H), E(H) \cup T(H))$. By Lemma 5.14, there must exist some minimal triangulation $T(G)$ of G such that H' is an induced subgraph of $G = (V, E \cup T(G))$. The conclusion follows. □

6. Augmenting AT-free Graphs. The purpose of this section is twofold. First, we exhibit a structural property of AT-free graphs that naturally allows one to "stretch" an AT-free graph to a new AT-free graph. This in turn provides a condition under which two AT-free graphs can be "glued together" to form a new AT-free graph (Corollary 6.10). Next, we provide a decomposition theorem for AT-free graphs.

To begin, we address the issue of creating new AT-free graphs out of old ones. Specifically, we show how to "augment" an arbitrary AT-free graph G to obtain a new AT-free graph. This augmentation will be accomplished by finding a particular dominating pair (x, y) and by adding new vertices x' and y' adjacent to x and y, respectively. This augmentation of G again confirms our intuition about the linear structure of AT-free graphs, since the dominating pair (x, y) has been stretched to a new dominating pair (x', y').

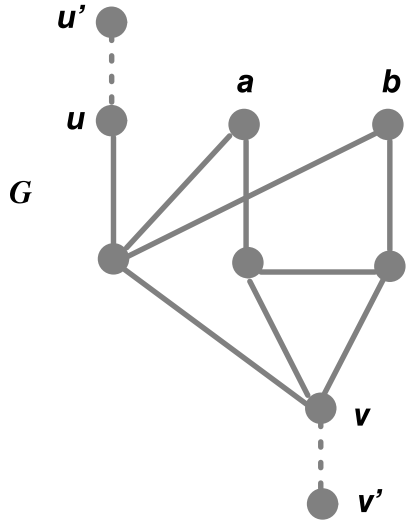

FIG. 6.1. *Illustrating pokable and unpokable vertices*

In preparation for stating the first main result of this section, we need to define a few terms. A vertex v of an AT-free graph G is called *pokable* if the graph G' obtained from G by adding a pendant vertex adjacent to v is AT-free; otherwise, it is called *unpokable*. For example, referring to Figure 6.1, vertex u is pokable since the addition of a pendant vertex u' does not create an AT in the graph. At the same time, vertex v is unpokable, for the addition of the vertex v' creates the AT $\{a, b, v'\}$. A dominating pair (x, y) is referred to as *pokable* if both x and y are pokable. For further reference, we take note of the following simple observation whose proof is routine.

OBSERVATION 6.1. *A vertex v of an asteroidal triple-free graph G is unpokable if and only if there exist vertices u and w in G such that u and w are unrelated with respect to v and there is a u, w-path in G that does not contain v.*

Whenever we have a vertex v for which there exist vertices u and w unrelated with respect to v, we shall refer to the following induced paths, which must exist by the definition of unrelated vertices:

a v,u-path $v = u_0, u_1, \ldots, u_p = u$ missed by w, and a v,w-path $v = w_0, w_1, \ldots, w_q = w$ missed by u. We are now in a position to make the previous discussion precise.

THEOREM 6.2. *Every connected asteroidal triple-free graph contains a pokable dominating pair; furthermore, every connected asteroidal triple-free graph which is not a clique contains a non-adjacent pokable dominating pair.*

Proof. The theorem is trivial for cliques. We shall, therefore, assume that G is not a clique. Now, Claim 4.1 guarantees the existence of a non-adjacent dominating pair (x, y_0) in G. Let F be the connected component of $N'(x)$ containing y_0, and let Y stand for the set of vertices y in F for which (x,y) is a dominating pair in G. The conclusion of Theorem 6.2 is implied by the following technical result that will be proved later.

LEMMA 6.3. *Y contains a vertex y such that G has no unrelated vertices with respect to y.*

Let us examine how Theorem 6.2 follows from Lemma 6.3. Note that Lemma 6.3, together with Observation 6.1, implies that Y contains a pokable vertex. Let β be a pokable vertex in Y and let X denote the set of vertices x' in the same component of $N'(\beta)$ as x, for which (β, x') is a dominating pair. Clearly x belongs to X, and so X is not empty. By applying Lemma 6.3 again, with β as the "anchor", we find a pokable vertex α in X. The proof of Theorem 6.2 is established by noting that (α, β) is the desired non-adjacent pokable dominating pair. □

Proof. (Lemma 6.3) The proof is by induction on the number of vertices in G. Assume that the lemma is true for all connected AT-free graphs with fewer vertices than G. We now present various facts that are used in the proof.

CLAIM 6.4. *Let v be a vertex in Y such that vertices u and w are unrelated with respect to v in G. Then, all vertices u_i and w_j $(1 \leq i \leq p; 1 \leq j \leq q)$, belong to F.*

Proof. Without loss of generality let i be the smallest subscript for which u_i lies outside F. Trivially, u_i must belong to $N(x)$. Since w cannot miss the v,x-path, $v = u_0, u_1, \ldots, u_i, x$ and since w is adjacent to no vertex on the path $v = u_0, u_1, \ldots, u_i$, it follows that w belongs to $N(x)$.

Similarly, since u cannot miss the v,x-path, $v = w_0, w_1, \ldots, w_q = w, x$ and since u is adjacent to no vertex on the path $v = w_0, w_1, \ldots, w_q$, it follows that u belongs to $N(x)$. But now, $\{u, v, w\}$ is an AT, contradicting G being AT-free. □

It is important to note that, by virtue of Claim 6.4, Lemma 6.3 is established as soon as we exhibit a vertex y in Y such that there are no unrelated vertices with respect to y in the subgraph of G induced by F. If F and Y coincide, then by the induction hypothesis such a vertex must exist. Therefore, from now on, we shall assume that

$$F \setminus Y \neq \emptyset. \tag{6.1}$$

Let $Y_1, Y_2, \ldots, Y_k$ $(k \geq 1)$ be the connected components of the subgraph of $\overline{G}$ induced by Y.

