Midterm Presentation - Problem L

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November 19, 2011

CS895 - Stochastic Modeling

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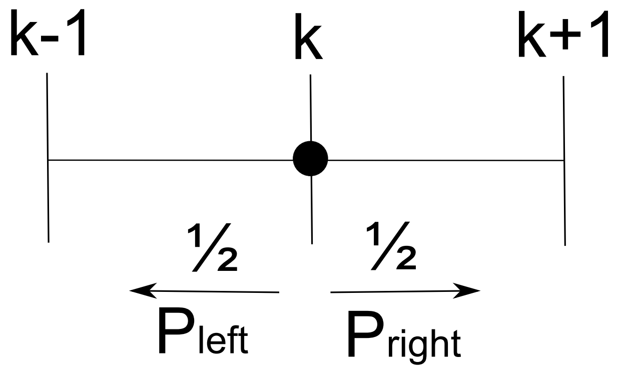
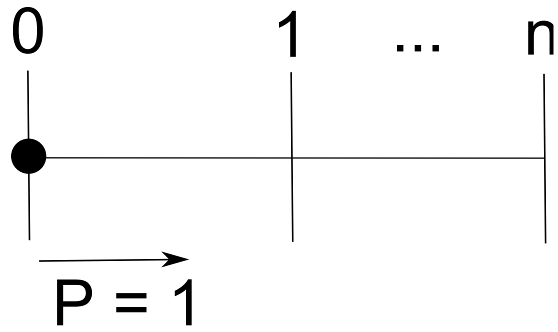
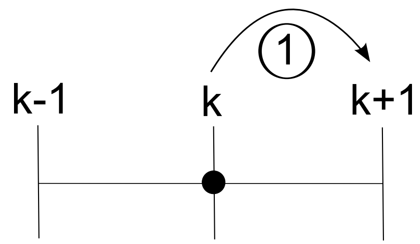
Problem L: Consider a particle that moves along the set $0, 1, ...n$ of integers so that at each step it is equally likely to move to any of its neighbors. Starting at 0, show that the expected number of steps it takes to reach n is n^2.





I_1 =*Number of iterations required for particule to go from* 0 *to* 1
I_k =*Number of iterations required for particule to go from* $k-1$ *to* k

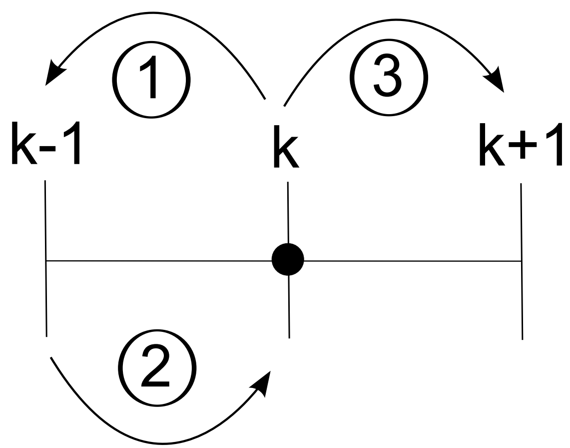
$$E[I_1] = 1$$



When $k > 1$, use Law of Total Expectation:

$$E[I_k] = \sum P(K=k) \times E(AtI_k|K=k)$$
$$= P(AtI_k|AtI_{k-1}) \times E[DirectlytoI_k|I_{k-1}] + P(AtI_k|AtI_{k-1}) \times E[IndirectlytoI_k|I_{k-1}]$$
$$= \frac{1}{2} \times 1 + \frac{1}{2}E[1 + I_{k-1} + I_k]$$
$$= \frac{1}{2} + \frac{1}{2}E[1] + \frac{1}{2}E[I_{k-1}] + \frac{1}{2}E[I_k]$$
$$= \frac{1}{2} + \frac{1}{2} \times 1 + \frac{1}{2}E[I_{k-1}] + \frac{1}{2}E[I_k] \rightarrow$$

$$\frac{1}{2}E[I_k] = 1 + \frac{1}{2}E[I_{k-1}] \rightarrow$$
$$E[I_k] = 2 + E[I_{k-1}]$$
$$E[I_k] - E[I_{k-1}] = 2$$



So:

$$\left.\begin{array}{l} E[I_2] - E[I_1] = 2 \\ E[I_3] - E[I_2] = 2 \\ E[I_4] - E[I_3] = 2 \\ ... \\ E[I_k] - E[I_{k-1}] = 2 \end{array}\right\} k - 1 Expectations \tag{1}$$

To go to I_k from $I_1 : E[I_k] - E[I_1] = 2(k-1)$, All integers between 1 and k must be visited to get from 1 to k.

$$
\begin{align}
E[I_k] = 2(k-1) + E[I_1] = 2k - 2 + 1 = 2k - 1 \tag{2}\\
E[\sum_{k=1}^{n} I_k] = \sum_{k=1}^{n} E[I_k] = \sum_{k=1}^{n}(2k-1) = \tag{3}\\
\frac{2n^2+2n}{2} - n \tag{4}\\
= \frac{2n(n+1)}{2} - n \tag{5}\\
= n(n+1) - n \tag{6}\\
= n^2 + n - n \tag{7}\\
= n^2 \tag{8}
\end{align}
$$