A cubic radial basis function in the MLPG method for beam problems

I. S. Raju[1] and D. R. Phillips[2]

Summary

A non-compactly supported cubic radial basis function implementation of the MLPG method for beam problems is presented. The evaluation of the derivatives of the shape functions obtained from the radial basis function interpolation is much simpler than the evaluation of the moving least squares shape function derivatives. The radial basis MLPG yields results as accurate or better than those obtained by the conventional MLPG method for problems with discontinuous and other complex loading conditions.

Introduction

Meshless methods are attractive as they overcome some of the disadvantages of the finite element method (FEM) such as discontinuous secondary variables across inter-element boundaries and the need for remeshing in large deformation problems [1-4]. Recent literature shows extensive research on meshless methods and, in particular, the meshless local Petrov-Galerkin (MLPG) method. Atluri, Cho, and Kim [4] present an analysis of thin beam problems using a Galerkin implementation of the MLPG method; a generalized moving least squares (GMLS) approximation is used to construct the trial functions, and the test functions are chosen from the same space. In references 5-7, a meshless Petrov-Galerkin implementation of the MLPG method is presented; the GMLS approximation is used to construct the trial functions, and the test functions are chosen from a different space. Closer scrutiny of these formulations shows that a large number of calculations are required to compute the first and second order derivatives of the moving least squares (MLS) trial functions that are required in the weak form. Hence, a computationally efficient alternative to the MLS trial functions is preferred.

In reference 8, the use of radial basis interpolation functions in the meshless local Petrov-Galerkin formulation for beam problems is explored. Both compactly and non-compactly supported radial basis functions (RBF) are considered. In addition, the RBFs are augmented with polynomial terms to increase the polynomial accuracy of the solutions. The resulting interpolation functions are simple, and the evaluation of the derivatives is simpler than for the traditional MLS approximations.

The purpose of this paper is to investigate the use of a non-compactly supported cubic radial basis function in the MLPG (RPG) method for beam problems. The method is evaluated by applying the formulation to patch test and mixed boundary value problems and problems with complex loading conditions.

Development of the Petrov-Galerkin Formulation

The notation of reference 8 is used in this paper for brevity and convenience in presentation. The MLPG equations are

$$\mathbf{K}^{(\text{node})}\mathbf{d} + \mathbf{K}^{(\text{bdry})}\mathbf{d} - \mathbf{f}^{(\text{node})} - \mathbf{f}^{(\text{bdry})} = \mathbf{0} \, , \tag{1}$$

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[1] Senior Technologist, Structures and Materials Competency, NASA Langley, Hampton, VA, USA
[2] Aerospace Engineer, Lockheed Martin Space Operations, NASA Langley, Hampton, VA, USA

where the superscript "bdry" denotes boundary,

$$\mathbf{d}^{\mathrm{T}} = \{w_1 \quad \theta_1 \quad w_2 \quad \theta_2 \quad \ldots \quad w_N \quad \theta_N\},$$ (2)

are the nodal values of deflections, w, and slopes, θ, at the N nodes of the model used to analyze the problem, and the matrices in Equation (1) are defined as in Equations (80b) – (80g) of reference 8. The radial basis functions used in this work possess the δ_{ij} property; therefore, the $\mathbf{d}^{\mathrm{T}}$ values in Equation (2) are actual nodal values rather than fictitious nodal values.

In the MLPG implementation, the trial functions used for beam problems are

$$w(x) = \sum_{j=1}^{N} \left(\psi_j^{(w)}(x) w_j + \psi_j^{(\theta)}(x) \theta_j \right),$$ (3)

where $\psi_j^{(w)}(x)$ and $\psi_j^{(\theta)}(x)$ are the shape functions for deflection and slope, respectively. In this work, the shape functions are derived using radial basis interpolations and are

$$\psi_l^{(w)}(x) = \sum_{k=1}^{N} \left(R_k(x) \cdot \eta_{(2k-1)(2l-1)} + S_k(x) \cdot \eta_{(2k)(2l-1)} \right)$$

$$\psi_l^{(\theta)}(x) = \sum_{k=1}^{N} \left(R_k(x) \cdot \eta_{(2k-1)(2l)} + S_k(x) \cdot \eta_{(2k)(2l)} \right),$$ (4)

