1. Let $L = \{ 0^i1^j : j = i \text{ or } j = 3i, \text{ where } i \text{ and } j \text{ are natural numbers} \}$. Prove by Myhill-Nerode that $L$ is non-regular. [15]
2. Let $S$ be the language over alphabet $\{a, b\}$ defined recursively as follows:

Basis Clause: $\Lambda \in S$
Inductive Clause: For all $x$, if $x \in S$, then $ax, axb \in S$
Exrremal Clause: Nothing is in $S$ unless it is obtained by the above two clauses.

Show by General Induction that $S \subseteq \{a^i b^j : i \geq j, \text{ and } i, j \text{ are natural numbers}\}$. [15]
3. Consider the following grammar:

\[
S \rightarrow TT \\
T \rightarrow aT \mid Ta \mid b
\]

(a) What kind of grammar is it? [2]

(b) What is the shortest string in the language generated by the grammar? [3]

(c) What kind of strings does the grammar generate? [5]

4. (a) Give a design/plan of a Turing machine that divides a given natural number by 2 and leaves the remainder in the second square from the left end of the tape. [5]
4 (b) Give the diagram of your Turing machine of (a) above. You may use basic Turing machines such as $T_a, T_R, T_{R\Delta}$, etc. [10]
5. Construct the $NFA - \Lambda$ for the regular expression $a^*b + (ba + b)^*$ using Kleene Theorem 1. Do not simplify your $NFA - \Lambda$. [15]
6. Consider the following two problems, Accepts and Accepts Λ:

Accepts: Given a string $x$ and a Turing machine $T$, this problem asks whether or not $T$ accepts $x$.

Accepts Λ: Given a Turing machine $T$, this problem asks whether or not $T$ accepts Λ.

The following questions are concerned with the unsolvability of those problems.
Let $x$ be a string and $T$ be a Turing machine.

(a) A Turing machine $T_1$ can be constructed by concatenating $T_x$ with $T$ (first $T_x$ then $T$), where $T_x$ is a Turing machine that writes $x$ and halts given Λ as an input.
Draw a diagram of $T_1$ as in 4 (b). [3]

(b) Suppose that $T_1$ accepts Λ. Does $T$ accept $x$? Give your reasons. [6]

(c) Suppose that Accepts is unsolvable. Can we conclude that Accept Λ is unsolvable? Give your reasons. [6]
7. For each of the following statements answer whether or not it is true. [15]

(a) Union of two context-free languages is context-free.
(b) \{a^n b^n : 0 \leq n \leq 10^5, n \text{ is an integer } \} \text{ is regular.}
(c) Union of infinitely many regular languages is regular.
(d) For any arbitrary language \( L, \Lambda \in L^+ \).
(e) If \( L_1 \) and \( L_2 \) are non-regular, then \( L_1 \cap L_2 \) is non-regular.
(f) If \( L_1 \) is regular and \( L_2 \) is non-regular, then \( L_1 \cup L_2 \) is non-regular.
(g) \((111^*)^* = (11 + 111)^*\).
(h) For a DFA \( \delta^*(q, xy) = \delta^*(\delta^*(q, y), x) \).
(i) \( \Lambda(S \cap T) = \Lambda(S) \cap \Lambda(T) \) for sets of states \( S \) and \( T \) of an NFA-\( \Lambda \).
(j) If the set of 'yes'-instances of a decision problem is accepted by some Turing machine, then that decision problem is solvable.
(k) If one Turing machine accepts the 'yes'-instances of a decision problem and another (different) Turing machine accepts the 'no'-instances of that decision problem, then that decision problem is solvable.
(l) The set of non-palindromes is non-regular.
(m) \( \{a^i b^j c^k : i = j + k, i, j, k \text{ are natural numbers} \} \text{ is context-free.} \)
(n) \( (L_1 \cup L_2)^* = L_1^* \cup L_2^* \).
(o) Let \( L_1 \) and \( L_2 \) be context-free languages generated by context-free grammars \( G_1 \) and \( G_2 \), respectively. Let \( S_1 \) be the start symbol of \( G_1 \) and \( S_2 \) be the start symbol of \( G_2 \), respectively. Also assume that \( G_1 \) and \( G_2 \) don’t have any non-terminals in common. Then \( S \rightarrow S_1 S_2 \) together with the productions of \( G_1 \) and \( G_2 \) generates \( L_1 L_2 \) and it is context-free.