A Local Coordinate Approach in the MLPG Method for Beam Problems

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A LOCAL COORDINATE APPROACH IN THE MLPG METHOD FOR BEAM PROBLEMS

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ABSTRACT

System matrices for Euler-Bernoulli beam problems for the meshless local Petrov-Galerkin (MLPG) method deteriorate as the number of nodes in the beam models are consistently increased. The reason for this behavior is explained. To overcome this difficulty and improve the accuracy of the solutions, a local coordinate approach for the evaluation of the generalized moving least squares shape functions and their derivatives is proposed. The proposed approach retains the accuracy of the MLPG methods.

INTRODUCTION

Meshless methods are increasingly being viewed as an alternative to the finite element method [1-3]. Recently, a meshless local Petrov-Galerkin (MLPG) method has been presented for C^0 and C^1 problems [3,4]. In these methods, moving least squares (MLS) interpolants [1] are used for C^0 problems and generalized MLS interpolants are used for C^1 problems [4]. References 3 and 4 showed excellent performance of the MLPG method for potential and elasticity problems and a good performance for beam problems.

When all the chosen parameters in the MLPG method are held constant and the number of nodes in the models are consistently increased, the error norms do not decrease; rather they show increases compared to coarser idealizations. The reasons for this behavior are studied. A local coordinate approach to the MLS interpolation is proposed. The proposed local coordinate approach is implemented and evaluated by applying it to three simple test cases.

BEHAVIOR OF THE MLPG METHOD WITH MESH REFINEMENT

The notation of reference 4 is used in this note for brevity and convenience in presentation. The MLPG equations are

$$\mathbf{K}_i^{(\text{node})}\mathbf{d} + \mathbf{K}_i^{(\text{bdy})}\mathbf{d} - \mathbf{f}_i^{(\text{node})} - \mathbf{f}_i^{(\text{bdy})} = \mathbf{0} \tag{1}$$

where

$$\{\mathbf{d}\} = \left\{\hat{u}^1, \quad \hat{\theta}^1, \quad \hat{u}^2, \quad \hat{\theta}^2, \quad \ldots\right\}^{\text{T}} \tag{2}$$

are the fictitious nodal values of deflections u and slopes θ, and the matrices in Eq. (1) are defined as in Eq. 35-36(g) of reference 4.

The MLPG equations are derived using a weighted residual weak form of the governing equations. The trial functions used for the beam problems are derived using the generalized MLS interpolation [4] as

$$u(x) = \sum_{i=1}^{N} \hat{u}^i \psi_i^{(u)}(x) + \hat{\theta}^i \psi_i^{(\theta)}(x) \tag{3}$$

where

$$\psi_i^{(u)}(x) = \sum_{j=1}^{m} p_j(x)\left[\mathbf{A}^{-1}\mathbf{P}^{\text{T}}\mathbf{w}\right]_{ji}$$

$$\psi_i^{(\theta)}(x) = \sum_{j=1}^{m} p_j(x)\left[\mathbf{A}^{-1}\mathbf{P}_x^{\text{T}}\mathbf{w}\right]_{ji} \tag{4}$$

with

$$\left[\mathbf{A}\right] = \mathbf{P}^{\text{T}}\mathbf{w}\mathbf{P} + \mathbf{P}_x^{\text{T}}\mathbf{w}\mathbf{P}_x \quad . \tag{5}$$

In Eq. (5) $\mathbf{P}$ is an (n,m) matrix and $\mathbf{w}$ is an (n,n) matrix defined as

$$[\mathbf{P}] = [\, \mathbf{p}(x_1) \quad \mathbf{p}(x_2) \quad \dots \quad \mathbf{p}(x_n) \,]^{\mathrm{T}} \; , \tag{6}$$

$$\mathbf{w} = \begin{bmatrix} w_1(\bar{x}) & & & \\ & w_2(\bar{x}) & & \\ & & \ddots & \\ & & & w_n(\bar{x}) \end{bmatrix} \tag{7}$$