where $R_k(x)$ are the radial basis functions,

$$S_k(x) = dR_k(x)/dx,$$ (5)

and η_{ij} are the elements of $[\mathbf{Q}_{\mathrm{B}}]^{-1}$, where

$$[\mathbf{Q}_{\mathrm{B}}] = \begin{bmatrix} R_1(x_1) & S_1(x_1) & R_2(x_1) & S_2(x_1) & \ldots & R_N(x_1) & S_N(x_1) \\ \dfrac{dR_1(x_1)}{dx} & \dfrac{dS_1(x_1)}{dx} & \dfrac{dR_2(x_1)}{dx} & \dfrac{dS_2(x_1)}{dx} & \ldots & \dfrac{dR_N(x_1)}{dx} & \dfrac{dS_N(x_1)}{dx} \\ R_1(x_2) & S_1(x_2) & R_2(x_2) & S_2(x_2) & \ldots & R_N(x_2) & S_N(x_2) \\ \dfrac{dR_1(x_2)}{dx} & \dfrac{dS_1(x_2)}{dx} & \dfrac{dR_2(x_2)}{dx} & \dfrac{dS_2(x_2)}{dx} & \ldots & \dfrac{dR_N(x_2)}{dx} & \dfrac{dS_N(x_2)}{dx} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ R_1(x_N) & S_1(x_N) & R_2(x_N) & S_2(x_N) & \ldots & R_N(x_N) & S_N(x_N) \\ \dfrac{dR_1(x_N)}{dx} & \dfrac{dS_1(x_N)}{dx} & \dfrac{dR_2(x_N)}{dx} & \dfrac{dS_2(x_N)}{dx} & \ldots & \dfrac{dR_N(x_N)}{dx} & \dfrac{dS_N(x_N)}{dx} \end{bmatrix}.$$ (6)

The derivatives of the trial functions are easy to evaluate directly from Equation (4). These derivatives are much simpler than the derivatives of the MLS trial functions, which involve numerous matrix inversion and multiplication operations (see references 1-7).

In this work, the radial basis function considered is the non-compactly supported cubic RBF [9],

$$R_k(x) = r^3, \text{ with } r = d_j/s_j, \tag{7}$$

where $d_j = \|x - x_j\|$, and s_j is some normalizing distance, usually chosen to be the entire problem domain, Ω (in this work, $0 \le x \le L$). As s_j covers the entire problem domain, $[\mathbf{Q}_B]$ is a $(2N, 2N)$ matrix that is evaluated and inverted once. As a result, the current RPG method involves two inversions of large matrices ($[\mathbf{Q}_B]^{-1}$ to obtain the shape functions, and $[\mathbf{K}]^{-1}$ to obtain the solution), in contrast to the conventional MLPG method, which involves the inversion of many smaller matrices to obtain the shape functions and their derivatives and the inversion of the large $[\mathbf{K}]$ matrix to obtain the solution. Because of this difference, the RPG method may not be more computationally efficient than the MLPG method *for large models*.

The test function, v, is assumed as in references 7 and 8 as

$$v(x) = \mu_i^{(w)} \chi_i^{(w)}(x) + \mu_i^{(\theta)} \chi_i^{(\theta)}(x). \tag{8}$$

In this paper, the test function components, χ_i, are chosen as in the conventional MLPG method as power weight functions [7],

$$\chi_i^{(w)}(x) = \left[1 - (d_i/s_o)^2 \right]^4 \text{ if } 0 \le d_i \le s_o, \quad = 0 \text{ if } d_i > s_o \tag{9}$$

with $d_i = \|x - x_i\|$. In Equation (9), s_o is a user-defined parameter that determines the extent of the test functions. The components of the test functions chosen for θ are the first derivatives of the components of the test functions chosen for the primary variable, w, as $\theta = (dw/dx)$ is also a primary variable:

$$\chi_i^{(\theta)} = d\chi_i^{(w)}/dx. \tag{10}$$

For this power function, the values of $\chi_i^{(w)}$, $\chi_i^{(\theta)}$, $(d\chi_i^{(w)}/dx)$, and $(d\chi_i^{(\theta)}/dx)$ are zero when $d_i \ge s_o$. As discussed in reference 7, when this test function is used, the $\mathbf{K}^{(node)}$ (Equation (1)) is simplified (see Equation (41) of reference 7).

