CS 281
Prof. Alex Pothen

Webbook:
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Lecture 20
6.1, 6.2 Relations
6.4 Operations on Relations
Next Lecture: 6.5 Equivalence Relations
6.6 Partial Orders
Cartesian Product

• Given two sets $S$, $T$, the Cartesian product $S \times T$ is the set of all pairs $(s, t)$, where $s \in S$ and $t \in T$.

\[ S \times T = \{(s, t) : s \in S, t \in T\} \]

E.g., $S$ is a set of students, $T$ is a set of courses. $S \times T$ represents the set of all possible course enrollments.

• A binary relation $R$ is a subset of the pairs in $S \times T$. If $(s, t) \in R$, we write $sRt$: $s$ is related to $t$.

• Let $A = \{1, 2\}$, and $B = \{1, 2, 3\}$. Then a relation $R$ is

\[ R = \{(1, 1), (1, 2), (2, 3)\} \]

• Represent relations pictorially by (bipartite) graphs, or tables.
Binary Relations

- A relation from a set $A$ to $A$ is called a relation on $A$.

- A directed graph representation: each element of $A$ is a vertex; an edge $(a, b)$ joins two vertices $a, b$ when $aRb$.

- E.g., the divisor relation.

- Let $U$ be a set, and consider the powerset of $U$, i.e., the collection of all subsets of $U$. Let $A, B$ denote any two subsets of $U$, and define a relation $R$ consisting of the pairs $(A, B)$, where $A \subseteq B$. This is a relation on the powerset of $U$: the $\subseteq_U$ relation.
Properties of Relations I

Reflexive Relations

- A relation $R$ on a set $A$ is reflexive if $(a,a)$ belongs to $R$ for every element $a \in A$.

- Let $\mathbb{Z}$ be the set of integers. Then $'$ \(\leq\)' and $'$ \(\geq\)' are reflexive relations on $\mathbb{Z}$. But $'$ \(<\)' and $'$ \(>\)' are not reflexive.

- In the directed graph of the relation $R$, there is a directed edge from a vertex $a$ to $a$: this is a “loop”.

- The relation $\subseteq_U$ is reflexive since $S \subseteq S$ for every set $S \subseteq U$. But $\subseteq_U$ is not reflexive since $A \not\subseteq A$ for any set $A$. 
Properties of Relations II
Symmetric Relations

- A relation $R$ on a set $A$ is symmetric if $a$ is related to $b$, then $b$ is related to $a$, for every pair $a, b \in A$.

- Let $Z$ be the set of integers. Then $' \neq'$ is a symmetric relation on $Z$.

- In the directed graph of $R$, if there is a directed edge from $a$ to $b$, then there is a directed edge from $b$ to $a$ as well, in a symmetric relation.
Properties of Relations III
Symmetric Relations II

• The inverse of a relation \( R = \{(a, b) : aRb\} \) is
  \[
  R^{-1} = \{(b, a) : (a, b) \in R\}.
  \]
The inverse of ' <' is '>'; that of ' <=' is '>='. 

• The inverse relation \( R^{-1} \) is obtained by reversing the direction of every edge in the graph of \( R \).

• A symmetric relation \( R \) is equal to its inverse \( R^{-1} \).
Properties of Relations IV
Antisymmetric Relations

- A relation $R$ on a set $A$ is antisymmetric if at most one of $aRb$ or $bRa$ is true for $a \neq b$.

- The relations ' $\leq$', ' $\geq$', ' $<$', ' $>$' are all antisymmetric. The subset relation $\subseteq_U$ is antisymmetric.

- In the directed graph of $R$, for distinct vertices $a$ and $b$, at most one of the edges $(a, b)$ or $(b, a)$ can be present. The loop $(a, a)$ may or may not be present.
Properties of Relations V
Transitive Relations

- A relation $R$ on a set $A$ is transitive if whenever $aRb$ and $bRc$ are true, then $aRc$ is also true, for all triples $a$, $b$, $c$ in $A$. Note that $a$, $b$, and $c$ need not be distinct.

