1. Prove that the sum of a rational number and an irrational number is irrational. [10]

Suppose that it is not true, that is, the sum of a rational number \( m \) and an irrational number \( n \) is rational.

Then there are integers \( p \) and \( q \) such that

\[
m + n = \frac{p}{q}.
\]

Since \( m \) is rational, there are integers \( s \) and \( t \) such that

\[
\frac{s}{t} + n = \frac{p}{q}.
\]

Hence

\[
n = \frac{p}{q} - \frac{s}{t}.
\]

\[
= \frac{pt - sq}{qt}
\]

Since \( p, q, s, t \) are integers, \( pt - sq \) and \( qt \) are integers. Hence \( \frac{pt - sq}{qt} \) is rational.

Hence \( n \) is rational, contradicting the hypothesis.

Hence the claim must be true.
2. Fill in the blanks with the **SHORTEST** string of characters so that the resultant proposition is valid. [15]

(a) \((P \land Q) \rightarrow P \iff \neg(P \land Q) \lor ? \underline{P}\)
\(\iff (\neg ? \underline{P} \lor \neg Q) \lor P \iff P \lor (\neg ? \underline{P} \lor \neg Q)\)
\(\iff (P \lor \neg ? \underline{P}) \lor \neg Q \iff \text{TRUE} \lor \neg Q \iff \text{TRUE}\)

(b) \(\neg \exists x \exists y \left[ (P(x) \land Q(y)) \rightarrow R(x, y) \right]\)
\(\iff \forall x \neg \exists y \left[ (P(x) \land Q(y)) \rightarrow R(x, y) \right]\)
\(\iff \forall x \forall y \neg [(P(x) \land Q(y)) \rightarrow R(x, y)]\)
\(\iff \forall x [P(x) \land \neg \forall y [Q(y) \land \neg R(x, y)]]\)
\(\iff \forall x P(x) \land \forall y Q(y) \land \neg R(x, y)\)

3. Express the assertions given below as propositions of predicate logic using the following predicates. The universe is the set of objects. [15]

- \(F(x)\): \(x\) is a flower.
- \(L(x, y)\): \(x\) likes \(y\).
- \(P(x)\): \(x\) is a person.
- \(R(x)\): \(x\) is red.

a) Some person likes some flower.

\[\exists x \exists y \left[ P(x) \land F(y) \land L(x, y) \right] \text{ or equivalent}\]

b) Some person likes a flower only if it is red.

\[\exists x \forall y \left[ (P(x) \land L(x, y) \land F(y)) \rightarrow R(y) \right] \text{ or equivalent}\]

c) Not every person likes a red flower.

\[\neg \forall x \forall y \left[ (P(x) \land F(y) \land R(y)) \rightarrow L(x, y) \right] \text{ or equivalent}\]
4. Restate the following assertions in "if ... then ..." form: [15]

(a) I am happy only if I am free.

\[
\text{If I am happy, then I am free.}
\]

(b) \( x = 1 \) is necessary for \( x^2 = 1 \).

\[
\text{If } x^2 = 1, \text{ then } x = 1.
\]

(c) Pythagoras’ theorem holds if a triangle is a right triangle.

\[
\text{If a triangle is a right triangle, then Pythagoras’ theorem holds.}
\]

5. For the following assertion answer the questions below: [15]

Assertion: If some printer is busy or out of service, then some printer job is lost.

(a) Find the converse in English.

\[
\text{If some printer job is lost, some printer is busy or out of service.}
\]

(b) Find the contrapositive in English.

\[
\text{If no printer job is lost, then no printer is busy or out of service.}
\]

(c) Find the negation in English. DO NOT simply say "It is not the case that ..." or something similarly trivial.

\[
\text{Some printer is busy or out of service and (but) no printer job is lost.}
\]
6. Indicate which ones of the following are true and which ones are false: [15]
(a) \( \emptyset \in \{\{\emptyset\}, \{\emptyset\}\}\)  
   \[ \text{F} \]
(b) \( \{\emptyset\} \subseteq 2^{\{0,1\}} \)  
   \[ \text{F} \]
(c) \( 0 \in \emptyset \)  
   \[ \text{F} \]
(d) \( \emptyset \in \{\emptyset, \emptyset\} \)  
   \[ \text{T} \]
(e) \( \{\emptyset\} \subseteq \{0\} \)  
   \[ \text{F} \]
(f) \( \emptyset \subseteq \{\emptyset\}, \{\emptyset, \emptyset\}\} \)  
   \[ \text{T} \]
(g) \( \{x\} \in \{x\} \)  
   \[ \text{F} \]
(h) \( A \subseteq A \times B \), where \( A \) and \( B \) are sets.  
   \[ \text{F} \]
(i) \( x \subseteq \{x, \{x\}\} \)  
   \[ \text{F} \]
(j) \( \emptyset \in \{0\} \)  
   \[ \text{F} \]
7. Prove that $B \subseteq A$ if $B - A = \emptyset$, where $A$ and $B$ are sets. [15]

Suppose that it is not true, that is, $B - A = \emptyset$ and $B \not\subseteq A$.

Then since $B \not\subseteq A$, there exists an element $x$ such that $x \in B$ and $x \not\in A$.

That means that $x \in B - A$.

Hence $B - A \neq \emptyset$, contradicting the hypothesis $B - A = \emptyset$.

Hence the claim must be true.