A PROBABILISTIC ANALYSIS OF MISPARKING IN RESERVATION BASED PARKING GARAGES

by

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ABSTRACT

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Parking in major cities is an expensive and annoying affair, the reason ascribed to the limited availability of parking space. Modern parking garages provide parking reservation facility, thereby ensuring availability to prospective customers. Misparking in such reservation based parking garages creates confusion and aggravates driver frustration. The general conception about misparking is that it tends to completely cripple the normal functioning of the system leading to chaos and confusion. A single mispark tends to have a ripple effect and therefore spawns a chain of misparks. The chain terminates when the last mispark occurs at the parking slot reserved by the first member of the misparking chain. Of interest is the probability distribution of the random variable that represents the length of the misparking chain. The most probable length of the chain as determined from the underlying probability distribution is therefore indicative of the extent of instability caused by a single mispark. In this thesis, reasonably tight bounds for the probability of occurance of a misparking chain of a given length are first determined. Next, the probability distribution of the length of misparking chain is approximated. It can be inferred from the mathematical derivations presented in this work that the most probable misparking chain length is very small compared to the size of the garage, thereby showing that a mispark has negligible effect on the stability of the parking system. Simulation results are also presented to validate the analytical solutions.

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I learnt a great lot working on this thesis and I'll cherish this experience throughout my life.

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CHAPTER I

INTRODUCTION

Parking assistance, currently is a hot topic of research. This can be attributed to the increasing demand for parking in major cities where parking is limited and costly. High contention for parking space spawns a need to deploy intelligent parking systems in order to provide parking assistance to customers. Without any parking assistance, locating a vacant parking slot in a heavily occupied parking garage is a time consuming and frustrating affair. As the contention for parking space is extremely high in major cities, a general and simple solution would be to provide reservation facility to customers. Reservation gaurantees parking space and avoids traffic congestion in parking garages. In addition, the driver is clearly aware of the slot assigned to him/her, which eliminates the trouble of locating an empty slot and therefore saves time. A wide variety of present day solutions rely on parking reservation as the basis.

One of the common characteristics of the present day parking systems is that they all assume ideal human behavior. In other words, customers are assumed to strictly adhere to the rules laid out by the parking assistance mechanism. This assumption is true in general but there can be exceptions. Misparking is one such exception that tends to destabilize any reservation based parking system. A customer may choose to park in a slot different from that reserved by him for his own convenience. However, it is to be noted that misparking is a rare event as there are often severe penalties(fine, towing, etc) associated with it. Nonetheless, it is still important to assess the damage or instability caused by misparking. Intuition tells us that misparking leads to chaos and confusion. Therefore, it of interest to determine both theoretically and experimentally if our intuition is correct. This requires a mathematical model to be defined to represent misparking.

In this work, the impact of misparking is analyzed in terms of length of the chain produced by a single initial mispark as this chain length is indicative of the extent of instability caused. Suppose, a driver A decides to park in a slot different from the slot reserved by him for his own convenience or purpose. This in turn triggers another mispark as one of the subsequent drivers B will find his assigned slot occupied and therefore is forced to park in a different slot. At this point, two possibilities exist. Either driver B can terminate the misparking chain by parking in the slot reserved by driver A or he can trigger another mispark by parking in a slot reserved by someone else, say driver C. Therefore, the misparking chain grows to a certain length with a certain probability before it is terminated. Of interest is the probabity distribution of the length of this chain and the most probable length of this chain. We address all these issues in the subsequent chapters.

CHAPTER II

STATE OF THE ART

In this chapter, numerous state of the art parking assistance systems based on reservation are briefly described to emphasize the importance of the misparking issue concerned with these mechanisms. [1] presents a car-park management system based on wireless sensor networks. The parking assistance mechanism in [1] finds a closet slot and guides a car to that slot. The authors of [2] evaluate the performance of a parking system in a situation where drivers have parking availability information. In [3], driver parking choice models are developed to study the impact of parking guidance systems on travel times of drivers. The simulation results presented in [3] clearly indicate a meagre reduction in the travelling times of drivers in presence of parking guidance systems.

[4] and [5] discuss a couple more parking assistance mechanisms. In particular, the parking system in [5] supports dynamic reservation of parking slots. Strategies to determine ideal parking slots for different customers based on their individual needs are proposed and evaluated in [6]. The authors of [7] propose a model to provide information regarding available parking slots to customers using VANET infrastructure. The effectiveness of the parking guidance mechanisms are analyzed in detail in [8], [9], [10] and [11]. However, any parking reservation policy is accompanied with the issue of misparking. None of the above cited parking mechanisms consider the aspect of human behaviour in their implementation. All the previously discussed mechanisms are deployed under the assumption that the protocols laid out by these mechanisms are strictly observed by all drivers, which is not completely true. As mentioned earlier, a driver may choose to violate the reservation policy by parking in a different slot for his/her convenience, thereby obstructing the normal functioning of the parking system.

Misparking in reservation based parking garages is therefore one of the primary concerns as it tends to destabilize the entire parking system. [12] provides a probabilistic approach to model misparking. The results presented in [12] depict that every reservation based parking system possesses the property of self recovery with respect to misparking. However, no explanation regarding the extent of instability caused by misparking is given in [12]. Quantifying the impact of misparking on stability of a reservation based parking system is the main contribution of this thesis.

CHAPTER III

PROBABILISTIC ANALYSIS OF MISPARKING

III.1 SYSTEM MODEL AND PROBLEM STATEMENT

III.1.1 System Model

The parking garage considered consists of n parking slots. Each of the n slots is reserved by one of the n drivers. The n drivers are denoted as P_1,P_2,P_3,...,P_n. The corresponding reserved slots are denoted as S_1,S_2,S_3,...,S_n. Cars enter the parking garage in a sequential manner and the occupation of slots is serialized in time. In other words, driver $i+1$ parks only after driver i has finished parking. A driver parks in the slot reserved by him if it is empty, otherwise he randomly parks in any one of the remaining slots (Fig. 1 and Fig. 2). Exactly one of the n drivers is assumed to initiate misparking. This assumption is reasonable as misparking is a rare event.

III.1.2 Problem Statement

Under the assumptions of the model previously described, define Z_i to be the event where P_i is the first driver to mispark. Also, let $A_{i,j}$ denote the event that P_i misparks in a slot reserved by P_j. Let B_k be the event that P_k is the last driver to mispark, ie. P_k closes the misparking chain by parking in slot S_i reserved by first member of the chain P_i. The length of the misparking chain is represented by a random variable L. Of interest is the probability density function of L, ie. $\Pr\{L = l\}$. Also, $\hat{L}$ represents the most probable value of L. The bounds on the probabilities for different values of L are determined next.

III.2 SOME IMPORTANT RESULTS

The proofs for some of the important results/inequalities that find use in subsequent derivations are presented next.

Lemma 1 *If H_k stands for harmonic summation of first k natural numbers and t is any positive integer, then*

$$\sum_{k=1}^{n} \frac{H_k^t}{k+t} \leq \frac{H_n^{t+1}}{t+1} \tag{1}$$

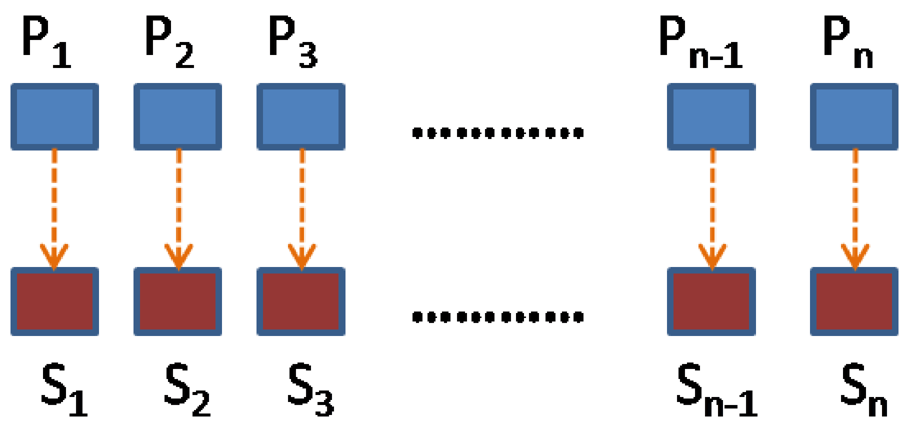

FIG. 1: System without misparking. The customers P_i strictly observe the reservation protocol by parking in their corresponding reserved slots S_i. A dashed arrow between a customer and a slot represents parking slot occupation.

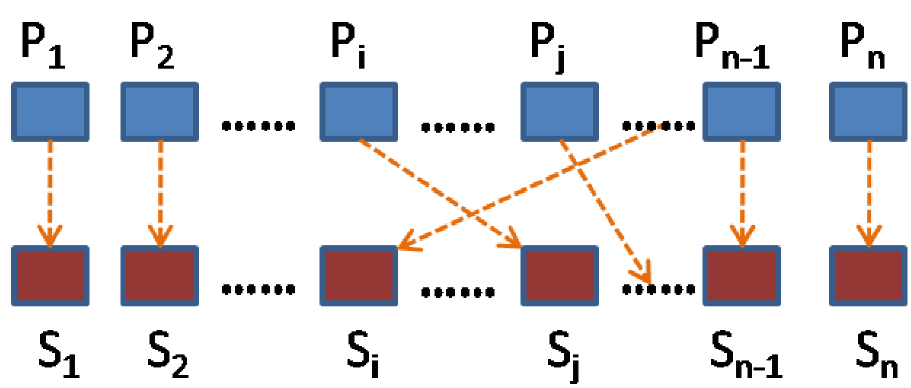

FIG. 2: System with misparking. Exactly one of the first $N-1$ customers say P_i initiates the misparking chain by parking in a slot S_j reserved by some other customer. Eventually, the chain terminates when S_i is occupied.

Proof: A simple mathematical induction on n validates the claimed result.

Base case: $n = 1$

$$\begin{aligned} \frac{H_1^t}{t+1} &\leq \frac{H_1^{t+1}}{t+1} \\ \frac{1}{t+1} &= \frac{1}{t+1} \end{aligned}$$

which is true.

Inductive case: Assume that (1) is true for $n = l$. It is to be shown that it is true for $n = l + 1$ as represented by inequality (2).

$$\sum_{k=1}^{l+1} \frac{H_k^t}{k+t} \leq \frac{H_{l+1}^{t+1}}{t+1} \tag{2}$$

or equivalently,

$$\sum_{k=1}^{l} \frac{H_k^t}{k+t} + \frac{H_{l+1}^t}{l+1+t} \leq \frac{H_{l+1}^{t+1}}{t+1} \tag{3}$$

By inductive hypothesis,

$$\sum_{k=1}^{l} \frac{H_k^t}{k+t} \leq \frac{H_l^{t+1}}{t+1} \tag{4}$$

Now, it is evident that inequality (5) implies inequality (3).

$$\frac{H_l^{t+1}}{t+1} + \frac{H_{l+1}^t}{l+1+t} \leq \frac{H_{l+1}^{t+1}}{t+1} \tag{5}$$

Multiplying (5) with $\frac{t+1}{H_l^{t+1}}$,

$$\begin{aligned} 1 + \left(\frac{H_{l+1}}{H_l}\right)^t \frac{1}{\left(1 + \frac{l}{t+1}\right) H_l} &\leq \left(\frac{H_{l+1}}{H_l}\right)^{t+1} \\ \Rightarrow \quad \left(\frac{H_{l+1}}{H_l}\right)^t \left(\frac{H_{l+1}}{H_l} - \frac{1}{\left(1 + \frac{l}{t+1}\right) H_l}\right) &\geq 1 \\ \Rightarrow \quad \left(1 + \frac{1}{(l+1)H_l}\right)^t \left(1 - \frac{lt}{(l+1)(l+t+1)H_l}\right) &\geq 1 \end{aligned}$$

Since $(1+x)^t \geq 1 + tx$, inequality (6) implies (5).

$$\left(1 + \frac{t}{(l+1)H_l}\right)\left(1 - \frac{lt}{(l+1)(l+t+1)H_l}\right) \geq 1 \tag{6}$$

$$\begin{aligned}
&\Rightarrow \quad 1+\frac{t}{(l+1)H_l}-\frac{lt}{(l+1)(l+1+t)H_l}\\
&\quad -\frac{lt^2}{(l+1)^2H_l^2(l+t+1)}\geq 1\\
&\Rightarrow \quad \frac{t}{(l+1)H_l}\left[1-\frac{l}{(l+1+t)}-\frac{lt}{(l+1)H_l(l+t+1)}\right]\geq 0\\
&\Rightarrow \quad \frac{t}{(l+1)H_l}\left[\frac{1+t}{(l+1+t)}-\frac{lt}{(l+1)H_l(l+t+1)}\right]\geq 0\\
&\Rightarrow \quad \frac{t}{(l+1)(l+t+1)H_l}\left[1+t-\frac{lt}{(l+1)H_l}\right]\geq 0\\
&\Rightarrow \quad \frac{t}{(l+1)(l+t+1)H_l}\left[1+t\left(1-\frac{l}{(l+1)H_l}\right)\right]\geq 0
\end{aligned}$$

Since,

$$\frac{t}{(l+1)(l+t+1)H_l}\geq 0, t\left(1-\frac{l}{(l+1)H_l}\right)>0 \tag{7}$$

Inequalities (5) and (2) are true.

This completes the proof.

Lemma 2 *If H_k stands for harmonic summation of first k natural numbers and t is any positive integer, then*

$$\sum_{k=1}^{n}\frac{H_k^t}{k+t}>\frac{H_n^{t+1}}{t+2}, t^2+t<n+1 \tag{8}$$

Proof: A simple mathematical induction on n validates the claimed result.

Base case: $n=1$

$$\begin{aligned}
\frac{H_1^t}{t+1} &> \frac{H_1^{t+1}}{t+2}\\
\frac{1}{t+1} &> \frac{1}{t+2}, \forall t
\end{aligned}$$

which is true.