where $\bar{x} = x - x_i$, and

$$\mathbf{p}^{\mathrm{T}}(x) = \begin{bmatrix} 1, & x, & x^2, & \dots & x^{m-1} \end{bmatrix} ,$$

$$\mathbf{p}_x^{\mathrm{T}}(x) = \frac{d\mathbf{p}^{\mathrm{T}}}{dx} = \begin{bmatrix} 0, & 1, & 2x, & \dots & (m-1)x^{m-2} \end{bmatrix} \tag{8}$$

with $(m-1)$ as the order of the basis function $\mathbf{p}(x)$ used in the MLS approximation. The weight functions $w_i(\bar{x})$ chosen are

$$w_i(\bar{x}) = \begin{cases} \left[\, 1 - d_i^2 / R_i^2 \,\right]^4 & \text{if} \quad d_i \le R_i \\ 0 & \text{if} \quad d_i > R_i \end{cases} \tag{9a}$$

and

$$w_i(\bar{x}) = \begin{cases} 1 - 6\left(\dfrac{d_i}{R_i}\right)^2 + 8\left(\dfrac{d_i}{R_i}\right)^3 - 3\left(\dfrac{d_i}{R_i}\right)^4 & \text{if} \quad 0 \le d_i \le R_i \\ 0 & \text{if} \quad d_i \ge R_i \end{cases} \tag{9b}$$

where $d_i = \| x - x_i \|$. The test function $v_i(x)$ in the MLPG weak form is chosen as

$$v_i(x) = \begin{cases} \left[\, 1 - d_i^2 / R_o^2 \,\right]^4 & \text{if} \quad d_i \le R_o \\ 0 & \text{if} \quad d_i > R_o \end{cases} . \tag{10}$$

Note that the lengths R_i and R_o in Eqs. (9) and (10) are user defined in the MLPG method.

In the current implementation a beam of length $4l$ is considered, as the choice of unit beam length l would mask numerical errors. Six models with 5, 9, 17, 33, 65, and 129 nodes uniformly distributed along the length of the beam are considered. The model with 17 nodes is presented in Figure 1. The distance between the nodes (Δ/l) in these models are 1.0, 0.5, 0.25, 0.125, 0.0625, and 0.03125 for the 5-, 9-, 17-, 33-, 65-, and 129-node models, respectively. The (R_o/l) in the test functions (Eq. 10) in each of these six models is different and is chosen equal to (2Δ). The (R_i/l) in Eq. (9) is chosen to be $(R_i/l = 3.5)$ for the 5-, 9-, and 17-node models and $(R_i/l = 16\Delta)$ for the 33-, 65-, and 129-node models. Two types of basis functions, quadratic basis $(1, x, x^2)$ and cubic basis $(1, x, x^2, x^3)$, are used. System matrices in Eq. (1) are developed with these parameters. The resulting system equations must be able to reproduce the constant, linear, and quadratic terms exactly when the quadratic basis is used, and additionally, the cubic term when the cubic basis is used. To evaluate the system matrices developed for the six models, two rigid body conditions and a constant-curvature condition were considered. These can be written as

$$u(x) = c_0, \qquad \theta = \frac{du}{dx} = 0 \quad ; \quad \text{Rigid body translation,}$$

$$u(x) = c_1 x, \qquad \theta = c_1 \qquad ; \quad \text{Rigid body rotation, and} \qquad (11)$$

$$u(x) = c_2 x^2, \quad \theta = 2c_2 x \qquad ; \quad \text{Constant - curvature,}$$

where c_0, c_1, and c_2 are arbitrary constants. The third condition in Eq. (11) corresponds to the problem of a cantilever beam with a moment, $M = EI \dfrac{d^2 u}{dx^2} = 2c_2$, applied at $x = 4l$. The problems described by Eq. (11) are simple test problems and should be reproduced exactly by the MLPG when quadratic or higher bases are used.

