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A TWO-TIMESCALE DISCRETIZATION SCHEME FOR COLLOCATION

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The development of a two-timescale discretization scheme for collocation is presented. This scheme allows a larger discretization to be utilized for smoothly varying state variables and a second finer discretization to be utilized for state variables having higher frequency dynamics. As such, the discretization scheme can be tailored to the dynamics of the particular state variables. In so doing, the size of the overall Nonlinear Programming (NLP) problem can be reduced significantly. Two two-timescale discretization architecture schemes are described. Comparison of results between the two-timescale method and conventional collocation show very good agreement. Differences of less than 0.5 percent are observed. Consequently, a significant reduction (by two-thirds) in the number of NLP parameters and iterations required for convergence can be achieved without sacrificing solution accuracy.

INTRODUCTION

In the collocation method, a single timescale discretization scheme is utilized for all the state variables. This approach works very well for solving a great variety of problems. However, if the dynamics of one or some of the state variables are at a high frequency (e.g., in six-degree-of-freedom trajectory optimization problems), a sufficiently small discretization timescale is required (i.e., greater number of segments needed) in order to capture the appropriate dynamics. However, with the present available collocation methodologies, even if only one state variable requires a small discretization timescale, all the state variables must utilize a small timescale due to the architecture of collocation. This limitation results in the size of the Nonlinear Programming (NLP) problem that must be solved becoming enormous due to the fine discretization employed. If a separate discretization scheme can be employed, one for the lower frequency state variables (using a lager

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timescale) and another one for the higher frequency state variables (at a finer timescale), the size of the NLP problem can be greatly reduced. Consequently, the finer discretization timescale can be tailored to only those state variables that have higher frequency dynamics in the governing equations, while the rest of the state variables can utilize a much larger discretization timescale. This two-timescale methodology is well suited for solving six-degree-of-freedom trajectory optimization problems, where the dynamics characterized by the translational motion state variables are at a low frequency, while the dynamics characterized by the rotational motion state variables are at a high frequency.[1]

This paper describes the development of a two-timescale discretization scheme for collocation. Before the discussion of the two-timescale collocation methodology, an overview of the standard collocation method is first provided. Finally, results from a sample test case using the two-timescale methodology are presented, and a comparison of the solutions obtained from this new method and from conventional collocation is provided.

STANDARD COLLOCATION METHODOLOGY

The optimal control problem minimizes a cost function

$$J = J(x, u, t) \tag{1}$$

subject to a set of state equations of motion for the dynamical system

$$dx/dt = \bar{f}(x,u,t) \tag{2}$$

having initial ($\bar{\psi}$) and final boundary ($\bar{\phi}$) and path constraints (g)

$$\bar{\psi}(x(t_0),t_0) = \bar{0} \tag{3}$$

$$\bar{\phi}(x(t_f),t_f) = \bar{0} \tag{4}$$

$$\bar{g}(x,u,t) \le \bar{0} \tag{5}$$

where x and u are vectors representing the state and control variables. In the method of Direct Collocation with Nonlinear Programming (DCNLP) the total time ($T = t_f - t_0$) is discretized into n segments (or nodes), where the length of each segment is $\Delta t = T/n$. The state differential equations (2) are approximated within each segment using an integration formula. The formulation for the approximate integration of the system equations transforms them into a set of discrete algebraic constraints (referred to as defect equations) imposed at each segment in the discretization. Within each segment, evaluation points are selected at which x and u must satisfy (2)-(5). When satisfied, an approximate solution to the system equations is obtained. In this manner, for the overall NLP problem, the state and control parameters are chosen to minimize the performance function while representing an approximate solution to the system equations. The discrete algebraic constraint equations can have different forms depending upon the implicit integration formulation utilized.[2,3]