Inductive case: Assume that (8) is true for $n=l$. It is to be shown that it is true for $n=l+1$ as represented by inequality (9).

$$\sum_{k=1}^{l+1}\frac{H_k^t}{k+t}>\frac{H_{l+1}^{t+1}}{t+2} \tag{9}$$

or equivalently,

$$\sum_{k=1}^{l}\frac{H_k^t}{k+t}+\frac{H_{l+1}^t}{l+1+t}>\frac{H_{l+1}^{t+1}}{t+2} \tag{10}$$

By inductive hypothesis,

$$\sum_{k=1}^{l} \frac{H_k^t}{k+t} > \frac{H_l^{t+1}}{t+2} \tag{11}$$

Now, it is evident that inequality (12) implies inequality (10).

$$\frac{H_l^{t+1}}{t+2} + \frac{H_{l+1}^t}{l+1+t} > \frac{H_{l+1}^{t+1}}{t+2} \tag{12}$$

Multiplying (12) with $\frac{t+2}{H_{l+1}^{t+1}}$,

$$\begin{aligned} 1 + \left(\frac{H_l}{H_{l+1}}\right)^{t+1} + \frac{t+2}{(l+t+1)H_{l+1}} &> 1 \\ \Rightarrow \left(1 - \frac{1}{(l+1)H_{l+1}}\right)^{t+1} + \frac{t+2}{(l+t+1)H_{l+1}} &> 1 \end{aligned}$$

Since $(1-x)^t \geq 1 - tx$, inequality (13) implies (12).

$$1 - \frac{t+1}{(l+1)H_{l+1}} + \frac{t+2}{(l+t+1)H_{l+1}} > 1 \tag{13}$$

$$\begin{aligned} \Rightarrow \quad \frac{t+2}{(l+t+1)H_{l+1}} - \frac{t+1}{(l+1)H_{l+1}} &> 0 \\ \Rightarrow \quad \frac{t+2}{(l+t+1)} - \frac{t+1}{(l+1)} &> 0 \end{aligned}$$

which is true $\forall t | \{t^2 + t < l + 1\}$. Therefore, inequalities (12) and hence (9) are true. This completes the proof.

Lemma 3 *If H_k stands for harmonic summation of first k natural numbers and t is any positive integer, then*

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2}{2!} + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k^2} \tag{14}$$

$$\frac{H_n^{t+1}}{(t+1)!} \leq \frac{1}{t!}\sum_{k=1}^{n} \frac{H_k^t}{k} < \frac{H_n^{t+1}}{(t+1)!} + \frac{e}{2} \tag{15}$$

$$\frac{1}{t!}\sum_{k=1}^{n} \frac{H_k^t}{k} - e < \frac{1}{t!}\sum_{k=1}^{n} \frac{H_k^t}{k+1} < \frac{1}{t!}\sum_{k=1}^{n} \frac{H_k^t}{k} \tag{16}$$

<u>Proof:</u> The complete proofs are provide in chapter 5.

III.3 UPPER BOUND (UB) ON PR$\{L = l\}$

Lemma 4 *The probability that the length of the misparking chain is 2 is bounded above as represented by inequality (17):*

$$\Pr\{L = 2\} < \frac{1}{n-1}\left(\frac{H_{n-1}^2}{2!} + \frac{e}{2}\right) \tag{17}$$

Proof: Based on the system model, the probablity is given by:

$$\Pr\{L = 2\} = \Pr\{Z_i\} \sum_{i=1}^{n-1} \Pr\{A_{i,j}\} \sum_{j=i+1}^{n} \Pr\{B_j\}$$

which is equivalent to

$$\begin{aligned}
\Pr\{L = 2\} &= \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{n-i} \sum_{j=i+1}^{n} \frac{1}{n-j+1} \\
&= \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{H_{n-i}}{n-i} \\
&= \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{H_k}{k} \\
&= \frac{1}{n-1}\left(\frac{H_{n-1}^2}{2!} + \frac{1}{2}\sum_{k=1}^{n-1}\frac{1}{k^2}\right) \\
&< \frac{1}{n-1}\left(\frac{H_{n-1}^2}{2!} + \frac{e}{2}\right) \qquad \text{Using Lemma 3}
\end{aligned}$$

Lemma 5 *The probability that the length of the misparking chain is 3 is bounded above as represented by inequality (18):*

$$\Pr\{L = 3\} < \frac{1}{n-1}\left(\frac{H_{n-2}^3}{3!} + \frac{e}{2}\right) \tag{18}$$

Proof: Again based on the system model, the probability is given by:

$$\Pr\{L = 3\} = \Pr\{Z_i\}\left[\sum_{i=1}^{n-2} \Pr\{A_{i,j}\} \sum_{j=i+1}^{n-1} \Pr\{A_{j,k}\} \sum_{k=j+1}^{n} \Pr\{B_k\}\right]$$

which is equivalent to

$$\begin{aligned}
& \frac{1}{n-1}\sum_{i=1}^{n-2}\frac{1}{n-i}\sum_{j=i+1}^{n-1}\frac{1}{n-j+1}\sum_{k=j+1}^{n}\frac{1}{n-k+1} \\
= & \frac{1}{n-1}\sum_{i=1}^{n-2}\frac{1}{n-i}\sum_{j=i+1}^{n-1}\frac{H_{n-j}}{n-j+1} \\
= & \frac{1}{n-1}\sum_{i=1}^{n-2}\frac{1}{n-i}\sum_{k=1}^{n-i-1}\frac{H_k}{k+1} \\
\leq & \frac{1}{n-1}\sum_{i=1}^{n-2}\frac{1}{n-i}\frac{H_{n-i-1}^2}{2!} \qquad \text{Using Lemma 1} \\
= & \frac{1}{n-1}\sum_{k=1}^{n-2}\frac{1}{k+1}\frac{H_k^2}{2!} \\
< & \frac{1}{n-1}\left(\frac{H_{n-2}^3}{3!}+\frac{e}{2}\right) \qquad \text{Using Lemma 3}
\end{aligned}$$

The upper bounds for chain lengths 4, 5 and 6 are obtained in a similar fashion.

Lemma 6 *The probability that the length of the misparking chain is 4, 5 and 6 are bounded above as given by inequalities (19), (20) and (21), respectively:*

$$\Pr\{L=4\} < \frac{1}{n-1}\left(\frac{H_{n-3}^4}{4!}+\frac{e}{2}\right) \tag{19}$$

$$\Pr\{L=5\} < \frac{1}{n-1}\left(\frac{H_{n-4}^5}{5!}+\frac{e}{2}\right) \tag{20}$$

$$\Pr\{L=6\} < \frac{1}{n-1}\left(\frac{H_{n-5}^6}{6!}+\frac{e}{2}\right) \tag{21}$$

Proof: The proof is similar to that provided in lemmas 4 and 5. The general inequality representing the upper bound on probability that the length of the misparking chain is any integer $l, l \geq 2$ is formally presented in theorem 1. Note that the expressions for trivial cases where length is 2 and 3 are already presented in lemmas 4 and 5 respectively.

Theorem 1 (Misparking Chain Length Upper Bound) *The probability that the length of the misparking chain is l is bounded above as represented by inequality (22):*

$$\Pr\{L=l\} < \frac{1}{n-1}\left(\frac{H_{n-l+1}^l}{l!}+\frac{e}{2}\right) \tag{22}$$

Proof: Following a similar approach as in preceding lemmas,

$$\Pr\{L=l\} = \Pr\{Z_i\}\left[\sum_{i=1}^{n-l+1}\Pr\{A_{i,j}\}\sum_{j=i+1}^{n-l}\Pr\{A_{j,k}\}\dots\sum_{y=x+1}^{n}\Pr\{B_y\}\right] \tag{23}$$

which is equivalent to

$$\begin{aligned}
\Pr\{L=l\} &= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\dots \\
&\quad \dots\sum_{y=z+1}^{n-1}\frac{1}{n-y+1}\sum_{x=y+1}^{n}\frac{1}{n-x+1} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\dots\sum_{y=z+1}^{n-1}\frac{H_{n-y}}{n-y+1} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\dots\sum_{z=w+1}^{n-2}\frac{1}{n-z+1}\sum_{p=1}^{n-z-1}\frac{H_p}{p+1}
\end{aligned}$$

Using Lemma 1,

$$\begin{aligned}
&\le \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\dots\sum_{z=w+1}^{n-2}\frac{H_{n-z-1}^2}{n-z+1}\frac{1}{2!} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\dots\sum_{p=1}^{n-w-2}\frac{H_p^2}{p+2}\frac{1}{2!} \\
&\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\frac{H_{n-i-(l-2)}^{l-1}}{(l-1)!} \\
&= \frac{1}{n-1}\sum_{p=1}^{n-l+1}\frac{1}{p+(l-2)}\frac{H_p^{l-1}}{(l-1)!} \\
&\le \frac{1}{n-1}\sum_{p=1}^{n-l+1}\frac{1}{p}\frac{H_p^{l-1}}{(l-1)!} \\
&< \frac{1}{n-1}\left(\frac{H_{n-l+1}^l}{l!}+\frac{e}{2}\right) \quad \text{(Using Lemma 3)}
\end{aligned} \tag{24}$$

This completes the proof.

III.4 LOWER BOUND (LB) ON PR$\{L = l\}$

The lower bound on the probabilities can be similarly found with the use of Lemma 2.

Theorem 2 (Misparking Chain Length Lower Bound) *The probability that the length of the misparking chain is l is bounded below as represented by inequality (25):*

$$\Pr\{L = l\} > \frac{2}{n-1}\left(\frac{H^{l}_{n-l+1}}{(l+1)!}\right) \tag{25}$$

Proof: From (24),

$$\begin{aligned}
\Pr\{L = l\} &= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\cdots \\
&\quad \cdots\sum_{y=z+1}^{n-1}\frac{1}{n-y+1}\sum_{x=y+1}^{n}\frac{1}{n-x+1} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\cdots\sum_{y=z+1}^{n-1}\frac{H_{n-y}}{n-y+1} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\cdots\sum_{z=w+1}^{n-2}\frac{1}{n-z+1}\sum_{p=1}^{n-z-1}\frac{H_p}{p+1}
\end{aligned}$$

Using lemma 2,

$$\begin{aligned}
&> \frac{2}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\cdots\sum_{z=w+1}^{n-2}\frac{H^{2}_{n-z-1}}{n-z+1}\frac{1}{3!} \\
&= \frac{2}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\sum_{j=i+1}^{n-l}\frac{1}{n-j+1}\cdots\sum_{p=1}^{n-w-2}\frac{H^{2}_{p}}{p+2}\frac{1}{3!}
\end{aligned}$$

Applying Lemma 2 iteratively in this fashion from right to left, we end up with

$$\begin{aligned}
\Pr\{L=l\} &> \frac{2}{n-1}\sum_{i=1}^{n-l+1}\frac{1}{n-i}\frac{H^{l-1}_{n-i-(l-2)}}{(l)!} \\
&= \frac{2}{n-1}\sum_{p=1}^{n-l+1}\frac{1}{p+(l-2)}\frac{H^{l-1}_{p}}{(l)!} \\
&> \frac{2}{n-1}\sum_{p=1}^{n-l+1}\frac{1}{p+(l-1)}\frac{H^{l-1}_{p}}{(l)!} \\
&> \frac{2}{n-1}\left(\frac{H^{l}_{n-l+1}}{(l+1)!}\right) \quad \text{(Using lemma 2)}
\end{aligned}$$

This completes the proof.

Combining Theorems 1 and 3,

Theorem 3 (Misparking Chain Length) *The probability that the length of the misparking chain is l is best represented by inequality (26):*

$$\frac{2}{n-1}\left(\frac{H^{l}_{n-l+1}}{(l+1)!}\right) < \Pr\{L=l\} < \frac{1}{n-1}\left(\frac{H^{l}_{n-l+1}}{l!}+\frac{e}{2}\right) \tag{26}$$

Proof: The result in (26) follows directly from theorems 1 and 3.

III.5 MOST PROBABLE MISPARKING CHAIN LENGTH

In this section, the upperbound on $\hat{L}$, the most probable chain length is determined. Inequalities (22) and (25) clearly establish bounds on probabilities of occurence of different lengths of misparking chain. Higher the difference between the upper and the lower bound, greater is the probability of occurence of the corresponding chain length. Let $g(l)$ be the function representing the difference between the bounds given by (22) and (25) respectively.

$$g(l) = \frac{1}{n-1}\left(\frac{l-1}{l+1}\frac{H^{l}_{n-l+1}}{l!}+\frac{e}{2}\right) \tag{27}$$

Also let,

$$f(l) = \frac{l-1}{l+1}\frac{H^{l}_{n-l+1}}{l!}. \tag{28}$$

Therefore, $max(f(l)) \Rightarrow max(g(l))$.

Also,$max(g(l)) = g(\hat{L})$.

Let us now study the properties of $f(l)$.

Lemma 7

$$f(l+1) < f(l), \forall l \geq \lceil H_{n-l} \rceil \tag{29}$$

Proof:

$$\begin{aligned} f(l+1) - f(l) &= \frac{l}{l+2}\frac{H_{n-l}^{l+1}}{(l+1)!} - \frac{l-1}{l+1}\frac{H_{n-l+1}^{l}}{l!} \\ &= \frac{l-1}{(l+1)!}\left(\frac{lH_{n-l}}{l^2+l-2}H_{n-l}^{l} - H_{n-l+1}^{l}\right) \\ &< 0, \forall l \geq \lceil H_{n-l} \rceil, \text{Since } 2 \leq l \leq n-1 \end{aligned}$$

or equivalently

$$f(l+1) - f(l) < 0, \forall l \geq \lceil H_n \rceil \tag{30}$$

Lemma 7 proves that $f(l)$ is decreasing in the range $[\lceil H_n \rceil, \infty]$, which implies $g(l)$ is decreasing in the range $[\lceil H_n \rceil, \infty]$. Since $g(\hat{L}) = max(g(l))$,

$$\hat{L} \leq \lceil H_n \rceil \tag{31}$$

It is a well known fact that $H_n = \Theta(\ln n)$. Therefore, the most probable chain length follows a $\Theta(\ln n)$ order of growth.