CLAIM 6.5. *Let t be a vertex in $F \setminus Y$. If some vertex z in Y_i satisfies $z \in D(t,x)$, then $Y_i \subset D(t,x)$.*

Proof. If the claim is false, then we find vertices z, z' in Y_i such that $z \in D(t,x)$ and $z' \notin D(t,x)$. Since Y_i is a connected subgraph of $\overline{G}$, there exists a chordless path $z = s_1, s_2, \ldots, s_r = z'$ joining z and z' in $\overline{G}$, with all internal vertices in Y_i.

Let j be the smallest subscript for which $s_j \notin D(t,x)$. Since $z' \notin D(t,x)$, such a subscript must exist. But now, in G, s_{j-1} and s_j are non-adjacent and s_j misses some t,x-path, while s_{j-1} intercepts all such paths. It follows that s_j misses a s_{j-1},x-path, a contradiction, since s_{j-1} belongs to Y. □

CLAIM 6.6. *Y induces a disconnected subgraph of $\overline{G}$.*

Proof. First, we claim that

$$|\, Y \,| \geq 2. \tag{6.2}$$

If (6.2) is false, then $Y = \{y_0\}$. Let U stand for the set of all vertices in F adjacent to y_0. Note that (6.1), along with the connectedness of F, guarantee that U is nonempty. But now, for every u in U, $Y = \{y_0\} \subset D(u, x)$. Thus, u is an attractor, contradicting Claim 4.2. Therefore, (6.2) holds. Note that by virtue of (6.2) it makes sense to talk about Y being disconnected in the complement.

We now continue the proof of Claim 6.6. If $Y = Y_1$, then (6.1) and the connectedness of F imply the existence of a vertex z in Y adjacent to some vertex t in $F \backslash Y$. Note, in particular, that z belongs to $D(t, x)$ and so, by Claim 6.5, $Y \subset D(t, x)$. However, now t is an attractor, a contradiction. With this, the proof of Claim 6.6 is complete. □

CLAIM 6.7. *Let v be a vertex in Y such that vertices u and w are unrelated with respect to v in G. Then,*
- *for all i, $(1 \leq i \leq p)$, v belongs to $D(u_i, x)$, and*
- *for all j, $(1 \leq j \leq q)$, v belongs to $D(w_j, x)$.*

Proof. Since v is adjacent to u_1, it follows that $v \in D(u_1, x)$. Let i be the smallest subscript for which v does not belong to $D(u_i, x)$. Let π be a u_i, x-path missed by v. Note that w must intercept π, for otherwise w would miss a v, x-path contained in $\{v, u_1, \ldots, u_i\} \cup \pi$. However, now $\{u, v, w\}$ is an AT. The proof that v belongs to $D(w_j, x)$ follows by a mirror argument. □

For every i, $(1 \leq i \leq k)$, let T_i stand for the set of vertices t in $F \setminus Y$ with the property that $Y_i \subset D(t, x)$. By renaming the Y_i's, if necessary, we ensure that

$$| T_1 | \leq | T_2 | \leq \ldots \leq | T_k | .$$

CLAIM 6.8. *Every vertex in T_1 is adjacent to all vertices in Y_1.*

Proof. The statement is vacuously true if T_1 is empty. Now assume that T_1 is nonempty and let t be a vertex in T_1 non-adjacent to some z in Y_1. Since, by Claim 4.2, t cannot be an attractor, we find a subscript j, $(j \geq 2)$, such that for some z' in Y_j, z' does not belong to $D(t, x)$. Thus $t \in T_1 \setminus T_j$. Now, $|T_1| \leq |T_j|$ implies that there must exist a vertex t' in $T_j \setminus T_1$. By Claim 6.5, z does not belong to $D(t', x)$. Note that t does not belong to $D(t', x)$, otherwise, by Claim 3.3, z would belong to $D(t', x)$, a contradiction.

Since z' does not belong to $D(t, x)$, in particular, z' is not adjacent to t. The fact that t does not belong to $D(t', x)$ implies the existence of a t', x-path π' missed by t. Since $z' \in D(t', x)$, z' intercepts π' and thus $\pi' \cup \{z'\}$ contains a z', x-path missed by t, contradicting that z' is in Y. □

We now continue the proof of Lemma 6.3. Let Z be a connected component of the subgraph of G induced by Y_1. By the induction hypothesis, Z contains a vertex v, such that Z has no unrelated vertices with respect to v. To complete the proof of Lemma 6.3, we only need show that F has no unrelated vertices with respect to v. Suppose u and w in F are unrelated with respect to v. By Claims 6.5 and 6.7, combined, all the vertices u_i and w_j $(1 \leq i \leq p;\ 1 \leq j \leq q)$ belong to Y or to T_1. By Claims 6.6 and 6.8, and the fact that the paths, $v = u_0, u_1, \ldots, u_p = u$ and $v = w_0, w_1, \ldots, w_q = w$, are chordless, it follows that at most u_1 and w_1 belong to $T_1 \cup Y \backslash Y_1$. However, if either u_1 or w_1 is in $T_1 \cup Y \backslash Y_1$ then, by Claims 6.6 and 6.8, the edge $u_1 w$ or the edge $w_1 u$ must be present, contradicting the fact that u and w are unrelated with respect to v. Thus, all the u_i's and w_j's belong to Y_1. In fact, since Z is a connected component of Y_1, all the u_i's and w_j's must belong to Z, a contradiction. This completes the proof of Lemma 6.3. □

Theorem 6.2 implies the following results that are interesting in their own right.

