CS 381 Solutions to Homework 8

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8. Basis Step: When $n = 0$, the left hand side(LHS) is equal to 2,
and the right hand side(RHS) is $(1 - (-7))/4 = 8/4 = 2$.
Hence LHS = RHS.
Inductive Step: Assume that it is true for n, that is
$2 - 2 \cdot 7 + ... + 2 \cdot (-7)^n = (1 - (-7)^{n+1})/4$
For $n+1$, $2 - 2 \cdot 7 + ... + 2 \cdot (-7)^n + 2 \cdot (-7)^{n+1} = (1 - (-7)^{n+1})/4 + 2 \cdot (-7)^{n+1}$
$= (1 - (-7)^{n+1} + 8 \cdot (-7)^{n+1})/4 = (1 + 7 \cdot (-7)^{n+1})/4 = (1 - (-7)(-7)^{n+1})/4 =$
$(1 - (-7)^{n+2})/4$. QED

22. We can prove by mathematical induction that if $n \geq 4$, then $n^2 \leq n!$.

Basis Step: If $n = 4$, then $4^2 = 16$ and $4! = 24$.
Hence $4^2 \leq 4!$.

Inductive Step: Assume that $n^2 \leq n!$.
For $n + 1$, we need to prove that $(n + 1)^2 \leq (n + 1)!$.
For that, it is sufficient to prove that $(n + 1) \leq n!$.
Since $n \geq 4$, $n + 1 \leq 2n$. But $2n \leq n!$ since $n \geq 4$.
Hence $n + 1 \leq n!$. Hence $(n + 1)^2 \leq (n + 1)!$.

36. Basis Step: If $n = 1$, then $4^2 + 5 = 21$, which is divisible by 21.
Inductive Step: Assume that $4^{n+1} + 5^{2n-1} = 21k$ for some integer k.
For $n + 1$, $4^{n+2} + 5^{2n+1} = 4 \cdot 4^{n+1} + 5^2 \cdot 5^{2n-1}$
$= 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} = 4 \cdot 21k + 21 \cdot 5^{2n-1}$
$=21(4k + 5^{2n-1})$, which is divisible by 21 because $4k$ and 5^{2n-1} are integers.

39. Basis Step: When $n = 1$, $\cap_{j=1}^{1} A_j = A_1$, and $\cap_{j=1}^{1} B_j = B_1$.
Since $A_1 \subseteq B_1$, $\cap_{j=1}^{n} A_j \subseteq \cap_{j=1}^{n} B_j$
Before proceeding further let us prove the following claim:
Claim: If $B \subseteq C$, then $B \cap A \subseteq C \cap A$.
Proof of Claim: For an arbitrary element x, $x \in B \cap A \leftrightarrow x \in B \wedge x \in A$
Since $B \subseteq C$, this implies that $x \in C \wedge x \in A$. Hence $x \in B \cap A \rightarrow x \in C \cap A$.

Inductive Step: Assume that $\cap_{j=1}^{n} A_j \subseteq \cap_{j=1}^{n} B_j$.
For $n+1$, $\cap_{j=1}^{n+1} A_j = (\cap_{j=1}^{n} A_j) \cap A_{n+1} \subseteq (\cap_{j=1}^{n} B_j) \cap A_{n+1} \subseteq (\cap_{j=1}^{n} B_j) \cap B_{n+1}$
by the claim applied twice.
But $(\cap_{j=1}^{n} B_j) \cap B_{n+1} = \cap_{j=1}^{n+1} B_j$. Hence $\cap_{j=1}^{n+1} A_j \subseteq \cap_{j=1}^{n+1} B_j$.

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8. It can be easily verified that only $25, 40, 50, 65, 75, 80, 90, 100, 105, 115, 120, 125$ and 130 can be obtained by using 25s and 40s if it is less thatn 140.
For any number not less than 140, we can show that if n is obtained by using 25s and 40s, then $(n + 5)$ is obtained by using them.
Proof: Basis Step: If $n = 140$, then $140 = 4 \cdot 25 + 40$.
Inductive Step: Assume that n can be expressed as $25k + 40m$ for some natural numbers k and m.
Then $n + 5 = 25k + 40m + 5 = 25(k - 3) + 40(m + 2)$ if $k \geq 3$.
Thus $k = 0, 1$, or 2. Let us consider them one by one.
(1) $k = 0$: In this case $n = 40m$. Since $n \geq 140$, $m > 3$.
Hence $n + 5 = 40m + 5 = 40(m - 3) + 125 = 40(m - 3) + 5 \cdot 25$
(2) $k = 1$: In this case $n = 25 + 40m$. Since $n \geq 140$, $m \geq 3$.
Hence $n + 5 = 40(m - 3) + 150 = 40(m - 3) + 6 \cdot 25$.
(3) $k = 2$: In this case $n = 50 + 40m$. Since $n \geq 140$, $m \geq 3$.
Hence $n + 5 = 40(m - 3) + 175 = 40(m - 3) + 7 \cdot 25$.
Hence in all cases $n + 5$ is expressed as $25k + 40m$ for some natural numbers k and m.

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4 (b) $f(2) = f(3) = f(4) = f(5) = 1$
(c) $f(2) = 2, f(3) = 5, f(4) = 33, f(5) = 1214$