III.6 SIMULATION RESULTS

This section presents the analysis based results followed by a camparison of the same with the results obtained from simulation. The conditions assumed for simulation is same as those assumed in theoretical analysis. Each of the n available parking slots is reserved by one of the n drivers. A driver P_i, in the range P_1 to P_n is randomly selected to mispark in a randomly selected slot different from the one reseved by him/her. The remaining drivers park in their reserved slots if available or else park in one of the remaining available slots, which also is randomly selected. The length of the misparking chain is noted when the last driver in the chain parks in the slot reserved by the first driver to mispark, P_i.

Two different sets of simulation, one with 100000 runs and other with 1000000 runs are performed to compute the probabilities $\Pr\{L = l\}$, for different values of l. Let s denote the number of simulation runs. The theoretical bounds together with simulation results for $n = 500$,$s = 100000$ are depicted in Fig. 3. It is observed that

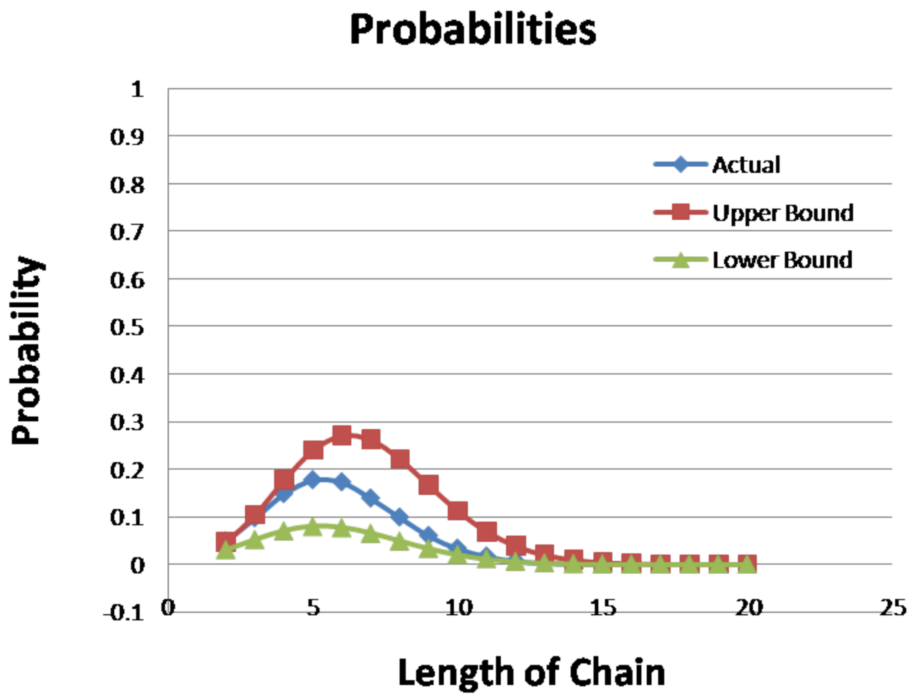

FIG. 3: Bounds vs simulation results, $n = 500$,$s = 100000$.

the curve representing the actual probability values obtained from simulation peaks at $\lfloor H_n \rfloor - 1 = \hat{L}$, the curve representing the theoretical upperbounds obtained from (22) peaks at $\lfloor H_n \rfloor$ and the curve representing the theoretical lowerbounds obtained from (25) peaks at $\lfloor H_n \rfloor - 1$. Thus, $\hat{L}$ is less than $\lceil H_n \rceil$ as claimed. It is clearly visible in Fig. 3 that the simulation based results never exceed the corresponding bounds. Next, s is increased to 1000000 keeping n constant at 500. Even in this case, the simulation results do not exceed the corresponding bounds as is evident from results of Fig. 4. The tightness of the proposed bounds, expressed as the difference between the upperbound and the lowerbound, is plotted in Fig. 5 for $n = 500$. An important observation made in Fig. 5 is that the maximum difference between the computed bounds is 0.196, which is 19.6% of the maximum possible difference since the probabilities are trivially bounded between 0 and 1. The average difference between the proposed bounds for $n = 500$ is 0.065 or 6.5% of the trivial maximum. Similar results for $n = 1000$, $n = 5000$ are presented next in figures 6, 7, 8. 9, 10 and 11. Even in these scenarios, it is observed that the curve representing the actual probabilities obtained through simulation peak at $\lfloor H_n \rfloor - 1$, the curve representing the

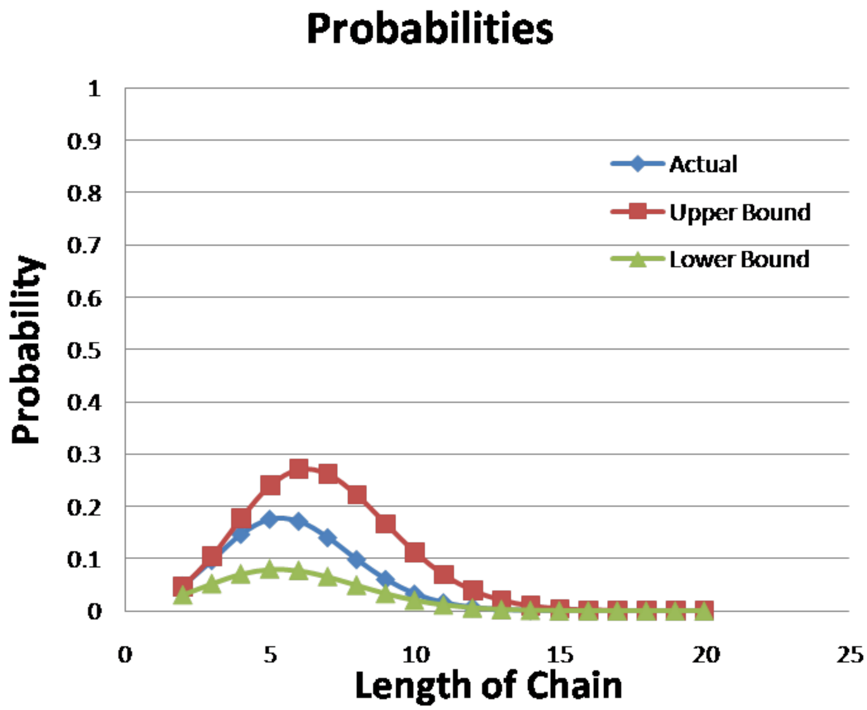

FIG. 4: Bounds vs simulation results, $n = 500$,$s = 1000000$.

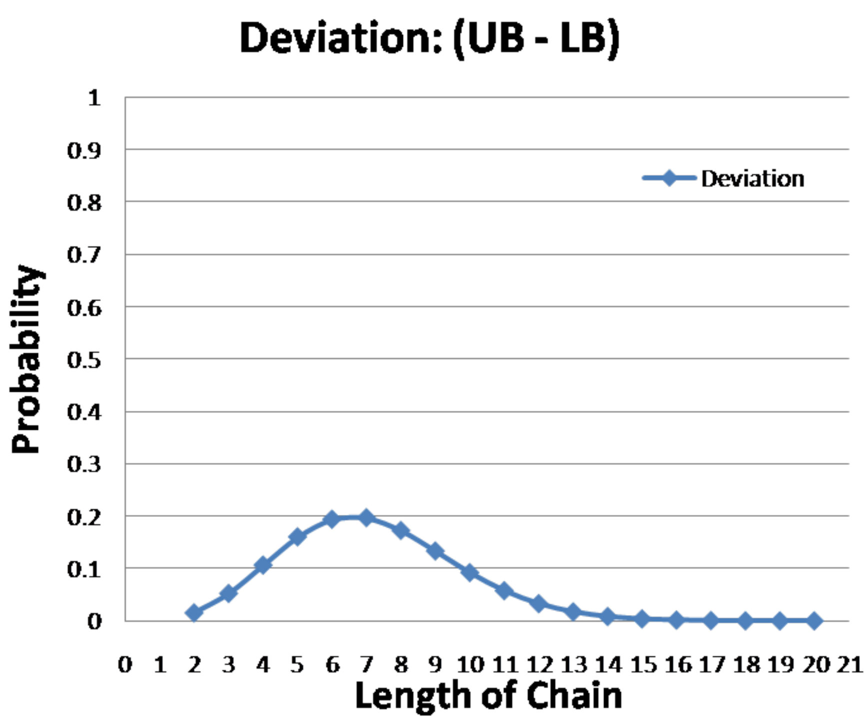

FIG. 5: Tightness of Bounds for $n = 500$, $s = 100000$ and $s = 1000000$.

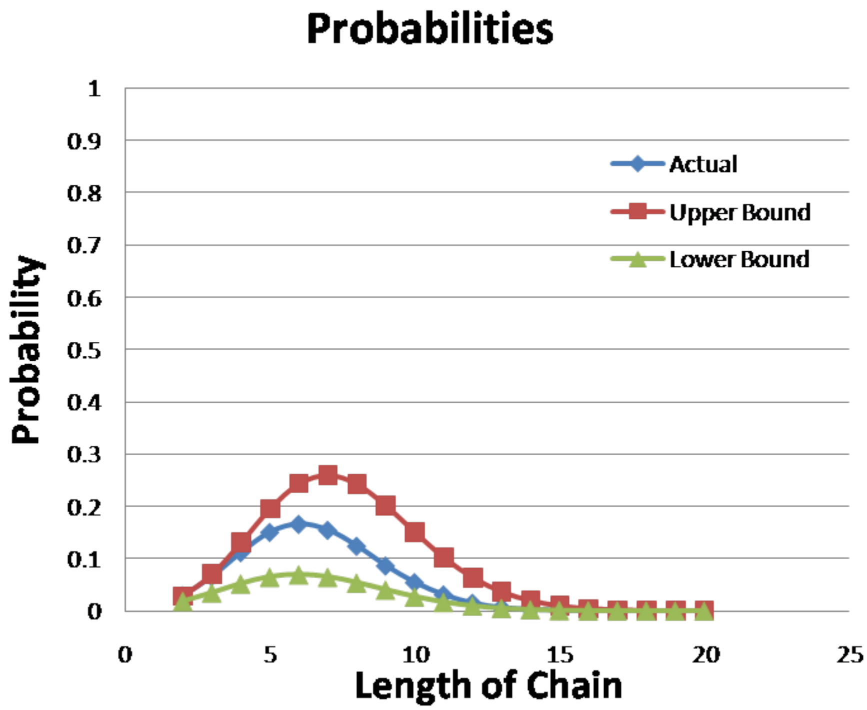

FIG. 6: Bounds vs simulation results, $n = 1000$,$s = 100000$.

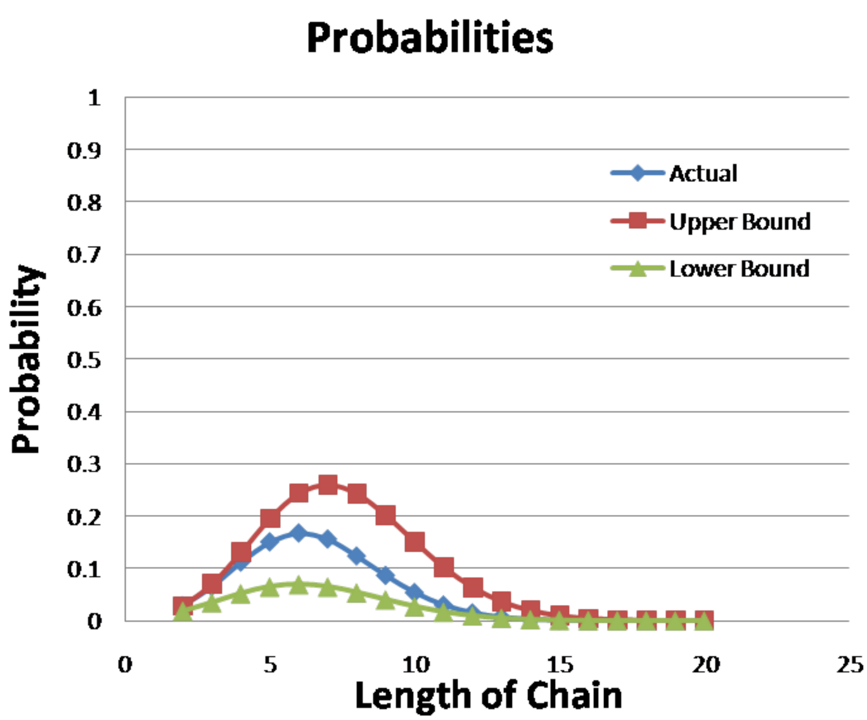

FIG. 7: Bounds vs simulation results, $n = 1000$,$s = 1000000$.