Beam Configurations and Models

A beam of constant flexural rigidity *EI* and a length of 4*l* is considered. The length 4*l* is specifically chosen to avoid scaling by a unit length, *l*. Five models with 5, 9, 17, 33, and 65 nodes uniformly distributed along the length of the beam are considered. Figure 1 shows a typical 17-node model. The distances between the nodes ($\Delta x / l$) in these models are 1, 0.5, 0.25, 0.125, and 0.0625 for the 5-, 9-, 17-, 33-, and 65-node models, respectively.
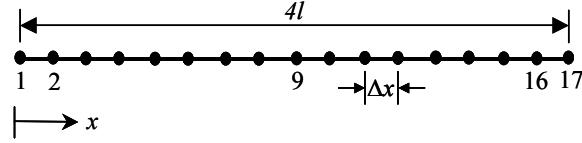
Figure 1: A 17-node model of the beam

Numerical Evaluations - Patch Tests

The radial basis MLPG (RPG) method was evaluated by applying the method to simple patch-test problems. The problems considered were (a) rigid body translation, (b) rigid body rotation, and (c) constant-curvature condition:

$$w(x) = \beta_0, \qquad \theta = dw/dx = 0 \ , \tag{11a}$$

$$w(x) = \beta_1 x, \qquad \theta = \beta_1 , \tag{11b}$$

$$w(x) = \beta_2 x^2/2, \quad \theta = \beta_2 x , \tag{11c}$$

where β_0, β_1, and β_2 are arbitrary constants. The third patch test is equivalent to the problem of a cantilever beam with a moment, $M = EI(d^2w/dx^2) = EI\beta_2$, applied at $x=4l$. The deflection, w, and the slope, θ, corresponding to problems (a), (b), and (c) were prescribed as essential boundary conditions (EBCs) at $x=0$ and $x=4l$. With these EBCs, the beam problems were analyzed using the RPG method. If the RPG method recovers the exact solution at all the interior nodes and at every arbitrary point of the beam, then the method passes the patch test. For all models considered, the method successfully reproduced the exact solutions to machine accuracy, thus passing all the patch tests.

Mixed Boundary Value Problems

Next, the RPG method was used to analyze mixed boundary value problems. The two problems considered were a cantilever beam with a tip load and a simply supported (SS) beam subjected to a uniformly distributed load (UDL). In the method, a 12-point Gaussian integration was used to integrate the weak form, the value of (s_o / l), which defines the extent of the test functions (see Equation (9)), was set as $(s_o / l =2\Delta x / l)$, and the value of (s_j / l), which defines the extent of the trial functions (Equation (7)), was set as $(s_j / l = L / l = 4)$. For the cantilever beam problem, the RPG method yielded excellent results. The simply supported beam with a uniformly distributed load was analyzed using 17-, 33-, and 65-node models. The maximum deflection values, i.e., the deflection at $(x = L / 2)$, for these three models obtained using the RPG method and using the conventional MLPG method with a quadratic polynomial basis function are compared in Table 1. In the MLPG method, a 20-point Gaussian integration was used, the value of (s_o / l) was set as $(s_o / l =2\Delta x / l)$, and the value of (s_j / l) was set as $(s_j / l =8\Delta x / l)$. From this table, it is seen that the RPG method performs as accurately as the conventional MLPG method. For each of the nodal models (17, 33, and 65 nodes), the RPG values for slope and moment are as accurate as the MLPG values and are in excellent agreement with the exact values, and the RPG values for shear converged with model refinement. The MLPG solution for the shear was erratic; the quadratic basis function is insufficient to accurately calculate the third derivatives for this problem, and the method could not recover the values with model refinement. The solution for

the shear converged only as the order of the basis function was increased to quartic [6]. The results discussed for this problem verify a perceived advantage of the RPG method over the MLPG method; namely, accurate solutions are obtained with easier evaluation of the shape function derivatives.

