

A Linear Bicharacteristic FDTD Method

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I. INTRODUCTION

The linear bicharacteristic scheme (LBS) was originally developed to improve unsteady solutions in computational acoustics and aeroacoustics [1]-[7]. It is a classical leapfrog algorithm, but is combined with upwind bias in the spatial derivatives. This approach preserves the time-reversibility of the leapfrog algorithm, which results in no dissipation, and it permits more flexibility by the ability to adopt a characteristic based method. The use of characteristic variables allows the LBS to treat the outer computational boundaries naturally using the exact compatibility equations. The LBS offers a central storage approach with lower dispersion than the Yee algorithm, plus it generalizes much easier to nonuniform grids. It has previously been applied to two and three-dimensional free-space electromagnetic propagation and scattering problems [3], [6], [7]. This paper extends the LBS to model lossy dielectric and magnetic materials. Results are presented for several one-dimensional model problems, and the FDTD algorithm is chosen as a convenient reference for comparison.

II. ONE-DIMENSIONAL IMPLEMENTATION

Maxwell's equations for linear, homogeneous and lossy media in the one-dimensional TE case (taking $\partial/\partial y = \partial/\partial z = 0$) are

$$\frac{\partial D_y}{\partial t} + \frac{\partial H_z}{\partial x} + \frac{\sigma}{\epsilon} D_y = 0 \quad (1)$$

$$\frac{1}{c^2} \frac{\partial H_z}{\partial t} + \frac{\partial D_y}{\partial x} + \frac{\sigma^*}{\mu c^2} H_z = 0 \quad (2)$$

where $c = 1/\sqrt{\mu\epsilon}$, and the electric and magnetic conductivities are given by σ and σ^* , respectively. The procedure for the LBS is to transform the dependent variables D_y and H_z to characteristic variables. To transform (1) and (2) into characteristic form, we first multiply (2) by c and then add and subtract from (1) to give

$$\frac{\partial (D_y + \frac{1}{c} H_z)}{\partial t} + c \frac{\partial (D_y + \frac{1}{c} H_z)}{\partial x} + \frac{\sigma}{\epsilon} D_y + \frac{\sigma^*}{\mu c} H_z = 0 \quad (3)$$

$$\frac{\partial (D_y - \frac{1}{c} H_z)}{\partial t} - c \frac{\partial (D_y - \frac{1}{c} H_z)}{\partial x} + \frac{\sigma}{\epsilon} D_y - \frac{\sigma^*}{\mu c} H_z = 0 \quad (4)$$

Now define $P = D_y + \frac{1}{c} H_z$ and $Q = D_y - \frac{1}{c} H_z$ to represent the right and left propagating solutions, respectively. P and Q are otherwise known as the characteristic variables. Using these definitions, (3) and (4) can be rewritten as

$$\frac{\partial P}{\partial t} + c \frac{\partial P}{\partial x} + \frac{1}{2} \left(\frac{\sigma}{\epsilon} + \frac{\sigma^*}{\mu} \right) P + \frac{1}{2} \left(\frac{\sigma}{\epsilon} - \frac{\sigma^*}{\mu} \right) Q = 0 \quad (5)$$

$$\frac{\partial Q}{\partial t} - c \frac{\partial Q}{\partial x} + \frac{1}{2} \left(\frac{\sigma}{\epsilon} - \frac{\sigma^*}{\mu} \right) P + \frac{1}{2} \left(\frac{\sigma}{\epsilon} + \frac{\sigma^*}{\mu} \right) Q = 0 \quad (6)$$

If we make the definitions $a = \sigma/\epsilon + \sigma^*/\mu$ and $b = \sigma/\epsilon - \sigma^*/\mu$, then (5) and (6) become

$$\frac{\partial P}{\partial t} + c \frac{\partial P}{\partial x} + \frac{a}{2} P + \frac{b}{2} Q = 0 \quad (7)$$

$$\frac{\partial Q}{\partial t} - c \frac{\partial Q}{\partial x} + \frac{b}{2} P + \frac{a}{2} Q = 0 \quad (8)$$

To develop the discretized algorithm for a one-dimensional system, the stencils of Figure 1 are proposed for the LBS. The stencil in Figure 1a is used for a right propagating wave and the stencil in Figure 1b is used for a left propagating wave. The upwind bias nature of these stencils is thus clearly evident. Using the stencils shown in Figure 1, the resulting finite difference equations are

$$\frac{(P_i^{n+1} - P_i^n) + (P_{i-1}^n - P_{i-1}^{n-1})}{2\Delta t} + c \left(\frac{P_i^n - P_{i-1}^n}{\Delta x} \right) + \frac{a}{2} P_i^{n+1} + \frac{b}{2} Q_i^n = 0 \quad (9)$$

$$\frac{(Q_i^{n+1} - Q_i^n) + (Q_{i+1}^n - Q_{i+1}^{n-1})}{2\Delta t} - c \left(\frac{Q_{i+1}^n - Q_i^n}{\Delta x} \right) + \frac{a}{2} P_i^n + \frac{b}{2} Q_i^{n+1} = 0 \quad (10)$$