Figure 1 illustrates the system constraint formulation using a higher-order integration formula based on a fifth-degree Gauss-Lobatto polynomial.[3] In this version of the DCNLP method, each state variable is represented by a quintic polynomial (in time) within each segment. Six parameters are required to define the polynomial uniquely; they can be determined using the values of the states (x_i, x_{i+1}, x_{i+2}) at the boundaries and at the center point, together with the values of the time derivatives f_i, f_{i+1}, and f_{i+2} which correspond to values of dx/dt (Eq. 2) evaluated at t_i, t_{i+1}, and t_{i+2}, respectively. Note that evaluation of f_i, f_{i+1}, and f_{i+2} requires specification of the control variables at the same times (i.e., u_i, u_{i+1}, u_{i+2}). Making use of x_i, x_{i+1}, x_{i+2}, u_i, u_{i+1}, u_{i+2}, f_i, f_{i+1}, and f_{i+2}, a polynomial representing the state time history between the endpoints can be constructed (as shown in Fig. 1).

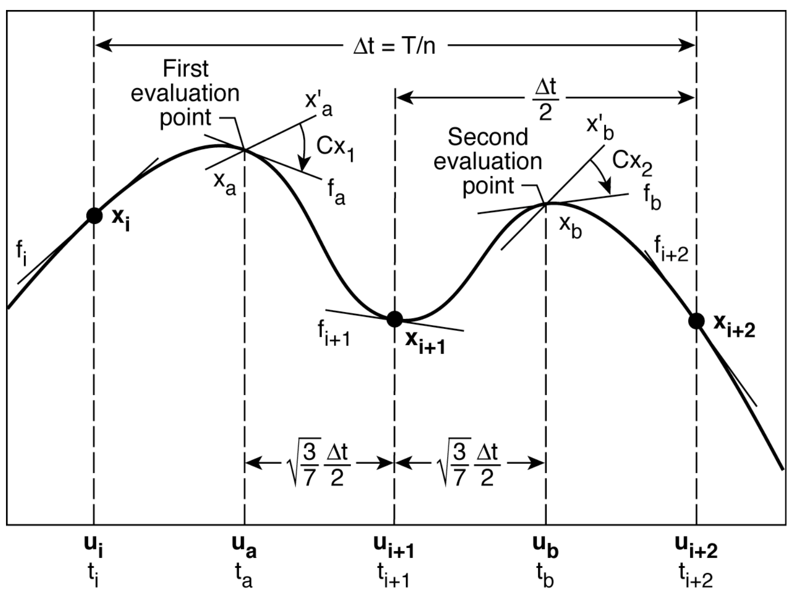

Figure 1 Fifth-degree Gauss-Lobatto System Constraint Formulation

In the fifth-degree Gauss-Lobatto polynomial, two evaluation points are required within each segment. The algebraic constraints take the form[3]

$$C x_1 = f_a - \dot{x}_a = \frac{1}{360}\Big[(32\sqrt{21}+180)\, x_i - 64\sqrt{21}\, x_{i+1} + (32\sqrt{21}-180)\, x_{i+2}$$

$$+ \Delta t\{(9+\sqrt{21})\, f_i + 98\, f_a + 64\, f_{i+1} + (9-\sqrt{21})\, f_{i+2}\}\Big] = 0 \qquad (6)$$

$$C x_2 = f_b - \dot{x}_b = \frac{1}{360}\Big[(-32\sqrt{21}+180)\, x_i + 64\sqrt{21}\, x_{i+1} + (-32\sqrt{21}-180)\, x_{i+2}$$

$$+ \Delta t\{(9-\sqrt{21})\, f_i + 98\, f_b + 64\, f_{i+1} + (9+\sqrt{21})\, f_{i+2}\}\Big] = 0 \qquad (7)$$

where $f_a = f(x_a, u_a, t_a)$ and $f_b = f(x_b, u_b, t_b)$. The states (x_i, x_{i+1}, x_{i+2}) and controls (u_i, u_a, u_{i+1}, u_b, u_{i+2}) shown in bold in Fig. 1 are discrete NLP parameters. The values of these states and controls are selected to force the algebraic constraints to zero. In so doing, the polynomial is made to satisfy the differential equation at the collocation points t_a and t_b of the segment (in addition to being implicitly satisfied at t_i, t_{i+1}, and t_{i+2}). When Eqs. (6) and (7) are satisfied, an approximate solution to the system equations is obtained. The method is discussed in much greater detail in Reference 3.