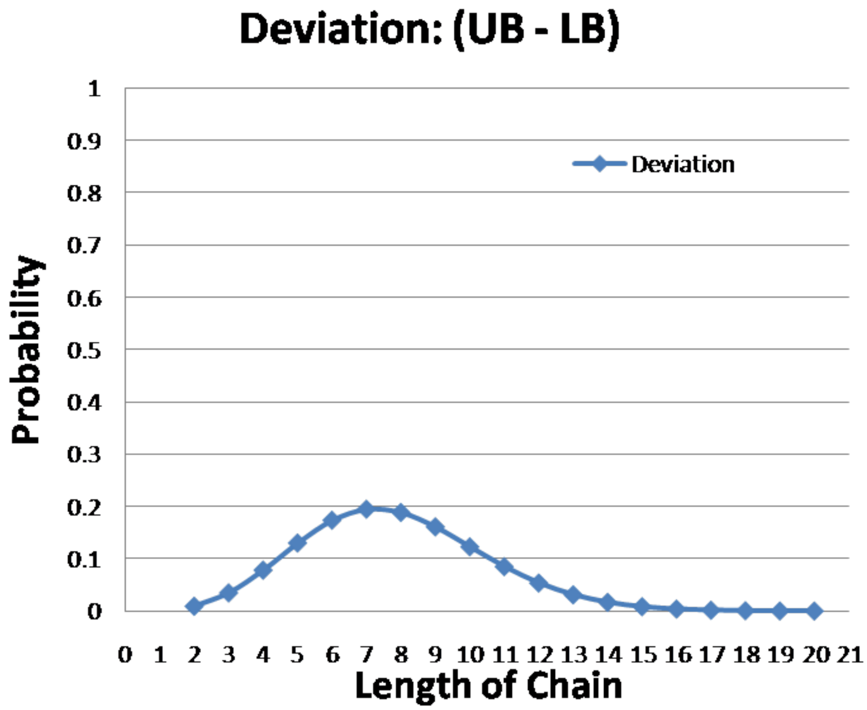

FIG. 8: Tightness of Bounds for $n = 1000$, $s = 100000$ and $s = 1000000$.

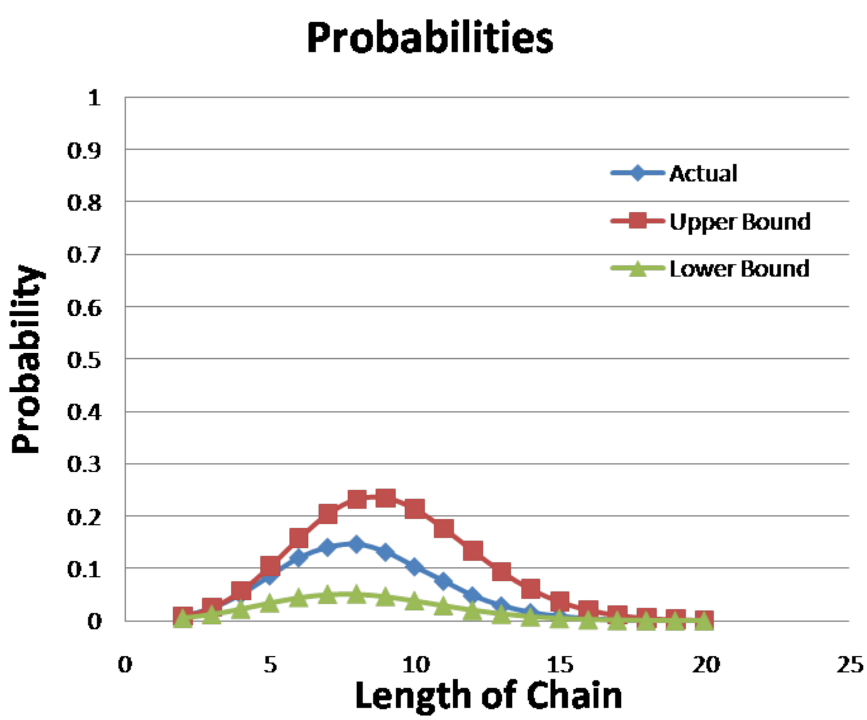

FIG. 9: Bounds vs simulation results, $n = 5000$,$s = 50000$.

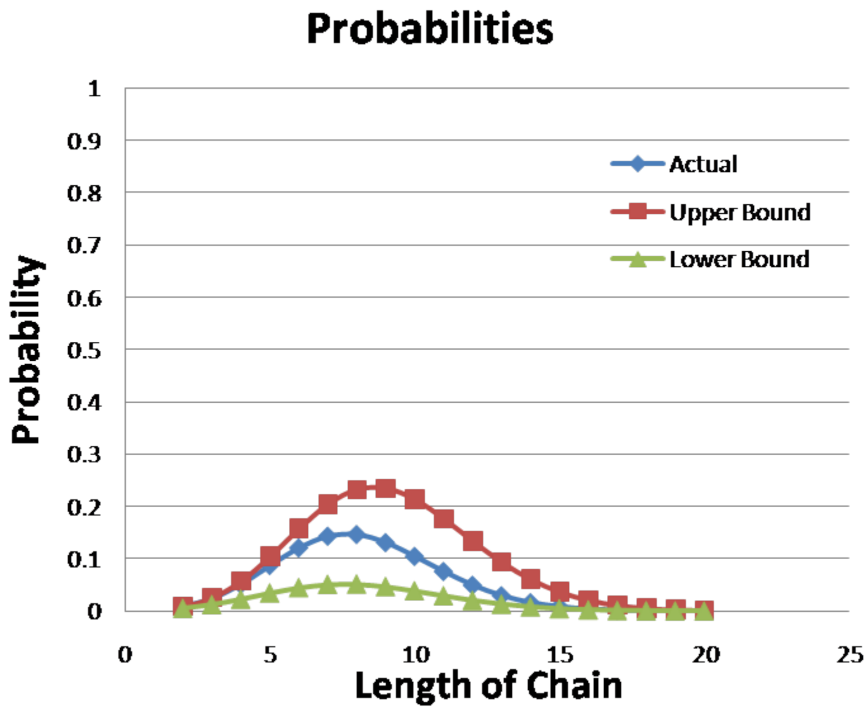

FIG. 10: Bounds vs simulation results, $n = 1000$,$s = 100000$.

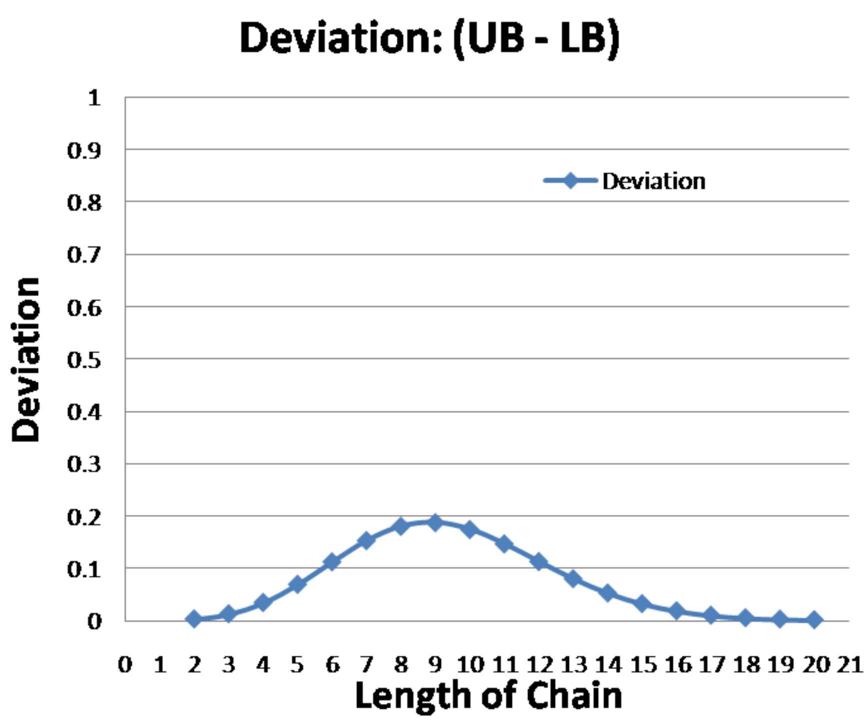

FIG. 11: Tightness of Bounds for $n = 5000$, $s = 50000$ and $s = 100000$.

computed upper bound peaks at $\lfloor H_n \rfloor$, whereas the curve representing the computed lower bound peaks at $\lfloor H_n \rfloor - 1$. In any case, $\hat{L} \leq \lceil H_n \rceil$ as claimed. The maximum and average difference between bounds for $n = 1000$ are 0.195(19.5%) and 0.068(6.8%) respectively. The maximum and average difference between bounds for $n = 5000$ are 0.187(18.7%) and 0.073(7.3%) respectively. In each of the above cases, it is observed that the range of chain lengths having significant probability of occurance is extremely small compared to the size of the garage. Overall, results depict that the impact of a mispark on the reservation based parking system is extremely low, considering the large size n of the parking garage. In other words, the system is self healing and does not require additional expensive infrastructure, thereby making reservation a reliable solution to the parking problem in large metros.

CHAPTER IV

A RECURSIVE SOLUTION

In this section, a recursive solution is proposed to compute the probability of occurence of any length p of the misparking chain. With the use of this recursive solution, an approximation to the actual probability is derived.

IV.1 FIRST DRIVER INITIATING MISPARK

Let $B_p(N)$ denote the event that a misparking chain of length p occurs in a garage of size N, given that the first driver to enter the garage misparks. Using the law of total probability, we get

$$\Pr\{B_p(N)\} = \sum_{j=2}^{N} \Pr\{B_p(N)|\{1 \to j\}\} \Pr\{1 \to j\} \tag{32}$$

where $p \geq 3$.

Now,

$$\Pr\{B_p(N)|\{1 \to j\}\} = \frac{N-j}{N-j+1} \Pr\{B_{p-1}(N-j+1)\} \tag{33}$$

where $\frac{N-j}{N-j+1}$ represents the probability that j does not close the chain. Therefore,

$$\Pr\{B_p(N)\} = \frac{1}{N-1} \sum_{j=2}^{N} \frac{N-j}{N-j+1} \Pr\{B_{p-1}(N-j+1)\} \tag{34}$$

Taking $k = N - j + 1$, we get (35).

$$\Pr\{B_p(N)\} = \frac{1}{N-1} \sum_{k=p-1}^{N-1} \frac{k-1}{k} \Pr\{B_{p-1}(k)\} \tag{35}$$

since $k \geq p - 1$.

Let $a_{p,N} = \Pr\{B_p(N)\}$. It is known from the previous section that,

$$a_{2,N} = \frac{H_{N-1}}{N-1} \tag{36}$$

Also from (35),

$$a_{p,N} = \frac{1}{N-1} \sum_{k=p-1}^{N-1} \frac{k-1}{k} a_{p-1,k}, p \geq 3 \tag{37}$$

Now,

$$\begin{aligned} a_{p,N+1} &= \frac{1}{N}\sum_{k=p-1}^{N}\frac{k-1}{k}a_{p-1,k} \\ &= \frac{1}{N}\sum_{k=p-1}^{N-1}\frac{k-1}{k}a_{p-1,k} + \frac{1}{N}\frac{N-1}{N}a_{p-1,N}, \end{aligned}$$

Therefore, from (35) and (38),

$$(N)a_{p,N+1} = (N-1)a_{p,n} + \frac{N-1}{N}a_{p-1,N} \tag{38}$$

or equivalently,

$$(N)a_{p,N+1} - (N-1)a_{p,N} = \frac{N-1}{N}a_{p-1,N}, p \geq 3 \tag{39}$$

IV.1.1 An Approximate Solution

In this subsection, it is proven that (39) is satisfied by

$$a_{p,N} = \frac{\ln^{p-1}(N-1)}{(p-1)!(N-1)}, \text{for large} N. \tag{40}$$

Some of the results specified in Apendix A are used by the prrofs provided in this section. Now, LHS of (39) after substituting (40) is given by,

$$N.\frac{\ln^{p-1}(N)}{(p-1)!N} - (N-1)\frac{\ln^{p-1}(N-1)}{(p-1)!(N-1)} = \frac{1}{(p-1)!}\bigg(\ln^{p-1}(N) - \ln^{p-1}(N-1)\bigg) \tag{41}$$

and RHS is given by,

$$\frac{N-1}{N}\frac{\ln^{p-2}(N-1)}{(N-1)(p-2)!} = \frac{\ln^{p-2}(N-1)}{(p-2)!N} \tag{42}$$

The goal is to show that $LHS \approx RHS$ for large N.

Theorem 4

$$\frac{1}{(p-1)!}\bigg(\ln^{p-1}(N) - \ln^{p-1}(N-1)\bigg) > \frac{\ln^{p-2}(N-1)}{(p-2)!N} \tag{43}$$

Proof:

To see this, observe that

$$\frac{1}{(p-1)!}\bigg(\ln^{p-1}(N) - \ln^{p-1}(N-1)\bigg) = \frac{1}{(p-1)!}\ln\left(\frac{N}{N-1}\right)\sum_{i=0}^{p-2}\ln^{p-2-i}(N)\ln^{i}(N-1) \tag{44}$$

by binomial expansion. Now,

$$\begin{aligned}
&\frac{1}{(p-1)!}\Bigg(\ln^{p-1}(N)-\ln^{p-1}(N-1)\Bigg)-\frac{\ln^{p-2}(N-1)}{(p-2)!N}\\
=\;& \frac{1}{(p-1)!N}\Bigg(N.\ln\left(\frac{N}{N-1}\right)\sum_{i=0}^{p-2}\ln^{p-2-i}(N)\ln^{i}(N-1)-(p-1)\ln^{p-2}(N-1)\Bigg)\\
>\;& \frac{1}{(p-1)!N}\Bigg(\sum_{i=0}^{p-2}\ln^{p-2-i}(N)\ln^{i}(N-1)-(p-1)\ln^{p-2}(N-1)\Bigg)\\
>\;& \frac{1}{(p-1)!N}\Bigg(\sum_{i=0}^{p-2}\ln^{p-2}(N-1)-(p-1)\ln^{p-2}(N-1)\Bigg)\\
=\;& 0
\end{aligned}$$

Hence proved.