COROLLARY 6.9. *Every asteroidal triple-free graph is either a clique or contains two non-adjacent pokable vertices.*

COROLLARY 6.10. (THE COMPOSITION THEOREM) *Given two asteroidal triple-free graphs G_1 and G_2, and pokable dominating pairs (x_1, y_1) and (x_2, y_2) in G_1 and G_2, respectively, let G be the graph constructed from G_1 and G_2 by identifying vertices x_1 and x_2. Then, G is an asteroidal triple-free graph.*

The reader is referred to Figure 6.2 for an illustration of the Composition Theorem.

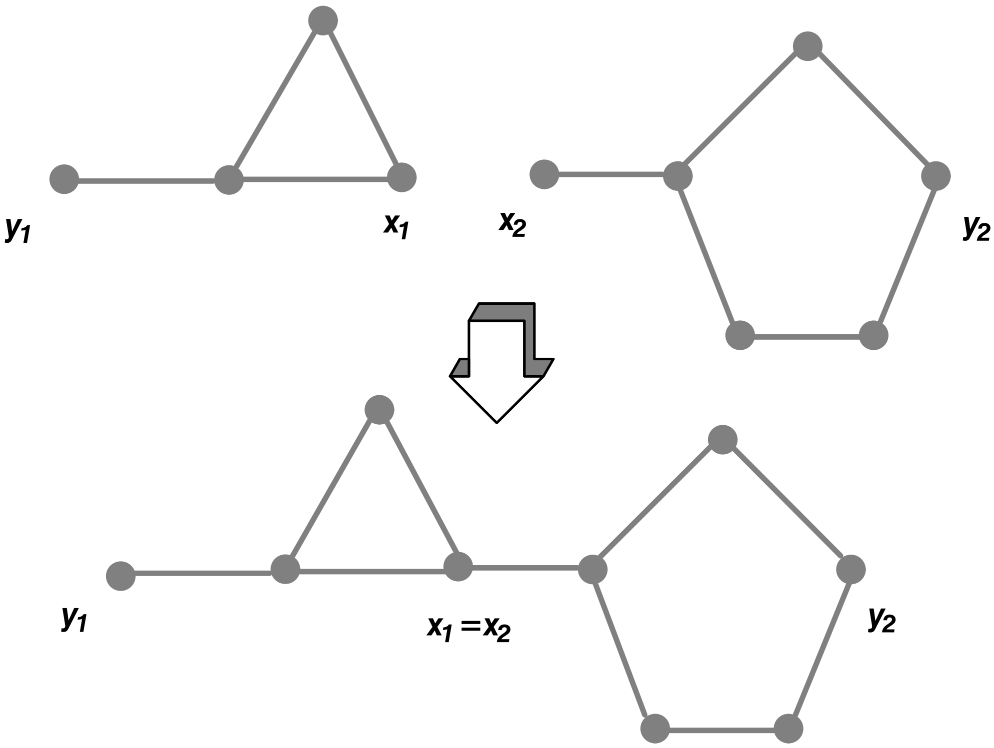

FIG. 6.2. *Illustrating the Composition Theorem*

We now show that the existence of a pokable dominating pair in a connected AT-free graph leads to a natural decomposition scheme. In preparation for stating the second main result of this section, we first give a necessary and sufficient condition for a vertex in a dominating pair to be pokable. Specifically, we have the following result.

CLAIM 6.11. *Let G be a connected asteroidal triple-free graph with a dominating pair (x, y). Then x is pokable if and only if there are no unrelated vertices with respect to x.*

Proof. The "if" part is easily seen. To prove the "only if" part, consider unrelated vertices u and v with respect to x. In particular, we find a v, x-path missed by u and a u, x-path missed by v. Since (x, y) is a dominating pair, u and v intercept every path joining x and y. Let π be such a path and let u' and v' be vertices on π adjacent to u and v, respectively. Trivially both u' and v' are distinct from x. But now, there exists a u, v-path in G that does not contain x (this path contains vertices u', v' and a subpath of π), implying that x is not pokable. ◻

Let $G = (V, E)$ be a connected AT-free graph with at least two vertices and let (x, y) be a pokable dominating pair in G. Define a binary relation R on G by writing for every pair u, v of vertices:

$$u \; R \; v \Longleftrightarrow D(u, x) = D(v, x). \tag{6.3}$$

Clearly, R is an equivalence relation; let $C_1, C_2, \ldots, C_k$ $(k \geq 1)$, be the equivalence classes of G/R. A class C_i is termed *non-trivial* if $\mid C_i \mid \geq 2$. The existence of non-trivial equivalence classes with respect to R is not immediately obvious. In what follows, we assume that the pokable dominating pair (x, y) is chosen to be non-adjacent whenever possible. The following result guarantees that non-trivial equivalence classes always exist.