TWO-TIME SCALE COLLOCATION METHODOLODY

A two-timescale discretization scheme for collocation utilizes a standard timescale discretization scheme for the smoothly-varying lower-frequency state variables and another finer timescale discretization scheme for the higher-frequency state variables. Two different two-timescale discretization schemes are presented: 1) using two segments to represent the higher frequency state variables for every one segment of the low frequency state variables (i.e., a two-to-one discretization architecture), and 2) using four segments to represent the higher frequency state variables for every one segment of the low frequency state variables (i.e., a four-to-one discretization architecture).

Two-To-One Discretization Architecture

In the two-timescale collocation architecture, the state variables are split into the lower frequency state variables represented by the vector x and the higher frequency state variables which are represented by the vector z. The control variables are defined by the vector u. Figure 2 illustrates the system constraint formulation for the two-to-one discretization architecture. Note, this figure has the same features as that of Figure 1, but has been simplified for clarity. As seen, there are two segments defined for the finer discretization state variables z for every one segment for the standard or lower frequency state variables x. The variables in bold are NLP parameters.

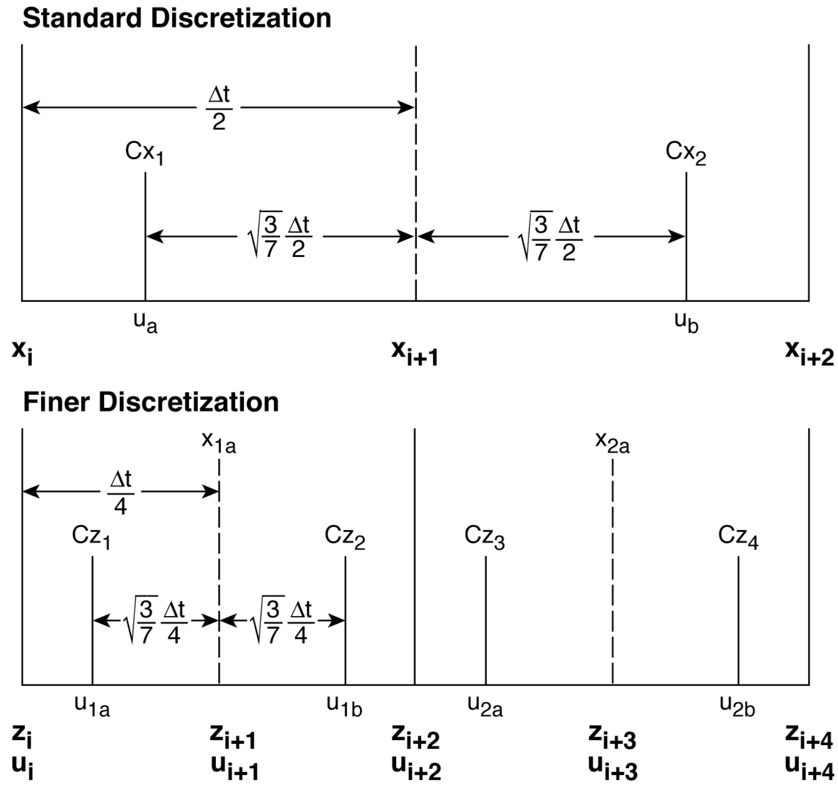

Figure 2 Two-to-one Two-timescale Discretization Scheme

The x state variables utilize the standard collocation discretization scheme. The "defect" algebraic constraint equations are formulated as described in the previous section. All the state and controls variables (x_i, x_{i+1}, x_{i+2}, z_i, z_{i+2}, z_{i+4}, u_i, u_{i+2}, u_{i+4}) at the left-hand side, center, and right-hand side of the segment that are necessary to evaluate the defect constraints equations (Cx_1 and Cx_2) are available (and shown in boldface), except for the control variables u_a and u_b at the left-of-center and right-of-center locations within the segment. In the standard collocation architecture, these variables would be NLP parameters. However, in the two-to-one two-timescale formulation, they have been eliminated. As a