Theorem 5

$$\frac{1}{(p-1)!}\Bigg(\ln^{p-1}(N)-\ln^{p-1}(N-1)\Bigg)-\frac{\ln^{p-2}(N-1)}{(p-2)!N}<\frac{1}{N-1} \tag{45}$$

Proof:

Let,

$$f(p,n)=\frac{1}{(p-1)!}\Bigg(\ln^{p-1}(N)-\ln^{p-1}(N-1)\Bigg)-\frac{\ln^{p-2}(N-1)}{(p-2)!N} \tag{46}$$

$$\begin{aligned}
=\;& \frac{1}{(p-1)!N}\Bigg(N(\ln^{p-1}(N)-\ln^{p-1}(N-1))-(p-1)\ln^{p-2}(N-1)\Bigg)\\
=\;& \frac{1}{(p-1)!N}\Bigg((N-1)(\ln^{p-1}(N)-\ln^{p-1}(N-1))+(\ln^{p-1}(N)-\ln^{p-1}(N-1))\\
& -(p-1)\ln^{p-2}(N-1)\Bigg)\\
=\;& \frac{1}{(p-1)!N}\Bigg((N-1)\ln\frac{N}{N-1}\sum_{i=0}^{p-2}\ln^{p-2-i}(N).\ln^{i}(N-1)+(\ln^{p-1}(N)\\
& -\ln^{p-1}(N-1))-(p-1)\ln^{p-2}(N-1)\Bigg)\\
\leq\;& \frac{1}{(p-1)!N}\Bigg(\sum_{i=0}^{p-2}\ln^{p-2-i}(N).\ln^{i}(N-1)-(p-1)\ln^{p-2}(N-1)\\
& +(\ln^{p-1}(N)-\ln^{p-1}(N-1))\Bigg)
\end{aligned} \tag{47}$$

Consider,

$$
\begin{aligned}
& \sum_{i=0}^{p-2} \ln^{p-2-i}(N).\ln^{i}(N-1) - (p-1)\ln^{p-2}(N-1) \\
= & \sum_{i=0}^{p-2} \ln^{i}(N-1)\Big(\ln^{p-2-i}(N) - \ln^{p-2-i}(N-1)\Big) \\
= & \sum_{i=0}^{p-2} \ln^{i}(N-1)\ln\frac{N}{N-1}\sum_{j=0}^{p-3-i} \ln^{p-3-i-j} N \ln^{j}(N-1) \\
< & \frac{1}{N-1}\sum_{i=0}^{p-2} \ln^{i}(N-1).(N-1)\ln(\frac{N}{N-1})\sum_{j=0}^{p-3-i} \ln^{p-3-i-j}(N)\ln^{j}(N) \\
< & \frac{1}{N-1}\sum_{i=0}^{p-2} \ln^{i}(N).(p-2-i)\ln^{p-3-i}(N) \\
= & \frac{1}{N-1}\ln^{p-3}(N)\sum_{i=0}^{p-2}(p-2-i) \\
= & \frac{1}{N-1}\frac{(p-2)(p-1)}{2}\ln^{p-3}(N) \\
< & \frac{1}{N-1}(p-2)(p-1)\ln^{p-3}(N) \qquad (48)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\ln^{p-1}(N) - \ln^{p-1}(N-1) &= \ln\left(\frac{N}{N-1}\right)\sum_{i=0}^{p-2} \ln^{p-2-i}(N).\ln^{i}(N-1) \\
&< \frac{1}{N-1}(N-1)\ln\left(\frac{N}{N-1}\right).(p-1)\ln^{p-2}(N) \\
&< \frac{1}{N-1}(p-1)\ln^{p-2}(N)
\end{aligned}
$$

Substituting (48) and (49) in (47), we get

$$
\begin{aligned}
& \frac{1}{N(N-1)(p-1)!}\Big((p-1)(p-2)\ln^{p-3} N + (p-1)\ln^{p-2} N\Big) \\
= & \frac{1}{N(N-1)}\left(\frac{\ln^{p-3} N}{(p-3)!} + \frac{\ln^{p-2} N}{(p-2)!}\right) \\
< & \frac{1}{N(N-1)}e^{\ln N} \\
= & \frac{1}{N-1}
\end{aligned}
$$

Hence proved. Therefore,

$$
0 < LHS - RHS < \frac{1}{N-1} \qquad (49)
$$

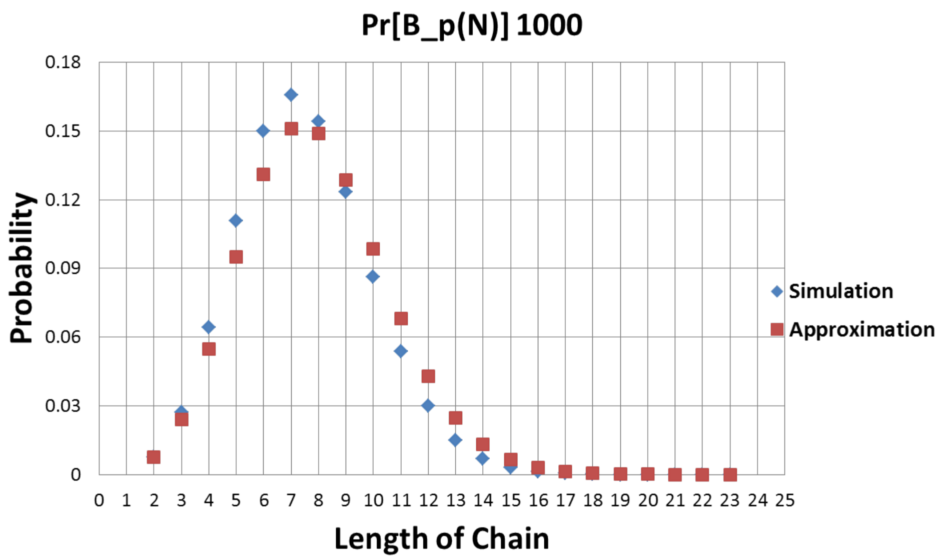

FIG. 12: Simulation vs Approximation, $n = 1000$.

Note that $\lim_{n\to\infty}\left(\frac{1}{N-1}\right) = 0$. Hence, $LHS \approx RHS$ for large N.

IV.1.2 Most Probable Chain Length Estimate

The proposed approximate solution $\frac{\ln^{p-1}(N-1)}{(p-1)!(N-1)}$ is a Poisson probability mass function with parameter $\lambda = \ln(N-1)$.
Since, λ is the expected value of a Poisson random variable, the most probable length of the misparking chain is approximately $\lambda = \ln(N-1)$.

IV.1.3 Results

In this subsection, the thoretical estimates proposed in the previous subsection are compared with the actual values obtained from simulation for various garage capacities. The number of simulation runs in all these experiments is one million. The graphs depicting the results are presented next. It can be observed from the previous graphs that the approximation is very close to the simulation values. Also, it can be observed that greater the value of N, better is the approximation. This fact is depicted in Fig. 16 and Fig. 17. It is clearly seen in Fig. 16 that the curve dampens towards the x axis with increase in N, thereby indicating that better approximations are obtained when the size of the garage is large. The average computed in each case

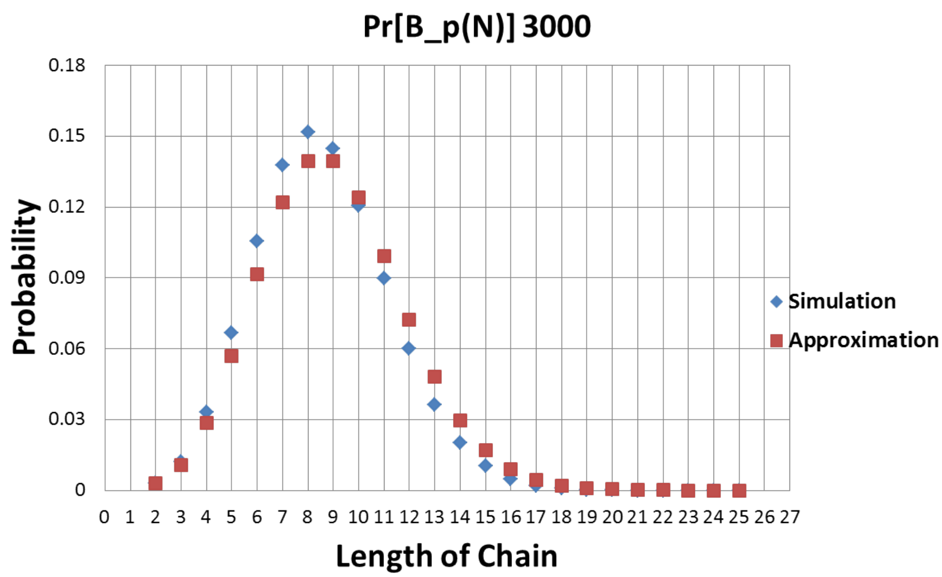

FIG. 13: Simulation vs Approximation, $n = 3000$.

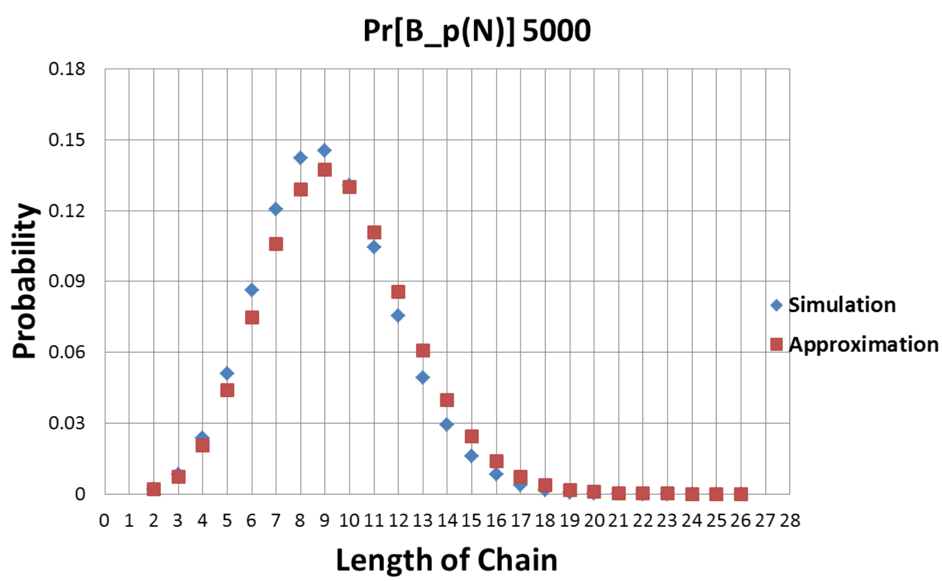

FIG. 14: Simulation vs Approximation, $n = 5000$.

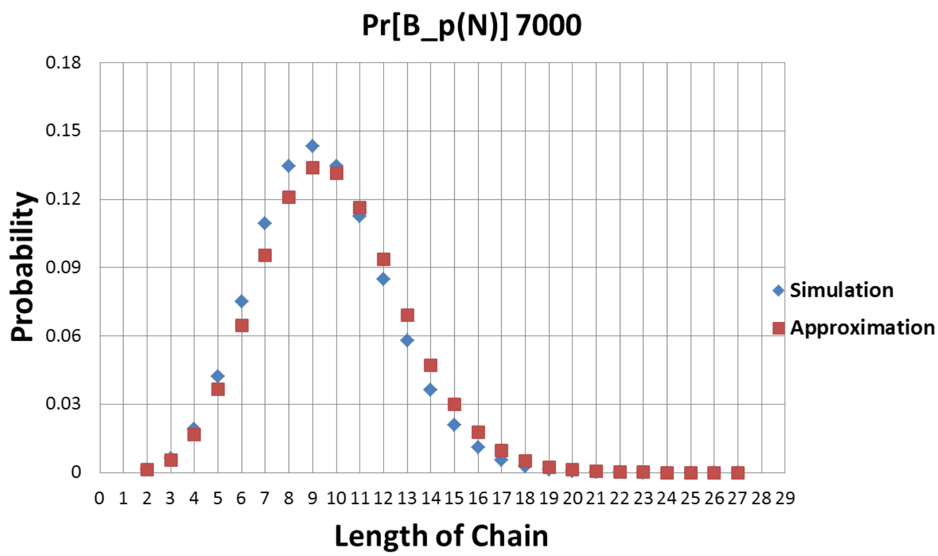

FIG. 15: Simulation vs Approximation, $n = 7000$.

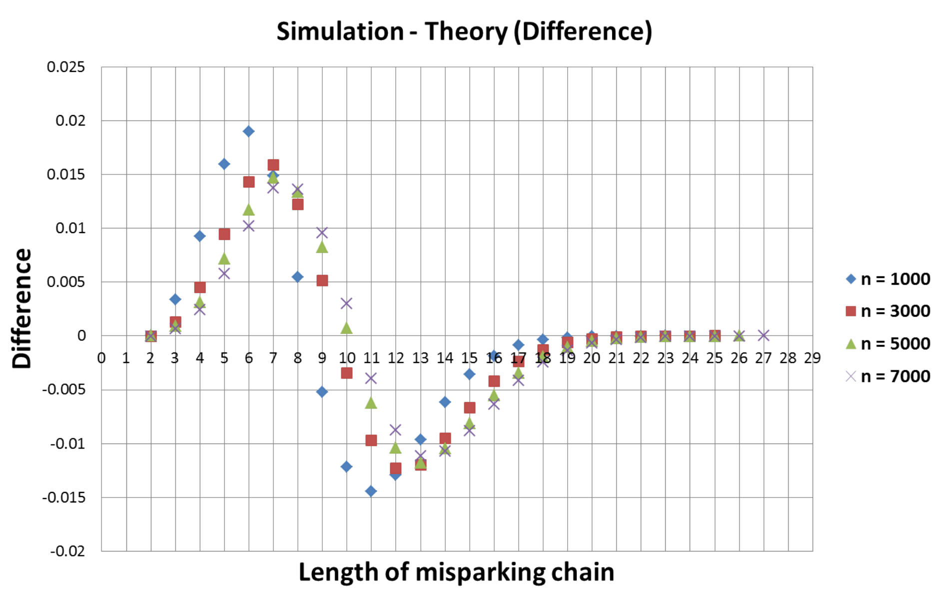

FIG. 16: Difference between simulation and approximation.