The $\{\mathbf{d}\}$ vectors that correspond to each of the conditions in Eq. (11) (and in the absence of any other loading) when used in Eq. (1) should result in a null right-hand vector if the $\mathbf{K}_i^{(\text{node})}$ is evaluated exactly. In general, the product results in a residual $\{\mathbf{r}\}$ vector as

$$\mathbf{K}_i^{(\text{node})}\{\mathbf{d}\}=\{\mathbf{r}\} \tag{12}$$

Each of the components of the vector $\{\mathbf{r}\}$ is nearly equal to machine zero if $\mathbf{K}_i^{(\text{node})}$ is evaluated accurately. To quantify the residual, an error norm of $\{\mathbf{r}\}$ is computed as

$$\|E\|_1 = \sqrt{\frac{1}{N_d}\sum_{k=1}^{N_d} r_k^2} \tag{13}$$

where r_k is the kth component of the vector $\{\mathbf{r}\}$ in Eq. (12) and N_d is the degrees of freedom in the model.

Table 1: Error norm $\|E\|_1$ of the residuals for six models and for two basis functions

Number of nodes in the model	$u=c_1$		$u=c_2 x$		$u=c_3 x^2$	
	Quadratic Basis	Cubic Basis	Quadratic Basis	Cubic Basis	Quadratic Basis	Cubic Basis
5*	0.5040e-14	0.1278e-12	0.2099e-14	0.4547e-13	0.5733e-14	0.9196e-13
9*	0.7515e-13	0.1496e-11	0.2362e-13	0.5514e-12	0.3321e-13	0.9747e-12
17*	0.2774e-10	0.8211e-10	0.1109e-10	0.3067e-10	0.1582e-10	0.5352e-10
33	0.3609e-9	0.1062e-5	0.1266e-9	0.4479e-6	0.2587e-10	0.9057e-6
65	0.1691e-6	0.1435e-2	0.7735e-7	0.5855e-3	0.1726e-6	0.1193e-2
129	0.1796e-4	0.5599e+0	0.8154e-5	0.2269e+0	0.1794e-4	0.4154e+0

$* R_i / l = 3.5$

Table 1 presents the $\|E\|_1$ for the three conditions in Eq. (11) when the weight function in Eq. (9b) is used. (Similar results are obtained when weight function (9a) is used and hence these results are not presented here.) As seen from the table, the $\|E\|_1$ deteriorates with model refinement and for higher order basis. Closer examination of the residuals for each of the six models showed that the residuals were of machine accuracy for nodes near the origin while the residuals were largest at nodes farthest from the

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origin. This observation was confirmed by running different cases with the origin at different locations along the length of the beam. Also, the residuals were largest for the models with the largest number of nodes.

Closer scrutiny of computations showed that the numerical values of the shape functions for nodes that are systematically located about the center of the beam (for example, nodes 3 and 15, 2 and 16, and 1 and 17 in the 17-node model of Figure 1) are not exactly identical as expected. These differences increased with model refinement and when a higher basis was used. The error norm in Table 1 can be improved by using higher precision computations or inversion routines. However, a much simpler alternative to improve the accuracy is presented below.

LOCAL COORDINATE APPROACH

In the MLS interpolation, the basis functions are in terms of the global coordinate x. The [**A**] matrix thus formed using this basis is generally of the form (see Eq. 16, ref. 4)

$$[\mathbf{A}] = \sum_{k=1}^{M} \left\{ w_k(\bar{x}) \mathbf{p} \cdot \mathbf{p}^{\mathrm{T}} + w_k(\bar{x}) \mathbf{p}_x \cdot \mathbf{p}_x^{\mathrm{T}} \right\} \tag{14}$$

where $\bar{x} = x - x_j$ and M are the number of nodes in the domain of definition of node j for which the [**A**] matrix is being computed. (For convenience in presentation, the [**A**] matrices thus formed will be referred to as the global method.) As the order of the polynomial basis increases the conditioning of the [**A**] matrix deteriorates. For example, the matrix [**A**] will have terms like 1, x^2, x^4, x^6 on the diagonal for a cubic basis function. The [**A**] matrices for nodes near the origin and the [**A**] matrices for nodes farthest from the origin will be different. The conditioning is worse for [**A**] matrices for nodes farthest from the origin. This explains the differences in the error norms observed in Table 1.