result, their values need to be determined. They are obtained by forming a quartic polynomial using the control variables u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} and evaluating this quartic polynomial at the specific times of u_a and u_b. Consequently, all the variables that are required for evaluating the defect constraints equations Cx_1 and Cx_2 using Eqs. (6) and (7) are available for the standard or lower frequency portion of the scheme.

Similarly, for the finer discretization portion of the scheme, the algebraic constraint equations Cz_1 through Cz_4 need to be formulated. Again, a few variables are needed which are not NLP parameters. They are the controls u_{1a}, u_{1b}, u_{2a}, and u_{2b}, along with two additional unknowns, the state variables x_{1a} and x_{2a} at the center points of the two segments. The values for these controls can be obtained by evaluating the same quartic polynomial, defined previously by using u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}, at the appropriate times to yield u_{1a}, u_{1b}, u_{2a}, and u_{2b}. The state variables x_{1a} and x_{2a} can be determined by forming a quintic polynomial using the NLP parameters at the left-hand (x_i, z_i, u_i), center (x_{i+1}, z_{i+2}, u_{i+2}), and right-hand (x_{i+2}, z_{i+4}, u_{i+4}) sides of the scheme, along with their function evaluations f_i, f_{i+2}, f_{i+4} using Eq. (2), and evaluating this polynomial at their respective times. Consequently, all the variables that are required for evaluating the defect constraint equations Cz_1 and Cz_2 in the first segment and Cz_3 and Cz_4 in the second segment utilizing Eqs. (6) and (7) are available for the higher frequency portion of the scheme. This process can be repeated for any number of segments.

Four-To-One Discretization Architecture

If additional refinement is required for the higher frequency state variables, a four-to-one discretization architecture can be utilized. Figure 3 illustrates the system constraint formulation. The overall scheme is very similar to that of the two-to-one architecture. However, there are four segments defined for the finer discretization state variables z for every one segment for the standard or lower frequency state variables x. The variables in bold are again NLP parameters.