These equations can be rewritten in the form

$$(1 + a\Delta t) P_i^{n+1} = R_1^n \quad (11)$$

$$(1 + a\Delta t) Q_i^{n+1} = R_2^n \quad (12)$$

where R_1^n and R_2^n are the residuals given by

$$R_1^n = P_{i-1}^{n-1} + (1 - 2\nu) (P_i^n - P_{i-1}^n) - b\Delta t Q_i^n \quad (13)$$

$$R_2^n = Q_{i+1}^{n-1} - (1 - 2\nu) (Q_i^n - Q_{i+1}^n) - b\Delta t P_i^n \quad (14)$$

and $\nu = c\Delta t/\Delta x$ is the Courant number. The final solution is then given by

$$P_i^{n+1} = R_1^n / (1 + a\Delta t) \quad (15)$$

$$Q_i^{n+1} = R_2^n / (1 + a\Delta t) \quad (16)$$

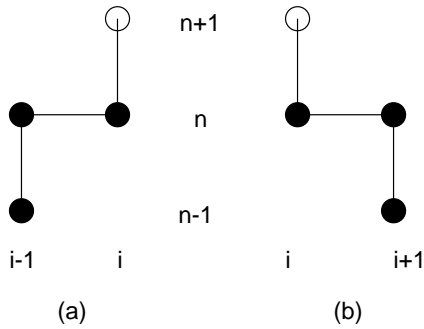


Fig. 1. One-dimensional upwind leapfrog computational stencils for right-going (a) and left-going (b) characteristics.

It is clear that the a and b coefficients can be precomputed and stored before time-stepping begins. Also, note from (15) and (16), as $\sigma \rightarrow \infty$, $P_i^{n+1} = Q_i^{n+1} = 0$ as required for the perfect conductor condition.

III. FOURIER ANALYSIS

A complete Fourier analysis of the LBS has already been completed [2], therefore, only the important results and conclusions from this analysis will be summarized. The stability condition for the LBS is $\nu \leq 1$ and the leading error term of the phase speed error for the LBS is $4\pi\nu(1-\nu)(1-2\nu)/(12N^2)$, where N is the grid resolution in cells/ λ . Results are shown in Figure 2 for the normalized phase speeds at different Courant numbers (ν) for the FDTD method and the LBS. To achieve less than 1% phase speed error requires about $N = 15$ for the FDTD method and about $N = 6$ for the LBS. Note that the LBS has zero dispersion error for $\nu = 1$ and $\nu = 0.5$. Based upon these results, the LBS method is about 2-3 times as economical as the FDTD method for the same level of accuracy.

IV. NUMERICAL RESULTS

The first problem is a free space propagation problem on a nonuniform grid with a mesh stretch ratio of 2 that is periodic every 10 cells. The problem space size is 1000 cells periodic boundary conditions. A characteristic based outer boundary condition similar to that in [12] was used and it is described in [8]. For the nonuniform grid, a base cell size of 1 cm is used, the time step is 0.33 ps and the Courant number $\nu = 1$. A Gaussian pulse was allowed to propagate for 724 meters, which leads to a time integration of 90,504 time steps. The Courant number $\nu = 0.8$, the time step was $\Delta t = 2.67$ ns and the Gaussian pulse had a FWHM pulse width of 2.26 ns. This pulse contained significant spectral content up to 1 GHz. Figure 3 shows the error in the electric field after $n = 1000$ time steps for both the FDTD method and the LBS. Note for this particular problem, the error for the LBS is exceptionally low. From further experimentation, it was demonstrated that the LBS provided excellent results (within 0.1% accuracy) up to a mesh stretch ratio of 3.