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The situation can be easily rectified if the MLS approximation is defined not in terms of a global basis, but rather in terms of a local basis. Figure 2 shows two identical shape functions, one centered at node j, and the other centered at node e. The global approximation for

$$u(x) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x})$$
$$= a_1 + a_2 x + a_3 x^2 + \ldots + a_m x^{m-1}$$

(15)

can be rewritten in the neighborhood of node j, recognizing that $x = x_j + \xi$ where ξ is a local coordinate measured from node j, as

$$u(x) = a_1 + a_2\left(x_j + \xi\right) + a_3\left(x_j + \xi\right)^2 + \ldots$$
$$= \left(a_1 + a_2 x_j + a_3 x_j^2\right) + \left(a_2 + 2a_3 x_j + \ldots\right)\xi + \left(a_3 + \ldots\right)\xi^2$$
$$= b_1 + b_2\xi + b_3\xi^2 + \ldots$$

(16)

where b_i, $i=1,\ldots, m\text{-}1$ are the new undetermined coefficients in the MLS approximation. (A similar local coordinate transformation can be affected for node e in Figure 2 as $x = x_e + \xi$.) The [**A**] matrix then is computed in a similar manner as in Eq. (14) but with

$$\mathbf{p}^{\mathrm{T}}(\xi) = \begin{bmatrix} 1, & \xi, & \xi^2, & \ldots & \xi^{m-1} \end{bmatrix}$$

and $\quad \mathbf{p}_x^{\mathrm{T}}(\xi) = \begin{bmatrix} 0, & 1, & 2\xi, & 3\xi^2, & \ldots & (m-1)\xi^{(m-2)} \end{bmatrix}$

(17)

as $\quad \dfrac{d}{dx}(\ ) = \dfrac{d}{d\xi}(\ )$.

LOCAL COORDINATE APPROACH RESULTS

The local coordinate approach is implemented in the evaluation of the shape functions and their derivatives for all the nodes in the six MLPG models of the beam. Table 2 compares the condition numbers of the [**A**] matrices at various locations on the

beam using global and local coordinate methods. The condition numbers are evaluated

using routines available in NAPACK and the procedure outlined in references 5 and 6.

When the global coordinate method is used, the condition numbers of the [**A**] matrices

for nodes farthest from the origin are much larger (suggesting poor conditioning) than the

nodes closest to the origin. The conditioning numbers of the [**A**] matrices vastly improve

when the local coordinate method is used, clearly demonstrating the advantages of the

local coordinate method.

Table 2: Comparison of the condition numbers of the [A] matrices at various locations on the beam using global and local coordinate methods

Location on the beam ($x/4l$)	Number of nodes in the model					
	5*	9*	17*	33	65	129
	Global Method Conditioning Number					
0.0	0.631e+3	0.106e+4	0.930e+3	0.271e+3	0.267e+3	0.189e+4
0.5	0.231e+5	0.268e+5	0.272e+5	0.785e+5	0.904e+6	0.131e+8
1.0	0.914e+6	0.771e+6	0.127e+7	0.422e+8	0.153e+10	0.365e+11
	Local Method Conditioning Number					
0.0	0.634e+3	0.106e+4	0.930e+3	0.271e+3	0.267e+3	0.189e+4
0.5	0.478e+3	0.496e+2	0.411e+2	0.111e+2	0.153e+2	0.141e+3
1.0	0.632e+3	0.106e+4	0.930e+3	0.271e+3	0.267e+3	0.189e+4

$* R_i / l = 3.5$

Table 3: Error norm $\|E\|_1$ of the residuals computed with the local coordinate approach