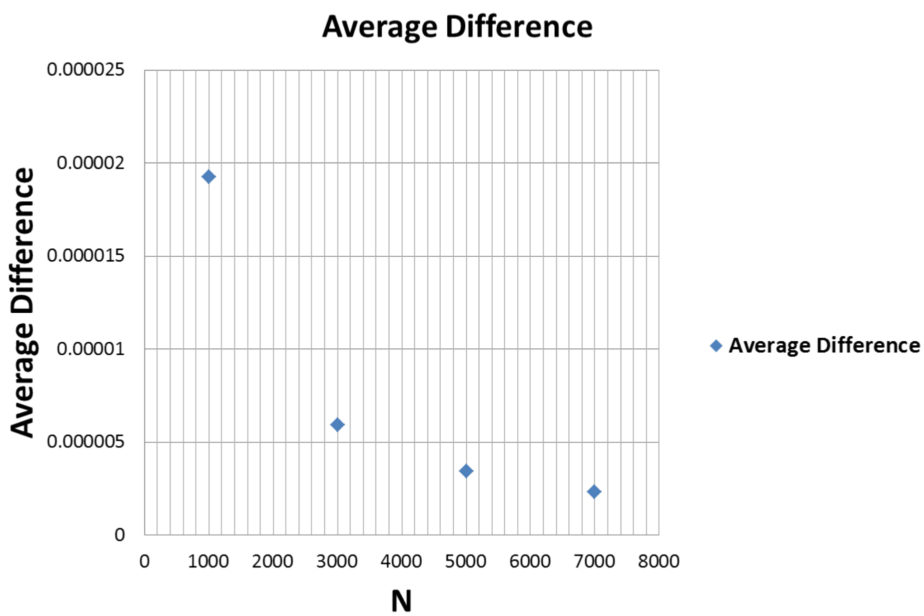

FIG. 17: Average difference between simulation and approximation.

is then plotted in Fig. 17. Even in this case, the average error decreases with increase in N. Also, notice in Fig. 12 that the curve representing simulation result peaks at 7, indicating that the most probable length of the misparking chain for $N = 1000$ is 7. Now, the curve representing the approximation also peaks at $\ln(1000 - 1) = 6.9 \approx 7$, thereby demonstrating the closeness of the approximation to the simulation result. Similar observations are made in Fig. 13, Fig. 14 and Fig. 15. The simulation curves in all these figures peak exactly at corresponding $\ln(N - 1)$. Therefore, it can be inferred that the most probable length of the misparking chain is extremely small compared to the size of the parking garage. In other words, misparking has negligible effect on the stability of a reservation based parking system and hence can be safely ignored while designing parking assistance mechanisms.

CHAPTER V

SUMMATION OF TERMS INVOLVING HARMONIC SERIES

In previous chapters, we encountered several complex expressions involving summations of terms involving harmonic sums. It is very important to determine the exact closed form equivalents to these terms or atleast good approximations when exact equivalents do not exist. This chapter deals with analysis of these complex terms. More specifically, a novel technique is proposed to determine the exact values of such complex expressions. In addition, certain interesting properties of these expressions are presented which were made use of in probabilistic analysis of misparking described in earlier chapters.

V.1 COMPUTATION OF $\sum_{K=1}^{N} \frac{H_K}{K}$ AND $\sum_{K=1}^{N} \frac{H_K^2}{K}$

First, the value of the expression $\sum_{k=1}^{n} \frac{H_k}{k}$ is computed. The proposed technique defines a custom matrix like arragement displayed below.

$$\begin{bmatrix} \frac{1}{1} \\ \\ \frac{1}{2} \\ \\ \frac{1}{3} \\ \\ \vdots \\ \\ \frac{1}{k} \\ \\ \vdots \\ \\ \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \ldots + \frac{1}{n} \\ \\ \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k} + \ldots + \frac{1}{n} \\ \\ \frac{1}{3} + \ldots + \frac{1}{k} + \ldots + \frac{1}{n} \\ \\ \vdots \\ \\ \vdots \\ \\ \frac{1}{n-1} + \frac{1}{n} \\ \\ \frac{1}{n} \end{bmatrix}$$

The above arrangement called ***Real*** is considered to be equivalent to the value given by the expression in (50). For each row of the right matrix, an intermediate value is generated by mutiplying the summation of all the values in that row with the corresponding row entry in the single column martix on the left side. The summation of all these intermediate values represents the value of ***Real***.

$$\frac{1}{1}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k}+\ldots+\frac{1}{n}*\frac{1}{n} \tag{50}$$

By rearraging the terms of the above equation, it can be observed that it is equivalent to (51).

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}=\sum_{k=1}^{n}\frac{H_k}{k} \tag{51}$$

The arrangement ***Real*** therefore represents $\sum_{k=1}^{n}\frac{H_k}{k}$. Next, ***Real*** is *completed* by filling the empty spaces as presented next. Let us refer to the new arrangement as the ***Rectangle***.

$$\begin{bmatrix}\frac{1}{1}\\ \\ \frac{1}{2}\\ \\ \frac{1}{3}\\ \\ \vdots\\ \\ \frac{1}{k}\\ \\ \vdots\\ \\ \frac{1}{n}\end{bmatrix}\begin{bmatrix}\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n}\\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n}\\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n}\\ \\ \vdots\\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n}\\ \\ \vdots\\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n}\end{bmatrix}$$

Following similar lines of interpretation as in ***Real***, the arrangement ***Rectangle*** is considered to be equivalent to (52).

$$\frac{1}{1}\sum_{k=1}^{n}\frac{1}{k}+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}+\ldots+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \tag{52}$$

(52) on simplification yields (53).

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{k=1}^{k}\frac{1}{k}=H_n^2 \tag{53}$$

The new arrangement ***Rectangle*** therefore represents H_n^2. Another arragement ***Lower Triangle*** consisting of all the terms present in ***Rectangle*** but not in ***Real*** is defined. The ***Lower Triangle*** arrangement is presented next.

$$\begin{bmatrix} \frac{1}{2} \\ \\ \frac{1}{3} \\ \\ \vdots \\ \\ \frac{1}{k} \\ \\ \vdots \\ \\ \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{1} \\ \\ \frac{1}{1}+\frac{1}{2} \\ \\ \vdots \\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k-1} \\ \\ \vdots \\ \\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n-1} \end{bmatrix}$$

The value corresponding to ***Lower Triangle*** is given by (54)

$$\sum_{k=2}^{n}\frac{1}{k}\sum_{i=1}^{k-1}\frac{1}{i}=\sum_{k=1}^{n}\frac{H_k}{k}-\sum_{k=1}^{n}\frac{1}{k^2} \tag{54}$$

It is evident that the relationship between the three arrangements described previously is given by

Rectangle=Real+Lower Triangle

or equivalently,

Real=Rectangle-Lower Triangle.

Therefore,

$$\sum_{k=1}^{n}\frac{H_k}{k}=H_n^2-\sum_{k=1}^{n}\frac{H_k}{k}+\sum_{k=1}^{n}\frac{1}{k^2} \tag{55}$$

which on simplification yields (56).

$$\sum_{k=1}^{n}\frac{H_k}{k}=\frac{1}{2}H_n^2+\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^2} \tag{56}$$

(56) provides an expression to determine the exact value of $\sum_{k=1}^{n}\frac{H_k}{k}$ Following a similar approach, the expression $\sum_{k=1}^{n}\frac{H_k^2}{k}$ is evaluated. The ***Real*** arrangement to

evaluate $\sum_{k=1}^{n} \frac{H_k^2}{k}$ is shown below.

$$\begin{bmatrix} \frac{1}{1} \\ \\ \frac{1}{2} \\ \\ \frac{1}{3} \\ \\ \vdots \\ \\ \frac{1}{k} \\ \\ \vdots \\ \\ \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{H_1}{1} + \frac{H_2}{2} + \frac{H_3}{3} + \ldots + \frac{H_k}{k} + \ldots + \frac{H_n}{n} \\ \\ \frac{H_2}{2} + \frac{H_3}{3} + \ldots + \frac{H_k}{k} + \ldots + \frac{H_n}{n} \\ \\ \frac{H_3}{3} + \ldots + \frac{H_k}{k} + \ldots + \frac{H_n}{n} \\ \\ \vdots \\ \\ \vdots \\ \\ \frac{H_{n-1}}{n-1} + \frac{H_n}{n} \\ \\ \frac{H_n}{n} \end{bmatrix}$$

Following a procedure similar to that used in evaluating $\sum_{k=1}^{n} \frac{H_k}{k}$, the following set of equations are obtained.

$$\textbf{\textit{Real}} = \sum_{k=1}^{n} \frac{H_k^2}{k} \tag{57}$$

$$\textbf{\textit{Rectangle}} = \sum_{k=1}^{n} \frac{1}{k} \sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} H_n^3 + \frac{1}{2} H_n \sum_{k=1}^{n} \frac{1}{k^2} \tag{58}$$

$$\textbf{\textit{Lower Triangle}} = \sum_{k=2}^{n} \frac{1}{k} \sum_{i=1}^{k-1} \frac{H_i}{i} = \frac{1}{2} \sum_{k=1}^{n} \frac{H_k^2}{k} + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i^2} - \sum_{k=1}^{n} \frac{H_k}{k^2} \tag{59}$$

Again, since ***Real*** = ***Rectangle*** - ***Lower Triangle***,

$$\sum_{k=1}^{n} \frac{H_k^2}{k} = \frac{1}{2} H_n^3 + \frac{1}{2} H_n \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{2} \sum_{k=1}^{n} \frac{H_k^2}{k} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i^2} + \sum_{k=1}^{n} \frac{H_k}{k^2} \tag{60}$$

which on simplification yields (61),

$$\sum_{k=1}^{n} \frac{H_k^2}{k} = \frac{1}{3} H_n^3 + \frac{1}{3} H_n \sum_{k=1}^{n} \frac{1}{k^2} - \frac{1}{3} \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i^2} + \frac{2}{3} \sum_{k=1}^{n} \frac{H_k}{k^2} \tag{61}$$

To proceed further it is necessary to evaluate $\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i^2}$. This is accomplished by repeating the same procedure for a different arrangement shown below.

$$\begin{bmatrix} \frac{1}{1^2} \\ \frac{1}{2^2} \\ \frac{1}{3^2} \\ \vdots \\ \frac{1}{k^2} \\ \vdots \\ \frac{1}{n^2} \end{bmatrix} \begin{bmatrix} \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n} \\ \frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n} \\ \frac{1}{3}+\ldots+\frac{1}{k}+\ldots+\frac{1}{n} \\ \vdots \\ \vdots \\ \frac{1}{n-1}+\frac{1}{n} \\ \frac{1}{n} \end{bmatrix}$$

From the above arrangement, it is clear that

$$\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i^2} = H_n\sum_{k=1}^{n}\frac{1}{k^2} - \sum_{k=1}^{n}\frac{H_k}{k^2} + \sum_{k=1}^{n}\frac{1}{k^3} \tag{62}$$

Substituting the value of $\sum_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{i^2}$ in (61) and solving, the result is obtained as shown in (63).

$$\sum_{k=1}^{n}\frac{H_k^2}{k} = \frac{1}{3}H_n^3 + \sum_{k=1}^{n}\frac{H_k}{k^2} - \frac{1}{3}\sum_{k=1}^{n}\frac{1}{k^3} \tag{63}$$

Similarly, the technique is applicable in evaluating $\sum_{k=1}^{n}\frac{H_k^3}{k}$, $\sum_{k=1}^{n}\frac{H_k^4}{k}$ and $\sum_{k=1}^{n}\frac{H_k^5}{k}$ by selecting the appropriate ***Real*** arrangements. The equations for the results corresponding to the above three expressions are shown next.

$$\sum_{k=1}^{n}\frac{H_k^3}{k} = \frac{1}{4}H_n^4 + \frac{3}{2}\sum_{k=1}^{n}\frac{H_k^2}{k^2} - \sum_{k=1}^{n}\frac{H_k}{k^3} + \frac{1}{4}\sum_{k=1}^{n}\frac{1}{k^4} \tag{64}$$

$$\sum_{k=1}^{n}\frac{H_k^4}{k} = \frac{1}{5}H_n^5 + 2\sum_{k=1}^{n}\frac{H_k^3}{k^2} - 2\sum_{k=1}^{n}\frac{H_k^2}{k^3} + \sum_{k=1}^{n}\frac{H_k}{k^4} - \frac{1}{5}\sum_{k=1}^{n}\frac{1}{k^5} \tag{65}$$

$$\sum_{k=1}^{n}\frac{H_k^5}{k} = \frac{1}{6}H_n^6 + \frac{5}{2}\sum_{k=1}^{n}\frac{H_k^4}{k^2} - \frac{10}{3}\sum_{k=1}^{n}\frac{H_k^3}{k^3} + \frac{5}{2}\sum_{k=1}^{n}\frac{H_k^2}{k^4} - \sum_{k=1}^{n}\frac{H_k}{k^5} + \frac{1}{6}\sum_{k=1}^{n}\frac{1}{k^6} \tag{66}$$

V.2 COMPTUTATION OF $\sum_{K=1}^{N} \frac{H_K^P}{K}, P > 0$

Next, a general equation to compute $\sum_{k=1}^{n} \frac{H_k^p}{k}, p > 0$ is determined.

Lemma 8

$$\sum_{k=1}^{n} \frac{H_k^p}{k} = a_0(p) H_n^{p+1} + a_1(p) \sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} + a_2(p) \sum_{k=1}^{n} \frac{H_k^{p-2}}{k^3} + \ldots + a_p(p) \sum_{k=1}^{n} \frac{1}{k^{p+1}} \quad (67)$$

with the condition that $\sum_{i=0}^{p} a_i(p) = 1$.