CLAIM 6.12. *G/R contains at least one non-trivial equivalence class.*

Proof. If $N'(x)$ is empty then the class containing y, $C(y)$, is equal to V and is therefore non-trivial. Otherwise, Theorem 6.2 and our choice of x and y, combined, guarantee that x and y are non-adjacent. Let F be the connected component of $N'(x)$ containing y and let Y stand for the subset of F consisting of all the vertices that are in a dominating pair with x. Clearly, $y \in Y$, and so Y is nonempty. If F contains at least two vertices then (6.2) guarantees that Y itself contains at least two vertices and so the equivalence class containing y is non-trivial.

We may, therefore, assume that $F = \{y\}$. Let y' be an arbitrary neighbor of y in $N(x)$. Clearly, $D(y', x) = V$, for otherwise if some vertex z does not belong to $D(y', x)$, then z must miss the y, x-path consisting of y, y', and x. Consequently, the equivalence class containing y is non-trivial and the proof of Claim 6.12 is complete. □

REMARK: In fact, the proof of Claim 6.12 also tells us that the class $C(y)$ containing y is always non-trivial as long as the original graph has at least two vertices.

A non-trivial class C of G/R is said to be *valid* if C induces a connected subgraph of G. As before, the existence of valid equivalence classes is not immediately obvious. As we shall prove next, such classes always exist. Specifically, we propose to show that $C(y)$ is valid. As it will turn out, *all* valid classes of G/R enjoy very interesting properties that will allow us to select an arbitrary one for the purpose of decomposing the original graph. This freedom of choice opens the door to parallel decomposition algorithms for AT-free graphs.

CLAIM 6.13. *G/R contains at least one valid equivalence class.*

Proof. If $N'(x)$ is empty, then $C(y) = V$ and there is nothing to prove. We may therefore assume that $N'(x)$ is nonempty. As before, we may also assume that y belongs to $N'(x)$. Let F be the connected subgraph of $N'(x)$ containing y, let Y stand for the subset of F consisting of all the vertices that are in a dominating pair with x, and let $C(y)$ be the equivalence class containing y.

Notice that every vertex w that belongs to $N(x)$ and to $C(y)$ must be adjacent to all the vertices in F. In particular, if such a vertex exists, then $C(y)$ which by Claim 6.12 is non-trivial, must be connected and, thus, valid.

We will assume, therefore, that $N(x)$ and $C(y)$ are disjoint. In turn, this implies that $C(y) = Y$. Recall that, by Claim 6.6, Y induces a disconnected subgraph of $\overline{G}$, confirming that $C(y)$ is connected as a subgraph of G. The conclusion follows. □

Let S be a set of vertices of G. The graph G' is said to arise from G by an *S-contraction* if G' contains all the vertices in $G \setminus S$ along with a new vertex s adjacent, in G', to all the vertices in $G \setminus S$ that were adjacent, in G, to some vertex in S. Our next result states a fundamental property of valid equivalence classes, namely that contracting any of them will result in an AT-free graph. The details are spelled out as follows.

LEMMA 6.14. *Let C be an arbitrary valid equivalence class of G/R. The graph G' obtained from G by a C-contraction is asteroidal triple-free.*

Proof. Let c be the vertex in G' obtained by contracting C. To begin, we claim that

$$\text{(6.4)} \qquad \text{there are no vertices } u,\ v \text{ in } G' \text{ such that } \{u, v, c\} \text{ is an AT.}$$

To justify (6.4) note that if $\pi(u, v)$ is a u, v-path missed by c, then the same path is missed, in G, by all the vertices in C. Let $\pi(u, c)$ be a u, c-path in G' missed by v. Then, there exists a vertex c_1 in C such that v misses the path $\pi(u, c) - c + c_1$. Similarly, let $\pi(v, c)$ be a v, c-path in G' missed by u. There must exist a vertex c_2 in C such that u misses the path $\pi(v, c) - c + c_2$. Since C induces a connected subgraph of G, there exists a path joining c_1 and c_2 all of whose internal vertices are in

C. By a previous observation, both u and v miss this path. Therefore, for a suitably chosen vertex c' in C, $\{u, v, c'\}$ is an AT in G, a contradiction. Thus, (6.4) must hold.

To complete the proof of Lemma 6.14, let $\{u, v, w\}$ be an arbitrary AT in G'. By (6.4), c is distinct from u, v, w. Let $\pi(u, v)$, $\pi(u, w)$, and $\pi(v, w)$ be paths in G' confirming that $\{u, v, w\}$ is an AT. If c belongs to none of these paths, then $\{u, v, w\}$ is an AT in G. We may therefore assume without loss of generality that c belongs to $\pi(u, v)$. Since w misses $\pi(u, v)$, it is clear that w is adjacent to no vertex in C.

We claim that there exists a path $\pi'(u, v)$ in G missed by w. This path contains the same vertices as $\pi(u, v)$ outside of C. Inside C it contains a path between two vertices c' and c'' of C such that:

- w misses a u, c'-path consisting of a subpath of $\pi(u, v)$;
- w misses a c'', v-path consisting of the remaining vertices in $\pi(u, v) - c$.

This completes the proof of Lemma 6.14. $\square$

The example in Figures 6.3 and 6.4 shows that the connectivity of the equivalence class C in Lemma 6.14 is required, if we are to guarantee that the resulting graph is AT-free. To wit, the graph G featured in Figure 6.3 is AT-free with a pokable dominating pair (x, e). The contraction of the equivalence class $\{a, b\}$ yields the graph G' in Figure 6.4, which has the AT $\{a', b', w'\}$. For the reader's benefit, the various values of the $D(*, x)$ sets, along with the equivalence classes corresponding to the graph in Figure 6.3 are summarized in Table 1 below.