- Let $Z$ be the set of integers. Then '$<$' is a transitive relation on $Z$. The '$\neq$' relation on $Z$ is NOT transitive.

- In the directed graph corresponding to $R$, if directed edges $(a, b)$ and $(b, c)$ are present, then the directed edge $(a, c)$ should exist. If there is a directed path from $a$ to some vertex $v$ in the graph, then there should be an edge $(a, v)$. 
Properties of Relations VI

- **Four definitions**
  - **transitivity:** if $aRb$ and $b Rc$, then $aRc$.
  - **reflexivity:** $aRa$ for every $a \in A$.
  - **symmetry:** if $aRb$ then $bRa$.
  - **antisymmetry:** for $a \neq b$, at most one of $aRb$ or $bRa$ is true.

- **These definitions should hold for every element in the set $A$.** If there is one element $a \in A$ for which $aRa$ is false, then the relation is not reflexive.

- **Transitivity:** For a given triple $a, b, c$, the condition given above is (trivially) true whenever $a$ is not related to $b$, or $b$ is not related to $a$.

- **Let $A$ be a nonempty set.** Then which of these properties does the empty relation on $A$ satisfy?
Operations on Relations

- Suppose we are given two relations $R : A \rightarrow B$, and $S : A \rightarrow B$ (from a set $A$ to a set $B$).

- Since these relations are subsets of the Cartesian product $A \times B$, we can combine them using set operations.

- Thus we can define new relations: $R \cup S$, $R \cap S$, $R \setminus S$, etc.
Composite Relations

• Suppose we are given two relations $R : A \to B$, and $S : B \to C$.

• The composite relation $S \circ R$ consists of ordered pairs
  \[
  \{(a, c) : \text{there is some } b \in B \text{ such that } (a, b) \in R, \text{ and } (b, c) \in S\}.
  \]

• If $R$ is a relation on a set $A$, then the relation $R^2$ on $A$ is $R \circ R$.
  In the directed graph of $R$, we add edges $(a_1, a_2)$ if there is a path of length two between $a_1$ and $a_2$.

• Similarly, we can define the relations $R^n = R^{n-1} \circ R$, for $n = 2, 3, \ldots$.
  In the directed graph of $R$, we add edges $(a_1, a_2)$ if there is a path of length $n$ between $a_1$ and $a_2$. 
Closures of Relations

- Let $R$ be a relation on a set $A$. The reflexive closure of $R$ is the relation $S = R \cup \Delta$, where $\Delta = \{(a, a) : a \in A\}$.

- The reflexive closure $S$
  1. is reflexive, and
  2. contains every other reflexive relation that contains $R$.

- Reflexive closure of '$<$' on the integers is '$\leq$'.

- Closure: take a relation and add as few pairs as possible to it to satisfy some property.
Symmetric Closure

- Let $R$ be a relation on a set $A$. Its symmetric closure is $S = R \cup R^{-1}$, where $R^{-1}$ is the inverse relation

  $$\{(b, a) : (a, b) \in R\}.$$

- In the graph, we add the reverse edge $(b, a)$ to $S$ for every edge $(a, b) \in R$.

- Transitive closure of $R$ is the relation $S$, where we add an edge $(a, v)$ to $S$ if there is a path from $a$ to $v$ in the graph of $R$.

- This is also the relation

  $$S = \bigcup_{i=1}^{\infty} R^i.$$
A relation \( R \) on a set \( A \) is an equivalence relation if it is reflexive, symmetric, and transitive.

Consider a set of cities of the world: two cities are related if they are connected by a road system. This is an equivalence relation.

Let the relation \( R \) on the set of integers consist of pairs \((a, b)\) if \( a \) and \( b \) leave the same remainder when divided by 2. This is an equivalence relation.

The set of all elements related to \( a \in A \) is the equivalence class of \( A \), and is denoted \([a]_R\). In the example, the equivalence classes are: \([0]_R\), the even numbers, and \([1]_R\), the odd numbers.