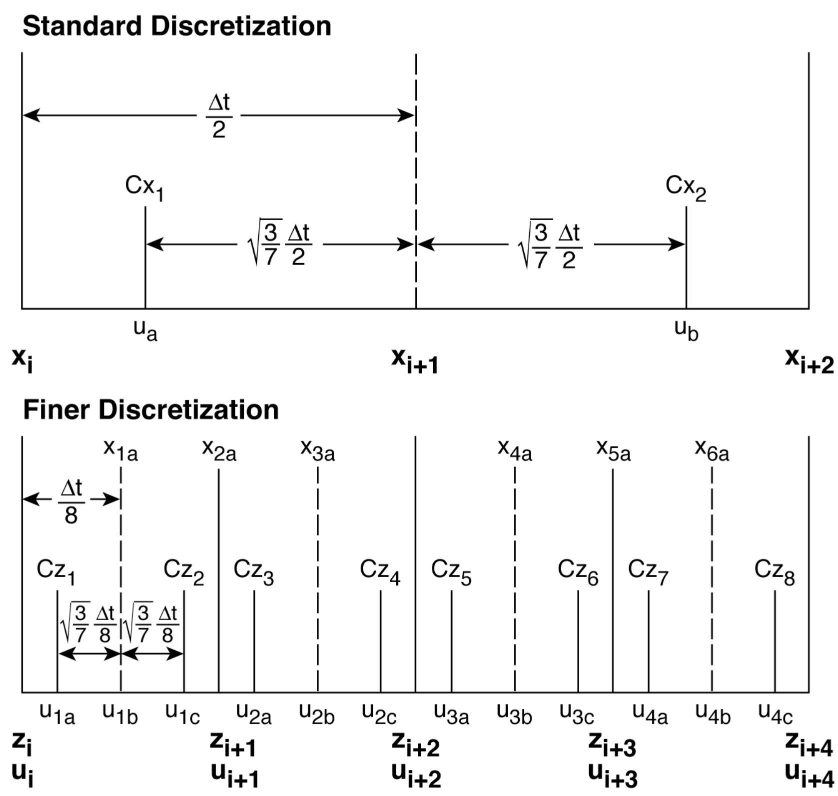

Figure 3 Four-to-one Two-timescale Discretization Scheme

The process for determining the defect constraints equations Cx_1 and Cx_2 in the standard or lower frequency portion of the scheme is the same as that described in the two-to-one two-timescale architecture section. The control variables u_a and u_b are obtained by forming a quartic polynomial using the control variables u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} and evaluating this quartic polynomial at the appropriate times to yield u_a and u_b. Consequently, all the variables that are required for evaluating the defect constraints equations Cx_1 and Cx_2 using Eqs. (6) and (7) are available.

For the finer discretization portion of the scheme, the algebraic constraints equations Cz_1 through Cz_8 are formulated as described in the two-to-one architecture section as well. However, since there are four segments for this architecture, there are additional variables that are unknown and need to be determined. All the controls (u_{ia}, u_{ib}, u_{ic}) interior to each of the four finer descretization segments are obtained by evaluating the same quartic polynomial defined by u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} at their respective times. Similarly, all the interior state variables (x_{ia}) are obtained by forming a quintic polynomial using the NLP parameters at the left-hand (x_i, z_i, u_i), center (x_{i+1}, z_{i+2}, u_{i+2}), and right-hand (x_{i+2}, z_{i+4}, u_{i+4}) sides of the interior segments, along with their function evaluations f_i, f_{i+2}, f_{i+4} using Eq. (2), and evaluating this polynomial at their respective times. Consequently, all the variables that are required for evaluating the defect constraint equations Cz_1 through Cz_8 utilizing Eqs. (6) and (7) within the four segments are available for the higher frequency portion of the scheme. This process can be repeated for any number of segments.

RESULTS

A problem similar to the lunar ascent problem of Bryson and Ho[4] is used to validate the two-timescale collocation architecture.

The two-dimensional lunar ascent problem has four state variables (X and Y Cartesian coordinates, U and V velocities) and one control variable, a thrust pointing angle (θ). The state equations are:

$$\dot{X} = U \tag{8}$$

$$\dot{Y} = V \tag{9}$$

$$\dot{U} = a * \cos\theta - 0.5 * g * \sqrt{U^2 + V^2} / \sqrt{X^2 + Y^2} \tag{10}$$

$$\dot{V} = a * \sin\theta - g - 0.5 * g * \sqrt{U^2 + V^2} / \sqrt{X^2 + Y^2} \tag{11}$$

where a is a constant thrust acceleration and g is gravity. Note that the last term in Eqs. (10) and (11) has been added to increase the coupling between the state equations in an effort to thoroughly exercise the two-timescale architecture. This term behaves similar to a drag force. The objective is to minimize the time to achieve a desired orbit altitude and velocity. The solution history using the standard collocation scheme is first provided. Then, a comparison is made with the results obtained utilizing both the two-to-one and four-to-one two-timescale architectures.

Figures 4-6 show the position, velocity, and control histories using the standard collocation scheme.

