CS 417/517 Computational Methods and Software

Spring 2003
Final Exam
Tues May 6, 2003

This examination has six questions (three pages). Please answer all of the questions. Write your answers only in the paper provided. Please number the pages in your answers, and on top of each page include your name and student identification number. On the first page, please write down the following honor pledge: " I have neither given nor received help from anyone while taking this examination.", and then sign it. Best wishes!

Any non-integer values to be computed in your answers should be computed to four digits of precision.

1. Are the following statements true or false? Give a reason for your answer. (Please give a correct reason for each answer to receive credit for it.)

   (a) An $n \times n$ matrix always has n eigenvalues, if we count the repeated eigenvalues multiple times.

   (b) An $n \times n$ matrix always has n linearly independent eigenvectors.

   (c) A least squares data fitting problem has a unique solution when the coefficient matrix has full column rank.

   (d) Orthogonal matrices are computationally useful because their inverses are easy to compute.

   (e) Matlab employs column pivoting to compute the QR factorization of a matrix when small diagonal elements occur in the matrix being factored.

   (f) The eigenvalues of a matrix can be computed accurately from the characteristic polynomial of the matrix.

2. List the following operations in increasing order of the work required to compute the operation, if A is an $n \times n$ matrix. In each case, give a rough estimate of the work required by each operation (a Big Oh estimate will do).

   (a) Solving a system of equations $A\,\underline{x} = \underline{b}$.

   (b) Computing a matrix vector product $A\,\underline{y}$.

   (c) Computing all eigenvalues and eigenvectors of A.

   (d) Computing the orthogonal factorization of A.

   (e) Computing the 1-norm of A.

3. If Q is a 2×2 orthogonal matrix such that

$$Q \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

what values can α have?

Show that Q can be computed by a Householder matrix H_1, and show the values of the elements in H_1, and the corresponding value of α.

4. Suppose that you are computing the QR factorization of the matrix A by Householder reflectors, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}.$$

   (a) How many Householder transformations will be required?

   (b) Compute the first Householder transformation H_1 as follows: you need compute only the elements of the Householder reflector, and then write down a formula that shows how the Householder matrix is obtained from the Householder reflector.

   (c) What are the values of the elements in the first column of the matrix $H_1\,A$?

   (d) How are the values of the elements in the first column of the matrix $H_2\,H_1\,A$ and $H_1\,A$ related? Here H_2 is the second Householder transformation used to zero the subdiagonal elements of the second column of $H_1 A$. (You do not need to compute either of the matrices $H1$ or $H2$ to answer these questions.)

   (e) Show that

$$H_1\,\underline{x} \;=\; \underline{x} - 2\,(\underline{w_1}^T\,\underline{x})\,\underline{w_1},$$

   where $\underline{w_1}$ is the Householder reflector from which the Householder matrix H_1 is obtained. Use this result to compute the product of the Householder matrix H_1 and the second column of A, using only the Householder reflector, without forming the matrix H_1.

5. Solve the least squares problem $A\,\underline{x} \approx \underline{b}$, where data is given below:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix},$$

and the vectors

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \qquad \underline{b} = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}.$$

Report the least squares solution, and the least squares residual. Find two column vectors that the least squares residual must be orthogonal to, and verify that the residual is indeed orthogonal to them.

6. Let A be a 4×4 matrix with eigenvalues equal to 1, 2, 3, and 4. What are the eigenvalues of the following matrices? Please give reasons for your answers to be given credit.

   (a) $A + 5I$

   (b) A^2

   (c) A^{-1}

   (d) $Q^T AQ$, where Q is a 4×4 orthogonal matrix.