Proof: By induction on p.
Base case: With reference to earlier derivations, the above hypothesis is easy verified for the base case $p = 1$.
Inductive step: Using the same procedure as before the value of $\sum_{k=1}^{n} \frac{H_k^{p+1}}{k}$ is determined using an arrangement shown below.

$$\begin{bmatrix} \frac{1}{1} \\ \\ \frac{1}{2} \\ \\ \frac{1}{3} \\ \\ \vdots \\ \\ \frac{1}{k} \\ \\ \vdots \\ \\ \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{H_1^p}{1} + \frac{H_2^p}{2} + \frac{H_3^p}{3} + \ldots + \frac{H_k^p}{k} + \ldots + \frac{H_n^p}{n} \\ \\ \frac{H_2^p}{2} + \frac{H_3^p}{3} + \ldots + \frac{H_k^p}{k} + \ldots + \frac{H_n^p}{n} \\ \\ \frac{H_3^p}{3} + \ldots + \frac{H_k^p}{k} + \ldots + \frac{H_n^p}{n} \\ \\ \vdots \\ \\ \vdots \\ \\ \frac{H_{n-1}^p}{n-1} + \frac{H_n^p}{n} \\ \\ \frac{H_n^p}{n} \end{bmatrix}$$

With respect to the arrangement shown above, the following set of equations are obtained.

$$\textbf{\textit{Real}} = \sum_{k=1}^{n} \frac{H_k^{p+1}}{k} \tag{68}$$

$$\textbf{\textit{Rectangle}} = H_n \sum_{k=1}^{n} \frac{H_k^{p}}{k} \tag{69}$$

$$\textbf{\textit{Lower triangle}} = \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p}}{i} - \sum_{k=1}^{n} \frac{H_k^{p}}{k^2} \tag{70}$$

Therefore,

$$\sum_{k=1}^{n} \frac{H_k^{p+1}}{k} = H_n \sum_{k=1}^{n} \frac{H_k^{p}}{k} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p}}{i} + \sum_{k=1}^{n} \frac{H_k^{p}}{k^2} \tag{71}$$

By inductive hypothesis, substituting the values of $\sum_{k=1}^{n} \frac{H_k^p}{k}$ and $\sum_{i=1}^{k} \frac{H_i^p}{i}$ in equation (71),

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^{p+1}}{k} &= H_n \Big[a_0(p) H_n^{p+1} + a_1(p) \sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} + a_2(p) \sum_{k=1}^{n} \frac{H_k^{p-2}}{k^3} + \ldots \\ &\quad + a_p(p) \sum_{k=1}^{n} \frac{1}{k^{p+1}}\Big] - \sum_{k=1}^{n} \frac{1}{k} \Big[a_0(p) H_k^{p+1} + a_1(p) \sum_{i=1}^{k} \frac{H_i^{p-1}}{i^2} \\ &\quad + a_2(p) \sum_{i=1}^{k} \frac{H_i^{p-2}}{i^3} + \ldots + a_p(p) \sum_{i=1}^{k} \frac{1}{i^{p+1}}\Big] + \sum_{k=1}^{n} \frac{H_k^{p}}{k^2}\end{aligned}$$

which implies,

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^{p+1}}{k} &= a_0(p) H_n^{p+2} + a_1(p) H_n \sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} + a_2(p) H_n \sum_{k=1}^{n} \frac{H_k^{p-2}}{k^3} + \ldots \\ &\quad + a_p(p) H_n \sum_{k=1}^{n} \frac{1}{k^{p+1}} - a_0(p) \sum_{k=1}^{n} \frac{H_k^{p+1}}{k} - a_1(p) \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-1}}{i^2} \\ &\quad - a_2(p) \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-2}}{i^3} - \ldots - a_p(p) \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i^{p+1}} + \sum_{k=1}^{n} \frac{H_k^{p}}{k^2}\end{aligned}$$

which on simplification gives,

$$\begin{aligned}\Big[1 + a_0(p)\Big] \sum_{k=1}^{n} \frac{H_k^{p+1}}{k} &= a_0(p) H_n^{p+2} + a_1(p) \Big[H_n \sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-1}}{i^2}\Big] \\ &\quad + a_2(p) \Big[H_n \sum_{k=1}^{n} \frac{H_k^{p-2}}{k^3} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-2}}{i^3}\Big] + \ldots \\ &\quad + a_p(p) \Big[H_n \sum_{k=1}^{n} \frac{1}{k^{p+1}} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{1}{i^{p+1}}\Big] + \sum_{k=1}^{n} \frac{H_k^{p}}{k^2}\end{aligned}$$

Let,

$$\alpha_j(n) = H_n \sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+1}} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-j}}{i^{j+1}} \tag{72}$$

It is clearly evident that $\alpha_j(p) = 0$.

Therefore,

$$\begin{aligned}
\alpha_j(n+1) &= H_{n+1} \sum_{k=1}^{n+1} \frac{H_k^{p-j}}{k^{j+1}} - \sum_{k=1}^{n+1} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-j}}{i^{j+1}} \\
&= \left[H_n + \frac{1}{n+1}\right]\left[\sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+1}} + \frac{H_{n+1}^{p-j}}{(n+1)^{j+1}}\right] \\
&\quad - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-j}}{i^{j+1}} - \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{H_i^{p-j}}{i^{j+1}} \\
&= \left[H_n \sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+1}} - \sum_{k=1}^{n} \frac{1}{k} \sum_{i=1}^{k} \frac{H_i^{p-j}}{i^{j+1}}\right] + \frac{H_n H_{n+1}^{p-j}}{(n+1)^{j+1}} \\
&\quad + \frac{1}{n+1} \sum_{k=1}^{n+1} \frac{H_k^{p-j}}{k^{j+1}} - \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{H_i^{p-j}}{i^{j+1}} \\
&= \alpha_j(n) + \frac{\left[H_{n+1} - \frac{1}{n+1}\right] H_{n+1}^{p-j}}{(n+1)^{j+1}} \\
&= \alpha_j(n) + \frac{H_{n+1}^{p-j+1}}{(n+1)^{j+1}} - \frac{H_{n+1}^{p-j}}{(n+1)^{j+2}}
\end{aligned}$$

On solving the recurrence,

$$\alpha_j(n) = \sum_{k=1}^{n} \frac{H_k^{p-j+1}}{k^{j+1}} - \sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+2}} \tag{73}$$

Therefore,

$$\begin{aligned}\left[1+a_0(p)\right]\sum_{k=1}^{n}\frac{H_k^{p+1}}{k} &= a_0(p)H_n^{p+2}+\sum_{k=1}^{n}\frac{H_k^p}{k^2}+a_1(p)\left[\sum_{k=1}^{n}\frac{H_k^p}{k^2}-\sum_{k=1}^{n}\frac{H_k^{p-1}}{k^3}\right]\\ &\quad +a_2(p)\left[\sum_{k=1}^{n}\frac{H_k^{p-1}}{k^3}-\sum_{k=1}^{n}\frac{H_k^{p-2}}{k^4}\right]+a_3(p)\left[\sum_{k=1}^{n}\frac{H_k^{p-2}}{k^4}\right.\\ &\quad \left.-\sum_{k=1}^{n}\frac{H_k^{p-3}}{k^5}\right]+\ldots+a_p(p)\left[\sum_{k=1}^{n}\frac{H_k}{k^{p+1}}-\sum_{k=1}^{n}\frac{1}{k^{p+2}}\right]\\ &= a_0(p)H_n^{p+2}+\left[1+a_1(p)\right]\sum_{k=1}^{n}\frac{H_k^p}{k^2}+\left[a_2(p)\right.\\ &\quad \left.-a_1(p)\right]\sum_{k=1}^{n}\frac{H_k^{p-1}}{k^3}+\ldots+\left[a_p(p)-a_{p-1}(p)\right]\sum_{k=1}^{n}\frac{H_k}{k^{p+1}}\\ &\quad -a_p(p)\sum_{k=1}^{n}\frac{1}{k^{p+2}}\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=1}^{n}\frac{H_k^{p+1}}{k} &= \frac{a_0(p)}{1+a_0(p)}H_n^{p+2}+\frac{1+a_1(p)}{1+a_0(p)}\sum_{k=1}^{n}\frac{H_k^p}{k^2}\\ &\quad +\frac{a_2(p)-a_1(p)}{1+a_0(p)}\sum_{k=1}^{n}\frac{H_k^{p-1}}{k^3}+\ldots+\frac{a_p(p)-a_{p-1}(p)}{1+a_0(p)}\sum_{k=1}^{n}\frac{H_k}{k^{p+1}}\\ &\quad -\frac{a_p(p)}{1+a_0(p)}\sum_{k=1}^{n}\frac{1}{k^{p+2}}\end{aligned}$$

The coefficients are $\frac{a_0(p)}{1+a_0(p)},\frac{1+a_1(p)}{1+a_0(p)},\frac{a_2(p)-a_1(p)}{1+a_0(p)},\frac{a_p(p)-a_{p-1}(p)}{1+a_0(p)},\ldots$ and $-\frac{a_p(p)}{1+a_0(p)}$.

It can be easily verified that sum of coefficients is 1.
This completes the proof.

V.2.1 Evaluating Coefficients

In this section, a closed form solution for evaluating the coefficients $a_i(p), 0 \leq i \leq p$ used in the previous section is provided.
The known values of $a_i(p), 0 \leq i \leq p$ as shown below are analyzed to frame a general

closed form solution for $a_i(p)$.

$$\begin{bmatrix} \\ p=1 \\ \\ p=2 \\ \\ p=3 \end{bmatrix} \begin{bmatrix} a_0(p) & a_1(p) & a_2(p) & a_3(p) & \dots \\ \\ \frac{1}{2} & \frac{1}{2} & & & \\ \\ \frac{1}{3} & 1 & -\frac{1}{3} & & \\ \\ \frac{1}{4} & \frac{3}{2} & -1 & \frac{1}{4} & \end{bmatrix}$$

The analysis begins with $a_0(p)$.

Lemma 9

$$a_0(p) = \frac{1}{p+1} \tag{74}$$

Proof: From the derivations in the preceding section, it is known that $a_0(p+1) = \frac{a_0(p)}{1+a_0(p)}$ If $a_0(p) = \frac{1}{p+1}$, then $a_0(p+1) = \frac{\frac{1}{p+1}}{\frac{p+2}{p+1}} = \frac{1}{p+2}$.
Hence proved.

Lemma 10

$$a_1(p) = \frac{p}{2} \tag{75}$$

Proof: Again, $a_1(p+1) = \frac{1+a_1(p)}{1+a_0(p)}$ If $a_0(p) = \frac{1}{p+1}, a_1(p) = \frac{p}{2}$, then $a_1(p+1) = \frac{\frac{p+2}{2}}{\frac{p+2}{p+1}} = \frac{p+1}{2}$.
Hence proved.

Following similar lines for proofs, the following solutions are obtained for coefficients $a_2(p), a_3(p) and a_4(p)$.

$$\begin{aligned} a_2(p) &= -\frac{p(p-1)}{6} = -\frac{p(p-1)}{3!} = -\frac{1}{3}\binom{p}{2} \\ a_3(p) &= \frac{p(p-1)(p-2)}{4!} = \frac{1}{4}\binom{p}{3} \\ a_4(p) &= -\frac{1}{5}\binom{p}{4} \end{aligned}$$

In general,

$$a_j(p) = (-1)^{j+1}\frac{1}{j+1}\binom{p}{j}. \tag{76}$$

It is required to verify that,

$$\sum_{j=1}^{p} a_j(p) = \frac{p}{p+1}. \tag{77}$$

$$\begin{aligned}
\sum_{j=1}^{p}(-1)^{j+1}\frac{1}{j+1}\binom{p}{j} &= \sum_{j=1}^{p}(-1)^{j+1}\frac{1}{p+1}\binom{p+1}{j+1} \\
&= \frac{1}{p+1}\sum_{j=1}^{p}(-1)^{j+1}\binom{p+1}{j+1} \\
&= \frac{1}{p+1}\sum_{k=2}^{p+1}(-1)^{k}\binom{p+1}{k} \\
&= \frac{1}{p+1}\left[\sum_{k=0}^{p+1}(-1)^{k}\binom{p+1}{k} - 1 + (p+1)\right] \\
&= \frac{1}{p+1}\left[0 - \frac{1}{p+1} + 1\right] \\
&= \frac{p}{p+1}
\end{aligned}$$

Hence the claim. Also,

$$\begin{aligned}
a_j(p+1) &= \frac{(-1)^{j+1}\frac{1}{j+1}\binom{p}{j} - (-1)^{j}\frac{1}{j}\binom{p}{j-1}}{\frac{p+2}{p+1}} \\
&= (-1)^{j+1}\frac{p+1}{p+2}\left[\frac{1}{j+1}\binom{p}{j} + \frac{1}{j}\binom{p}{j-1}\right] \\
&= (-1)^{j+1}\frac{p+1}{p+2}\frac{1}{j+1}\left[\binom{p}{j} + \frac{j+1}{j}\binom{p}{j-1}\right] \\
&= (-1)^{j+1}\frac{p+1}{p+2}\frac{1}{j+1}\left[\binom{p}{j} + \binom{p}{j-1} + \frac{1}{j}\binom{p}{j-1}\right] \\
&= (-1)^{j+1}\frac{p+1}{p+2}\frac{1}{j+1}\left[\binom{p+1}{j} + \frac{1}{p+1}\frac{p+1}{j}\binom{p}{j-1}\right] \\
&= (-1)^{j+1}\frac{p+1}{p+2}\frac{1}{j+1}\left[\binom{p+1}{j} + \frac{1}{p+1}\binom{p+1}{j}\right] \\
&= (-1)^{j+1}\frac{p+1}{p+2}\frac{1}{j+1}\binom{p+1}{j}\left[1 + \frac{1}{p+1}\right] \\
&= (-1)^{j+1}\frac{1}{j+1}\binom{p+1}{j}.
\end{aligned}$$

The previous derivation shows that

$$a_j(p) = (-1)^{j+1}\frac{1}{j+1}\binom{p}{j}, 1 \le j \le p \tag{78}$$

Using the above derivations, (67) can be rewritten as (79).

$$\sum_{k=1}^{n} \frac{H_k^p}{k} = \frac{H_n^{p+1}}{p+1} + \sum_{j=1}^{p} (-1)^{j+1} \frac{1}{j+1} \binom{p}{j} \sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+1}} \tag{79}$$

(79) can be further simplified to (80).

$$\sum_{k=1}^{n} \frac{H_k^p}{k} = \frac{1}{p+1} \left[H_n^{p+1} + \sum_{j=1}^{p} (-1)^{j+1} \binom{p+1}{j+1} \sum_{k=1}^{n} \frac{H_k^{p-j}}{k^{j+1}} \right] \tag{80}$$

V.3 PROPERTIES OF $\sum_{K=1}^{N} \frac{H_K^P}{K}$

Some of the interesting properties of $\sum_{k=1}^{n} \frac{H_k^p}{k}$ are highlighted in this section.