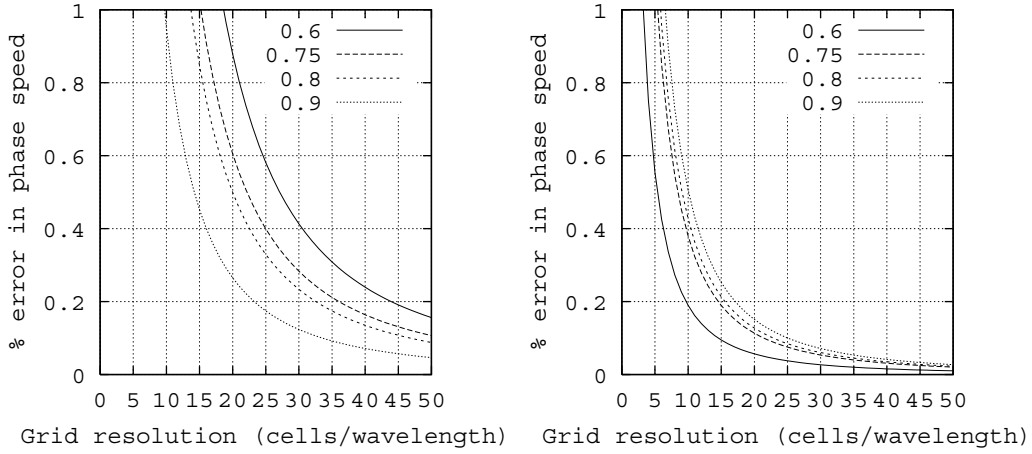


Fig. 2. Percentage error in phase speed versus grid resolution for the FDTD method (left) and the LBS (right). Plot parameter is ν , the Courant number.

The next problem involved reflection and transmission for a lossy dielectric half-space using a dielectric surface boundary condition implementation outlined in [8]. A nonuniform grid was used with the dielectric half-space for $5 \leq x \leq 10$ m with material parameters $\epsilon_{r2} = 4$, $\sigma_2 = 0.02$, $\sigma_2^* = 0$ and $\mu_{r2} = 1$. This problem tests the implementation of the dielectric surface boundary condition which exists in the grid at cell $i = 501$ which puts the x coordinate for the boundary at 7.24 meters. Electric field data was recorded at $x = 5$ meters (i.e. $i = 346$). Figure 4 shows the reflection coefficient magnitude results.

For clarity, FDTD results were not shown in the figure, but the maximum error was near 100%. We clearly see that the LBS is superior on a nonuniform grid, with a reflection coefficient accuracy level within 2%. Further results for uniform and nonuniform grids and perfect conductors can be found in [14].

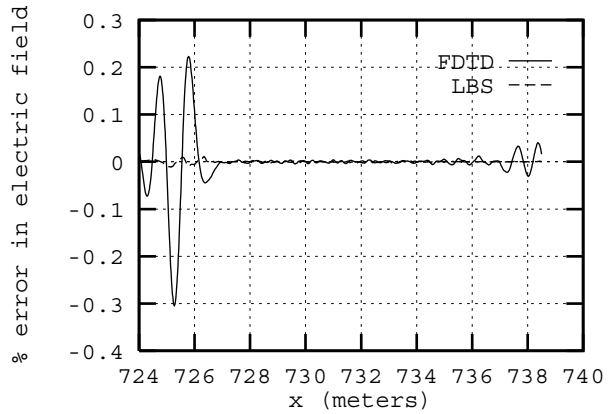


Fig. 3. Percent error in electric field for a free space propagation problem on a nonuniform grid using the FDTD method and the LBS.

V. CONCLUSIONS

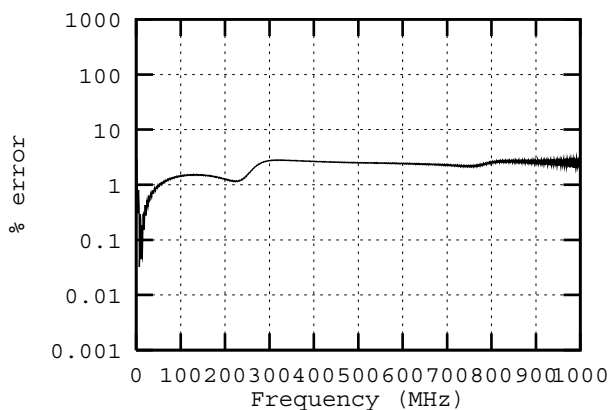


Fig. 4. Percent error in reflection coefficient magnitude versus frequency for reflection from a lossy dielectric half-space using FDTD and the LBS on a non-uniform grid.

ular in fluid dynamics. Third, the LBS provides more accurate results on nonuniform grids. The upwind biasing provides a more flexible generalization to unstructured grids. A dielectric surface boundary condition was implemented in a separate paper [8] and results were provided for two one-dimensional model problems involving lossy dielectric materials and free space. The results indicate that the LBS is a superior algorithm for treatment of dielectric materials, especially its performance on nonuniform grids. Based upon these results, the LBS is a very promising alternative to a conventional FDTD algorithm for many applications. Extensions to two and three-dimensional problems should be straightforward.

This paper has extended the Linear Bicharacteristic Scheme for computational electromagnetics to model homogeneous and heterogeneous lossy dielectric and magnetic materials. It was demonstrated that the LBS has several distinct advantages over conventional FDTD algorithms. First, the LBS is a second-order accurate algorithm which is about 2-3 times as economical. The LBS can also be made to have zero dispersion error in certain instances. Second, the LBS provides a more natural and flexible way to implement surface boundary conditions and outer radiation boundary conditions by using characteristics and an upwind bias technique pop-

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