Table 1: Comparison of maximum deflection for the SS beam with UDL

Model	Maximum deflection (at $x = L / 2$)		
	Exact	RPG	MLPG
17-node	-3.3333e-7	-3.2739e-7	-3.3106e-7
33-node	-3.3333e-7	-3.3407e-7	-3.3735e-7
65-node	-3.3333e-7	-3.3420e-7	-3.3848e-7

## Problems with Complex Loading Conditions

The RPG method was next applied to problems with complex loading conditions. The problems considered were (a) a cantilever beam with a uniformly distributed load on a portion of the beam (not shown here, see reference 8), (b) a continuous beam with the additional support in the center of the beam (reference 8), and (c) a cantilever beam with a hinge shown in Figure 2. The RPG solution (with $(s_o / l = 4\Delta x / l)$) for the cantilever beam problem (problem (a)) exhibited convergence with model refinement. These results are consistent with those reported in reference 7, where this problem was studied using the conventional MLPG method (with parameters as reported above). The RPG method handled the load discontinuity well and yielded results in overall agreement with the exact solutions. For the continuous beam problem (problem (b)), the RPG method yielded very accurate results for both the primary and secondary variables and handled the discontinuity caused by the additional support well. For this problem, the MLPG method required a large number of nodes to obtain an accurate solution [7], but the RPG method yielded accurate results when the smaller nodal (17-node and 33-node) models were used. These results again verify perceived advantages of the RPG method over the MLPG method; comparable results can be obtained by the RPG method for comparable computing efforts. For the hinge problem (problem (c)), the RPG and exact solutions for deflection and slope at key locations along the length of the beam for three nodal refinements are presented in Table 2. As the hinge at ($x = l$) cannot admit the moment, there are two separate slopes, θ^L and θ^R, at the left ($x = l^-$) and right ($x = l^+$) segments on either side of the hinge, respectively. Accurate solutions were obtained with the coarse models, and the solutions improved as the models were refined. The RPG and exact solutions for the deflection obtained from the 50-node model for this problem are compared in Figure 3.
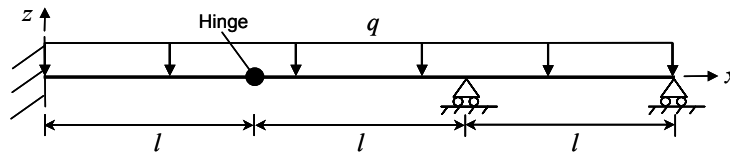
Figure 2: Cantilever beam with a hinge

Table 2: Comparison of RPG results and exact solutions at several locations along the beam

	$w / w_{Exact} (x = l)$	$\theta^L / \theta^L_{Exact} (x = l)$	$\theta^R / \theta^R_{Exact} (x = l)$	$\theta / \theta_{Exact} (x = 2l)$	$\theta / \theta_{Exact} (x = 3l)$
14-node	0.9769	0.9748	0.9679	0.9774	0.8944
26-node	0.9814	0.9789	0.9798	0.9817	0.9656
50-node	0.9880	0.9867	0.9868	0.9882	0.9755
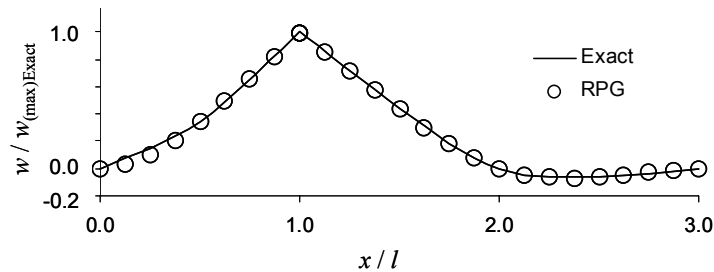
Figure 3: Comparison of normalized deflection obtained with the RPG method and the exact solution

Concluding Remarks

A radial basis function implementation of the MLPG method for beam problems was presented. The use of radial basis functions (RBF) rather than the traditional moving least squares interpolations reduced the computing effort required to solve problems; substantially fewer matrix inversion and multiplication operations were required by the radial basis MLPG (RPG) to evaluate the derivatives of the shape functions. When non-compactly supported RBFs were used, the RPG method involved only two inversions of large matrices. The non-compactly supported cubic RBF was found to yield accurate solutions for the problems studied; for mixed boundary value problems, the RPG method achieved the same accuracy in results as the MLPG method with comparable computing effort. The RPG method yielded very good results for problems with discontinuous and other complex loading conditions.

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