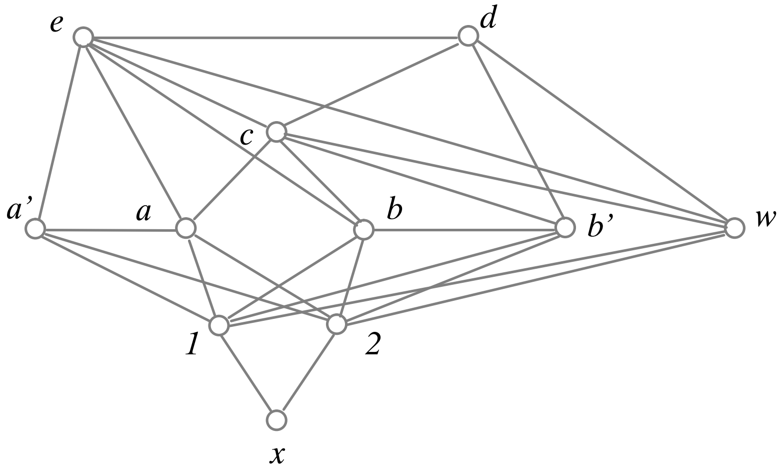

FIG. 6.3. *A graph G*

Let $C(y)$ be the equivalence class containing y. Let G' be the graph obtained from G by a $C(y)$-contraction. Recall that the proof of Claim 6.13 guarantees that $C(y)$ is valid and so, Lemma 6.14 asserts that the graph G' is also AT-free. Let y' be the vertex of G' obtained by contracting $C(y)$. We now show that, in fact, more can be said about G'. Specifically, we have the following result.

LEMMA 6.15. *(x, y') is a pokable dominating pair in G'.*

Proof. To begin, we establish that (x, y') is a dominating pair in G'. For this purpose, suppose that there exists some path $\pi(x, y')$ joining x and y', missed by a vertex w. Clearly, w is adjacent, in G, to no vertex in $C(y)$. In particular, w is not adjacent to y. Since $C(y)$ is valid, w misses, in G, a y, x-path consisting of all the vertices in $\pi(x, y')$, along with a suitable path in $C(y)$. Therefore, (x, y') must be a dominating pair in G'.

Equivalence Class	$D(*, x)$
x	$\{x, 1, 2\}$
1,2	$V \setminus \{c, d, e\}$
a'	$V\setminus\{c, d\}$
a, b:	$V\setminus\{d\}$
b'	$V\setminus\{e\}$
w, c, d, e	V

TABLE 6.1
Illustrating the various equivalence classes

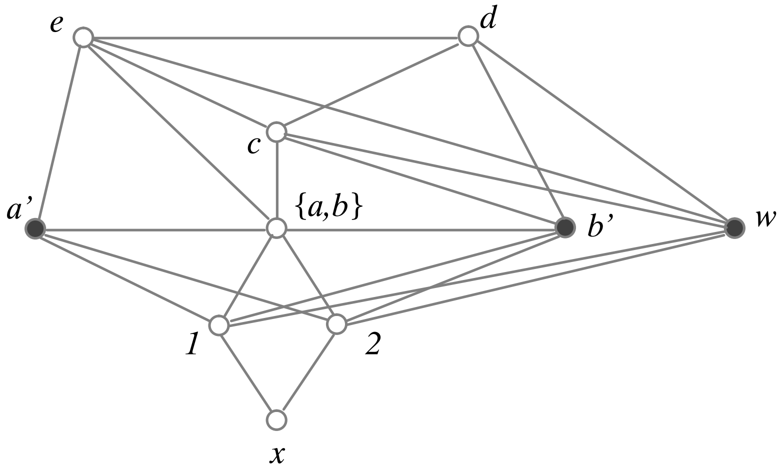

FIG. 6.4. *The graph G' obtained by contracting $\{a, b\}$*

Next, we show that both x and y' are pokable vertices of G'. Suppose that x is not pokable. Now Claim 6.11 guarantees the existence of unrelated vertices u and v (with respect to x). This, in turn, implies the existence of paths $\pi(v, x)$ and $\pi(u, x)$ in G', missed by u and v, respectively. Since (x, y') is a dominating pair in G', y' belongs to neither of these paths. But now, these paths must have been paths in G, a contradiction.

Finally, suppose that y' is not pokable. By virtue of Claim 6.11 this implies the existence of vertices u and v and of paths $\pi(v, y')$ and $\pi(u, y')$ in G', missed by u and v, respectively. In particular, neither u nor v is adjacent to y'. In turn, this implies that neither u nor v is adjacent to a vertex in $C(y)$. But now, in G, there exists a u, y-path missed by v and a v, y-path missed by u, contradicting that y is pokable. This completes the proof of Lemma 6.15. $\square$

At this stage, the reader may wonder whether the class $C(y)$ is the only one whose contraction leaves x pokable. The answer is provided by the following result that complements Lemma 6.15.

LEMMA 6.16. *Let C be an arbitrary valid equivalence class in an asteroidal triple-free graph G, and let G' be the graph obtained from G by a C-contraction. If C is distinct from $C(x)$ and $C(y)$, then (x, y) is a pokable dominating pair in G'.*

Proof. Let c be the vertex of G' obtained by the C-contraction. By assumption, c is distinct from y and from x. We begin by showing that (x, y) is a dominating pair in G'. Suppose that there exists some path $\pi(x, y)$ joining x and y in G', missed by a vertex w. Clearly, c must belong to $\pi(x, y)$. Notice that w is adjacent, in G, to no vertex in C. Since C is valid, w misses, in G, a

y, x-path consisting of all the vertices in $\pi(x, y) - c$, along with a suitable path in C. Thus, (x, y) must be a dominating pair in G'.