Number of nodes in the model	$u=c_1$		$u=c_2 x$		$u=c_3 x^2$	
	Quadratic Basis	Cubic Basis	Quadratic Basis	Cubic Basis	Quadratic Basis	Cubic Basis
5*	0.1173e-14	0.3500e-13	0.2342e-15	0.1201e-14	0.3174e-14	0.3853e-13
9*	0.2521e-13	0.4900e-13	0.8357e-14	0.1699e-13	0.3659e-13	0.4146e-13
17*	0.1392e-12	0.2169e-12	0.4764e-13	0.1680e-12	0.2126e-12	0.8124e-12
33	0.4389e-12	0.1390e-11	0.1876e-12	0.5060e-12	0.4084e-12	0.2183e-11
65	0.4196e-11	0.3890e-11	0.1142e-11	0.1879e-11	0.2548e-11	0.5930e-11
129	0.4029e-10	0.2778e-10	0.1240e-10	0.8191e-11	0.2400e-10	0.2166e-10

$* R_i / l = 3.5$

The error norms shown in Table 1 are recomputed and the results are presented in

Table 3. As expected, all models and the quadratic and cubic basis functions produced

the error norms close to machine accuracy, suggesting that the local coordinate approach produces accurate results compared to the global coordinate approach.

COMPUTATIONAL ADVANTAGE OF THE LOCAL COORDINATE APPROACH

In the conventional MLPG implementation, the [**A**] matrix is calculated and inverted at every node in the model. When using the local coordinate methodology with uniform nodal spacing, the shape functions are exactly identical for nodes whose R_i places the entire shape function in the interior of the domain of the problem. Hence, for those nodes the [**A**] matrices are identical. As such, considerable reduction in computational effort and cost can be achieved by the proposed local coordinate approach thus eliminating a perceived disadvantage of the MLPG method.

CONCLUDING REMARKS

The MLPG method for beam problems (C^1 problems) showed that the solutions deteriorated as the number of nodes in the models were progressively increased. Closer examination revealed that the moving least squares (MLS) shape function calculations involved the computation of the [**A**] matrix and that this matrix became ill conditioned for nodes farthest from the origin. To overcome this difficulty a local coordinate approach for the MLS basis functions was proposed. The proposed approach restored the accuracy of the MLPG method for beam problems.

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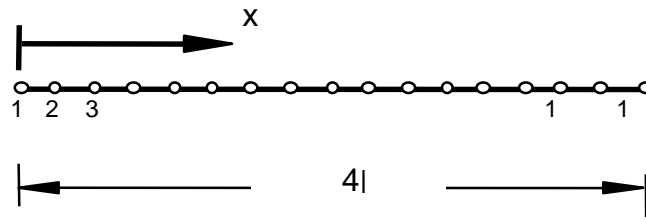[6] S.C. Chapra and R.P. Canale, *Numerical Methods for Engineers*, Second Edition, 1988. McGraw-Hill Book Co. New York, p. 257. (See also http://www.netlib.org/napack/)
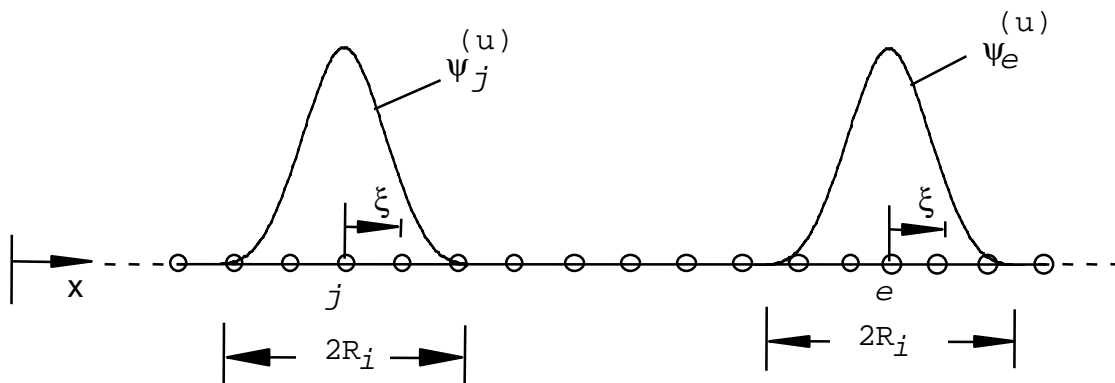
Figure 1: Beam and a17-node

Figure 2: Local Coordinate

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