Lemma 11

$$\sum_{k=1}^{n} \frac{H_k^p}{k} \geq \frac{1}{p+1} H_n^{p+1} \tag{81}$$

Proof: By induction on n.

Base case: For $n = 1$, we get $H_1^p = 1 \geq \frac{1}{p+1} H_1^{p+1} = \frac{1}{p+1}$, which is true for all $p \geq 0$.

Inductive step: Let the hypothesis be true for any arbitrary integer n.

It has to be proved that

$$\sum_{k=1}^{n+1} \frac{H_k^p}{k} \geq \frac{1}{p+1} H_{n+1}^{p+1}. \tag{82}$$

Now,

$$\sum_{k=1}^{n+1} \frac{H_k^p}{k} = \sum_{k=1}^{n} \frac{H_k^p}{k} + \frac{H_n^p}{n+1} \geq \frac{H_n^{p+1}}{p+1} + \frac{H_n^p}{n+1}. \tag{83}$$

It follows that (82) is established as soon as it is proven that

$$\frac{H_n^{p+1}}{p+1} + \frac{H_n^p}{n+1} \geq \frac{1}{p+1} H_{n+1}^{p+1}. \tag{84}$$

Multiplying (84) by $\frac{p+1}{H_{n+1}^{p+1}}$, we get

$$\begin{aligned} \left(\frac{H_n}{H_{n+1}}\right)^{p+1} + \frac{p+1}{n+1} . \frac{1}{H_{n+1}} &\geq 1 \\ => \left(\frac{H_{n+1} - \frac{1}{n+1}}{H_{n+1}}\right)^{p+1} + \frac{p+1}{n+1} . \frac{1}{H_{n+1}} &\geq 1 \\ => \left(1 - \frac{1}{(n+1)H_{n+1}}\right)^{p+1} + \frac{p+1}{n+1} . \frac{1}{H_{n+1}} &\geq 1 \end{aligned}$$

It is straightforward to show that

$$(1-x)^n \geq 1-nx, \forall x \epsilon R, n \epsilon N \tag{85}$$

With the above inequality (85),

$$\begin{aligned}
&\left(1-\frac{1}{(n+1)H_{n+1}}\right)^{p+1} + \frac{p+1}{n+1}.\frac{1}{H_{n+1}} \\
&\geq 1-\frac{p+1}{n+1}.\frac{1}{H_{n+1}} + \frac{p+1}{n+1}.\frac{1}{H_{n+1}} \\
&\geq 1
\end{aligned}$$

This completes the proof. The above result clearly shows that the tail of (79) is greater than 0. That is,

$$\sum_{j=1}^{p}(-1)^{j+1}\frac{1}{j+1}\binom{p}{j}\sum_{k=1}^{n}\frac{H_k^{p-j}}{k^{j+1}} \geq 0. \tag{86}$$

Next an upper bound for the tail is established. In order to establish an upper bound for the tail the following claims are put forth.

Claim 1:

$$(1-x)^n \leq 1-nx+\frac{n(n-1)}{2}x^2, 0 \leq x \leq 1, n \geq 1 \tag{87}$$

Proof:

Proof for Claim 1 is provided by induction next.

Basis: It is staightforward that the inequality holds when $n = 1$.

Inductive step:

Let n be any arbitrary integer for which the hypothesis holds. It is to be shown that,

$$(1-x)^{n+1} \leq 1-(n+1)x+\frac{n(n+1)}{2}x^2, 0 \leq x \leq 1, n \geq 1. \tag{88}$$

Now,

$$\begin{aligned}
(1-x)^{n+1} = (1-x)(1-x)^n &\leq (1-x)\left(1-nx+\frac{n(n-1)}{2}x^2\right) \\
&= 1-nx+\frac{n(n-1)}{2}x^2-x+nx^2-\frac{n(n-1)}{2}x^3 \\
&= 1-(n+1)x+\frac{n(n+1)}{2}x^2-\frac{n(n-1)}{2}x^3 \\
&\leq 1-(n+1)x+\frac{n(n+1)}{2}x^2
\end{aligned}$$

Hence proved. Let the tail of (79) be represented as shown in (89).

$$T(n,p) = \frac{1}{p+1}\sum_{i=2}^{p+1}(-1)^i\binom{p+1}{i}\sum_{k=1}^{n}\frac{H_k^{p+1-i}}{k^i} \tag{89}$$

<u>Claim 2:</u>

$$T(n,p) - T(n-1,p) \leq \frac{p}{2}\frac{H_n^{p-1}}{n^2} \tag{90}$$

<u>Proof:</u>

$$\begin{aligned}
T(n,p) - T(n-1,p) &= \frac{1}{p+1}\sum_{i=2}^{p+1}(-1)^i\binom{p+1}{i}\frac{H_n^{p+1-i}}{n^i} \\
&= \frac{1}{p+1}\left[\sum_{i=0}^{p+1}(-1)^i\binom{p+1}{i}\frac{H_n^{p+1-i}}{n^i} - H_n^{p+1} + (p+1)\frac{H_n^p}{n}\right] \\
&= \frac{1}{p+1}\left[\left(H_n - \frac{1}{n}\right)^{p+1} - H_n^{p+1} + (p+1)\frac{H_n^p}{n}\right] \\
&= \frac{H_n^{p+1}}{p+1}\left[\left(1 - \frac{1}{nH_n}\right)^{p+1} - 1 + (p+1)\frac{1}{nH_n}\right] \\
&\leq \frac{H_n^{p+1}}{p+1}\left[1 - \frac{p+1}{nH_n} + \frac{p(p+1)}{2}\frac{1}{n^2H_n^2} - 1 + \frac{p+1}{nH_n}\right] \\
&= \frac{p}{2}\frac{H_n^{p-1}}{n^2}
\end{aligned}$$

Hence the proof.

By Claim 2 we have

$$\begin{aligned}
T(n,p) - T(n-1,p) &\leq \frac{p}{2}\frac{H_n^{p-1}}{n^2} \\
T(n-1,p) - T(n-2,p) &\leq \frac{p}{2}\frac{H_{n-1}^{p-1}}{(n-1)^2} \\
T(n-2,p) - T(n-3,p) &\leq \frac{p}{2}\frac{H_{n-2}^{p-1}}{(n-2)^2} \\
&\vdots \\
T(2,p) - T(1,p) &\leq \frac{p}{2}\frac{H_2^{p-1}}{2^2}
\end{aligned}$$

It is known that $T(1,p) = \frac{p}{p+1}$. Summing up the above equations, we get

$$T(n,p) - T(1,p) \leq \frac{p}{2}\sum_{k=2}^{n}\frac{H_k^{p-1}}{k^2}. \tag{91}$$

It follows that

$$
\begin{aligned}
T(n,p) &\le \frac{p}{p+1} + \sum_{k=2}^{n} \frac{H_k^{p-1}}{k^2} \\
&\le \frac{p}{p+1} + \sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} - \frac{p}{2} \\
&\le \frac{p}{2}\left[\sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} - \frac{1-p}{1+p}\right] \\
&\le \frac{p}{2}\left[\sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2}\right]
\end{aligned}
$$

In one of the subsequent derivations, it is shown that

$$\sum_{k=1}^{n} \frac{H_k^{p-1}}{k^2} < e.(p-1)!. \tag{92}$$

Therefore,

$$T(n,p) < \frac{e.p!}{2}. \tag{93}$$

Or equivalently,

$$\frac{T(n,p)}{p!} < \frac{e}{2}. \tag{94}$$

Theorem 1 represents the consolidated view of the above results.

Theorem 6

$$\frac{H_n^{p+1}}{(p+1)!} \le \frac{1}{p!}\sum_{k=1}^{n} \frac{H_k^p}{k} < \frac{H_n^{p+1}}{(p+1)!} + \frac{e}{2} \tag{95}$$

Next, bounds for $\frac{1}{p!}\sum_{k=1}^{n} \frac{H_k^p}{k+1}$ are established.
Let,

$$
\begin{aligned}
T(n,p) &= \frac{1}{p!}\left(\sum_{k=1}^{n} \frac{H_k^p}{k} - \sum_{k=1}^{n} \frac{H_k^p}{k+1}\right) \\
&= \frac{1}{p!}\sum_{k=1}^{n} \frac{H_k^p}{k(k+1)} \\
&< \frac{1}{p!}\sum_{k=1}^{n} \frac{H_k^p}{k^2} \\
&< \frac{1}{p!}\int_{k=1}^{n} \frac{H_k^p}{k^2} dt
\end{aligned}
$$

Now consider $\int_{k=1}^{n} \frac{H_k^p}{k^2} dt$. On integration by substitution, taking $\ln k + 1 = t$, we get

$$\begin{aligned}\int_{k=1}^{n} \frac{H_k^p}{k^2} dt &= \int_{t=1}^{\ln n+1} t^p e^{1-t} dt \\ &= e \int_{t=1}^{\ln n+1} t^p e^{-t} dt\end{aligned}$$

Therefore,

$$T(n,p) < \frac{e}{p!} \int_{t=1}^{\ln n+1} t^p e^{-t} dt. \tag{96}$$

Also let,

$$\begin{aligned}\alpha(p) &= \int_{t=1}^{\ln n+1} t^p e^{-t} dt \\ &= \frac{1}{e} - \frac{(\ln n+1)^p}{en} + p.\alpha(p-1) \\ \alpha(1) &= \frac{2}{e} - \frac{\ln n+1}{en} - \frac{1}{en}\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha(p) &= \frac{1}{e} + \frac{p}{e} + \frac{p(p-1)}{e} + \ldots + \frac{p!}{e} \\ &\quad -\Bigg(\frac{(\ln n+1)^p}{en} + \frac{p(\ln n+1)^{p-1}}{en} + \frac{p(p-1)(\ln n+1)^{p-2}}{en} \\ &\quad + \ldots + \frac{p!(\ln n+1)}{en} + \frac{1}{en}\Bigg) \\ &< \frac{1}{e}\Big(1 + p + p(p-1) + \ldots + p!\Big) \\ &< \frac{p!}{e}\left(\frac{1}{p!} + \frac{1}{(p-1)!} + \ldots + 1\right) \\ &< \frac{p!}{e}\left(\sum_{i=1}^{p} \frac{1}{i!}\right) \\ &< \frac{p!}{e}\left(\sum_{i=1}^{\infty} \frac{1}{i!}\right) \\ &< \frac{p!}{e} e \\ &< p!\end{aligned}$$

Substituting the value of $\alpha(p)$ in (96), we get

$$T(n,p) < \frac{e}{p!} p! = e \tag{97}$$

The above observations are formally stated in theorem 2 presented next.

Theorem 7

$$\frac{1}{p!}\sum_{k=1}^{n}\frac{H_k^p}{k} - e < \frac{1}{p!}\sum_{k=1}^{n}\frac{H_k^p}{k+1} < \frac{1}{p!}\sum_{k=1}^{n}\frac{H_k^p}{k} \tag{98}$$

Proof: The proofs directly follow from the above dervied results.

CHAPTER VI

CONCLUSION

The work in this thesis is directed towards mathematically proving that any reservation based parking system is capable of self recovery with respect to misparking and that the instability caused by a single mispark is negligible considering the large capacity of the parking garage. In other words, it is shown both theoretically and experimentally that the most likely value of length of the chain resulting from a single mispark follows a $\Theta(\ln n)$ order of growth which is very small compared to garage capacity n. First, tight bounds for the probability of misparking chain length are established. Using these bounds, the behavior of the most probable misparking chain length is studied. The simulation results clearly agree with the theoretical solutions. As a next step, a recursive solution is proposed. Then, based on the proposed recursive solution, an approximate solution is proposed. The validity and accuracy of the approximation is demonstrated through results obtained from both theory and simulation. In all the experiments, the misparking chain length obtained is extremely small compared to the size of the parking garage. All the analyses made in this work are applicable to any reservation based parking system in general. Tighter bounds, multiple misparks, misparking prevention and design of recovery mechanisms are scope of future work.

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APPENDIX A

REVIEW OF SOME IMPORTANT FORMULAE

A.1 E^X

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{99}$$

Therefore,

$$N = e^{\ln(N)} = 1 + \frac{\ln(N)}{1!} + \frac{(\ln(N))^2}{2!} + \frac{(\ln(N))^3}{3!} + \dots \tag{100}$$

A.2 $A^N - B^N$

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1} a^{n-1-i}b^i \tag{101}$$

A.3 LIMITS

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n &= e \\
\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} &= e
\end{aligned}$$

Therefore,

$$\begin{aligned}
(n-1)\ln\left(\frac{n}{n-1}\right) &= (n-1)\ln\left(1+\frac{1}{n-1}\right) \\
&= \ln\left(1+\frac{1}{n-1}\right)^{n-1} < \ln(e) = 1 \\
(n)\ln\left(\frac{n}{n-1}\right) &= (n-1+1)\ln\left(1+\frac{1}{n-1}\right) \\
&= \ln\left(1+\frac{1}{n-1}\right)^{n-1+1} > \ln(e) = 1
\end{aligned}$$

A.4 POISSON RANDOM VARIABLE

For a given Poisson random variable X, the probability mass function p is given by,

$$p(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \tag{102}$$

where λ is expected value of X.

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