Next, we show that both x and y are pokable vertices of G'. If x is not pokable, Claim 6.11 guarantees the existence of vertices u and v unrelated with respect to x. In turn, this implies the existence of paths $\pi(v, x)$ and $\pi(u, x)$ in G', missed by u and v, respectively. Since x is pokable in G, c must belong to (at least) one of these paths. Symmetry allows us to assume, with no loss of generality, that c belongs to $\pi(u, x)$. The fact that v misses $\pi(u, x)$ guarantees that v is adjacent, in G, to no vertex in C. But now, we have reached a contradiction: v misses a u, x-path in G consisting of all the vertices of $\pi(u, x)$ outside C along with a suitably chosen path in C. Thus, x must be pokable in G'.

A perfectly similar argument, whose details are omitted, asserts that y is also pokable. With this, the proof of Lemma 6.16 is complete. $\square$

Lemmas 6.15 and 6.16, combined, set the stage for a decomposition theorem for AT-free graphs. Consider an AT-free graph $G = (V, E)$ and let (x, y_0) be a pokable dominating pair in G. Let $G_0, G_1, \ldots, G_k$ be a sequence of graphs defined as follows:

i) $G_0 = G$;

ii) For all i, $(0 \leq i \leq k-1)$, let R_i be the equivalence relation defined on G_i by setting $uR_iv \iff D(u, x) = D(v, x)$, and let C be an arbitrary valid equivalence class of G_i/R_i. Let G_{i+1} be the graph obtained from G_i by a C-contraction; (i.e. G_{i+1} contains all the vertices in $G_i \backslash C$ as well as a new vertex c which is adjacent to all vertices in $G_i \backslash C$ that were adjacent to at least one vertex in C);

iii) G_k consists of a single vertex.

Such a sequence $G_0, G_1, \ldots, G_k$ is called *involutive*. The reader is referred to Figure 6.5 featuring the first five graphs in an involutive sequence of the given graph. Note that in the transition from G_2 to G_3 in Figure 6.5 two equivalence classes could be contracted, namely $\{a, b\}$ and $\{d, efgh\}$. We have selected to contract the class $C = \{a, b\}$.

The obvious question is whether every connected AT-free graph has such an involutive sequence. This fundamental question is answered in the affirmative in the following theorem.

THEOREM 6.17. *Every connected asteroidal triple-free graph G has an involutive sequence.*

Proof. We shall assume that G is not a clique, since otherwise there is nothing to prove. By Theorem 6.2, we find a non-adjacent pokable dominating pair (x, y_0) in G. Consider the transition from G_i to G_{i+1} for some i $(0 \leq i \leq k-1)$. Let C be an arbitrary valid equivalence class in G_i/R_i, and let (x, y_i) be a pokable dominating pair in G_i. Define y_{i+1} to be y_i in case C is distinct from $C(y_i)$ and to be the vertex obtained by contracting $C(y_i)$, otherwise. Clearly, G_{i+1} is connected whenever G_i is. By Lemmas 6.14, 6.15, and 6.16, combined, G_{i+1} is AT-free and (x, y_{i+1}) is a pokable dominating pair in G_{i+1}. This completes the proof of Theorem 6.17. $\square$

We close with the obvious question: can such an involutive sequence be constructed efficiently?

7. Dominating Pairs in High Diameter AT-free Graphs. The purpose of this section is to show that, in a connected AT-free graph with diameter larger than three, the set of vertices that can be in dominating pairs is restricted to two disjoint sets. Specifically, we have the following result.

THEOREM 7.1. *Let G be a connected asteroidal triple-free graph with diameter at least four. There exist nonempty, disjoint sets X and Y of vertices of G such that (x, y) is a dominating pair if and only if $x \in X$ and $y \in Y$.*

We note that Theorem 7.1 is best possible in the sense that for AT-free graphs of diameter less than four, the sets X and Y are not guaranteed to exist. To wit, C_5 and the graph of Figure 7.1

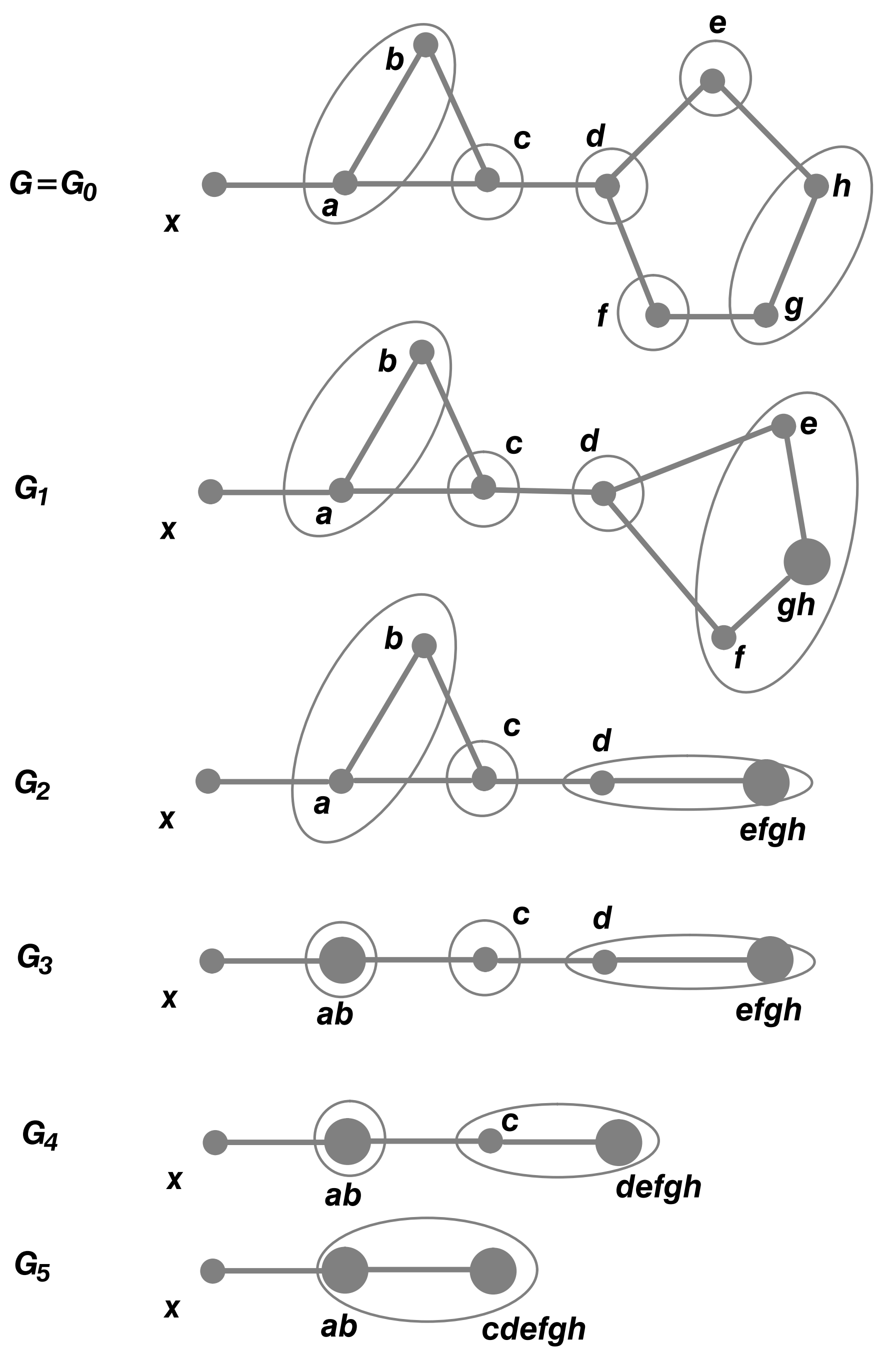

FIG. 6.5. *Illustration of an involutive sequence*

provide counterexamples of diameter two and three, respectively.

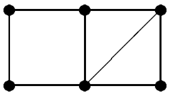

FIG. 7.1. *An AT-free graph of diameter three*

Proof. Let (x_0, y_0) be a dominating pair in G achieving the diameter. (The existence of such a pair follows from Theorem 4.3.) Let Y stand for the set of all the vertices y in G such that (x_0, y) is a dominating pair, and let X be the set of all the vertices x in G for which (x, y_0) is a dominating pair. We propose to show that X and Y are the sets with the property specified in Theorem 7.1. Our proof relies on a number of intermediate results that we present next.
To begin, we note that

(7.1) $$x_0 \in X \text{ and } y_0 \in Y.$$

In addition, by Claim 6.6,

(7.2) $$\text{both } X \text{ and } Y \text{ are disconnected in } \overline{G}.$$

Our choice of x_0 and y_0 guarantees that

(7.3) $$x_0 \text{ (respectively } y_0\text{) is adjacent to no vertices in } Y \text{ (respectively } X\text{).}$$

Otherwise, (7.1) and (7.2) would imply that $d(x_0, y_0) \leq 3$.

Note that (7.2) and (7.3) combined guarantee that

(7.4) $$X \text{ and } Y \text{are disjoint.}$$

The following argument justifies (7.4). If $z \in X \cap Y$ then, in particular, $z \in X$ and so (z, y_0) is a dominating pair. By (7.2), there exists a z, y_0-path contained in Y. By (7.3), x_0 misses this path, contradicting the fact that (z, y_0) is a dominating pair.

Let x and y be arbitrary vertices in X and Y, respectively. We claim that

(7.5) $$(x, y) \text{ is a dominating pair.}$$

To justify (7.5), suppose that some vertex u misses an x, y-path π. Observe that (7.2) guarantees the existence of an x_0, y-path contained in $\pi \cup X$. Since (x_0, y) is a dominating pair, this path is dominating. By (7.3), y_0 must be adjacent to a vertex of $\pi \setminus \{x\}$. Thus, $\pi \cup \{y_0\}$ contains an x, y_0-path. This path must be dominating and so u must be adjacent to y_0. A perfectly similar argument shows that u is adjacent to x_0, contradicting that x_0 and y_0 achieve the diameter.

Next, let x be an arbitrary vertex in X. We claim that

(7.6) $$\text{if } (x, z) \text{ is a dominating pair then } z \in Y.$$

Trivially, $z \notin X$; since $\text{diam}(G) \geq 4$, x and z are not adjacent. If $z \notin Y$, there exists an x_0, z-path π missed by some vertex u. Note that $\pi \cup X$ contains an x, z-path. Since, by assumption, (x, z) is a dominating pair, this path is dominating and so y_0 must intercept it. By (7.3) y_0 intercepts $\pi \setminus \{x_0\}$. Since (x_0, y_0) is a dominating pair it follows that u is adjacent to y_0. Trivially, u is not adjacent to x, otherwise the path y_0, u, x which is dominating implies that x and x_0 are adjacent and so $d(x_0, y_0) \leq 3$. Further, u and x being non-adjacent guarantees that x and x_0 are also non-adjacent, else u misses the x, z-path contained in $\pi \cup \{x\}$. Now, (7.2) guarantees that some x' in X is adjacent to both x_0 and x. Since (x, z) is a dominating pair, u must be adjacent to x'. However, this implies that $d(x_0, y_0) \leq 3$, a contradiction.

Let y be an arbitrary vertex in Y. As above, we can prove that

(7.7) $$\text{if } (y, z) \text{ is a dominating pair then } z \in X.$$

Note that by virtue of (7.4), (7.5), (7.6), and (7.7), to complete the proof of Theorem 7.1 we only need to prove that if (v, w) is a dominating pair then $v \in X$ and $w \in Y$ (or $v \in Y$ and $w \in X$). Suppose not.

By (7.5), (7.6), and (7.7) it must be that $v \notin X \cup Y$ and $w \notin X \cup Y$. Let F be the component of $N'(x_0)$ that contains Y. (Observe that $\text{diam}(G) \geq 4$ guarantees that Y is restricted to a unique component of $N'(x_0)$.) We claim that

(7.8) $$v \text{ or } w \text{ belongs to } F.$$

To justify (7.8), consider a shortest v, w-path in G. By assumption, this path is dominating and so both x_0 and y_0 must intercept it. Assume, without loss of generality, that y_0 intercepts the path "closer" to w than x_0, at a vertex t. Trivially, x_0 is adjacent to no vertex on this path from t to w, and the conclusion follows.

Let H be the component of $N'(y_0)$ that contains X. By virtue of (7.8) we may assume, without loss of generality, that $w \in F$ and that $v \in H$. Now, observe that y_0 can miss no w, x_0-path since such a path extends inside H to a w, v-path missed by y_0. Similarly, no vertex $y \in Y$ non-adjacent to y_0 can miss a w, x_0-path; otherwise, y would miss a y_0, x_0-path, a contradiction. Let $y \in Y$ be a vertex that misses some w, x_0-path π. By the previous argument, y and y_0 are adjacent. However, since (w, v) is a dominating pair, y must intercept every w, v-path contained in $\pi \cup H$, implying that y is adjacent to some neighbor x' of x_0. But now we have reached a contradiction: x_0 and y_0 are joined by a path of length three.

With this the proof of Theorem 7.1 is complete. $\square$

8. Concluding Remarks and Open Problems. Many families of graphs including interval graphs, permutation graphs, trapezoid graphs, and cocomparability graphs demonstrate a type of linear ordering on their vertex sets. It is precisely this linear order that is exploited, in one form or another, in a search for efficient algorithms for these classes of graphs. The classes mentioned are known to have wide-ranging practical applications. In addition, they are all subfamilies of the class of graphs called *asteroidal triple-free graphs* (AT-free, for short).

This work is the first attempt[2], known to us, at investigating structural properties of the AT-free graphs. In this direction our contributions are as follows:

1. We showed that every connected AT-free graph has a dominating pair, that is, a pair of vertices such that every path joining them is a dominating set;
2. We provided properties of dominating pairs in AT-free graphs related to the concept of connected domination and diameter;
3. We provided a characterization of AT-free graphs in terms of dominating pairs;
4. We provided a characterization of AT-free graphs in terms of minimal triangulations;
5. We provided a decomposition theorem for AT-free graphs.

The authors have also addressed some algorithmic questions with respect to asteroidal triple-free graphs. Specifically, in [9], $O(|V| + |E|)$ time algorithms are given for finding a pokable dominating

[2] A preliminary version of this work has appeared in [8]

pair in a connected AT-free graph $G = (V, E)$, and for finding all dominating pairs in a connected AT-free graph $G = (V, E)$ of diameter greater than three. Included in the latter algorithm is an efficient procedure for computing all of the "D" sets, with respect to a particular pokable dominating pair vertex. An extended abstract of [9] can be found in [11]. Some preliminary results and an alternative approach to the dominating pair problem can be found in [10] and [12], respectively.

Many other questions are still open. For example, it is well known [17] that cocomparability graphs have a linear ordering; this ordering exemplifies the linear structure we observe in interval graphs, permutation graphs and trapezoid graphs. It would be interesting to see whether the AT-free graphs also possess some linear ordering. Such an ordering could, conceivably, be exploited for algorithmic purposes.

A further natural question to ask is: "What are the roles of dominating pairs and pokable vertices in the subfamilies of AT-free graphs?" It is clear that the extreme vertices of any intersection representation, for a connected graph in any of the subfamilies, form a dominating pair. Some additional partial answers to this question have been given, in a slightly different setting, in [21] and [22]. Investigating further properties of dominating pairs and pokability in each of these particular families promises to be a fruitful area for further research.

Recently Möhring [20] has added to the understanding of the linear structure of AT-free graphs by showing that the pathwidth of an AT-free graph equals its treewidth.

Just as there are many families of perfect AT-free graphs, one would expect to see a rich hierarchy of families of non-perfect AT-free graphs. So far nothing is known here. Since perfect AT-free graphs strictly contain cocomparability graphs, it would be interesting to study the perfect AT-free graphs.

The fastest recognition algorithm known to us runs in $O(n^3)$ time with an n-vertex graph as input. It is a tantalizing open problem to produce a recognition algorithm that is more efficient, perhaps, even optimal.

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