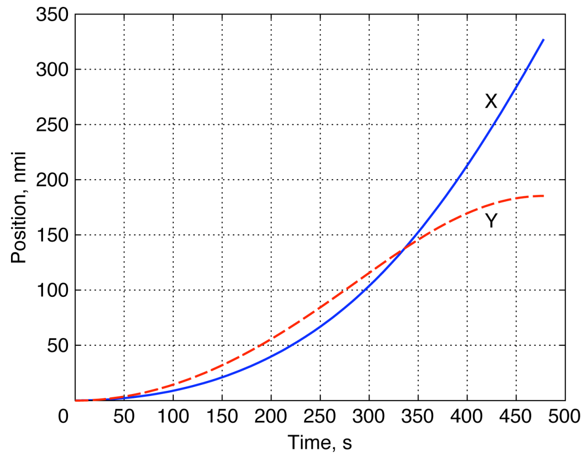

Figure 4 Time History of X And Y Position

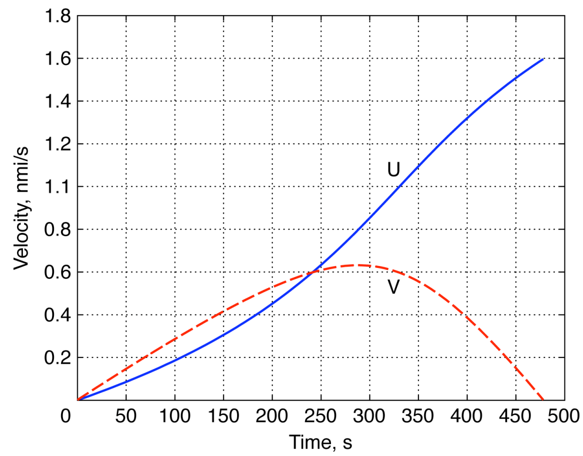

Figure 5 Time History of U and V Velocities

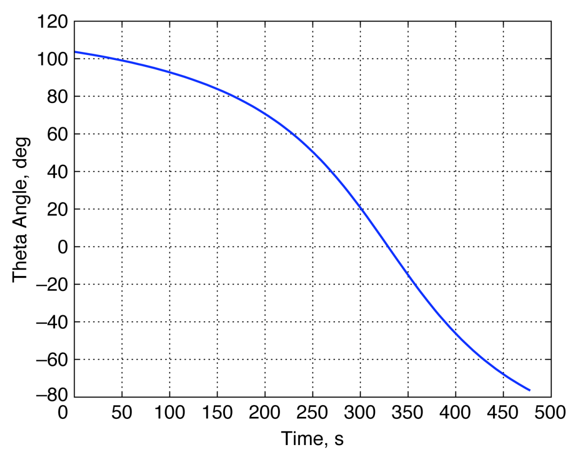

Figure 6 Time History of Theta Angle

Two cases are used to illustrate the two-timescale discretization architecture. Case 1 exercises the two-to-one two-timescale architecture. In this case, the X and Y state variables utilize a standard discretization, while U and V state variables utilize a finer discretization. Specifically, two segments are employed for states U and V for every one segment for states X and Y. Note that for this test case U and V are chosen arbitrarily to act as the “high-frequency” states for the two-timescale discretization; there is no significant difference in the rate at which U and V change in comparison to the other states. The intention is simply to show how well the solution of the problem can be accomplished using two timescales rather than just one. For an equivalent comparison to the two-timescale solution, the standard one-timescale collocation architecture utilizes 20 segments. Table 1 summarizes the results.

Table 1
COMPARISON OF THE STANDARD AND TWO-TO-ONE
TWO-TIMESCALE ARCHITECTURES

	# of Segments	# of NLP Parameters	# of Defects Equations	# of Iterations	Max Difference
Standard	20	246	160	104	— N.A. —
Case 1	10/20	166	120	58	0.07%, 0.5%

As seen, the number of NLP parameters is significantly reduced from 246 for the standard one-timescale collocation architecture to 166 for the two-to-one two-timescale discretization collocation architecture. The number of algebraic constraint equations is reduced as well. Consequently, the resulting number of iterations required for solution convergence is significantly reduced from 104 to 58, respectively. Furthermore, nearly identical results are obtained with the two-timescale collocation architecture. The maximum difference observed in any of the state variables is less than 0.07 percent between the two methods. The difference in the control variable is only slightly higher (less than 0.5 percent). These results illustrates that a significant reduction in the number of NLP parameters, and hence solution convergence, can be achieved without sacrificing solution accuracy.

Case 2 exercises the four-to-one two-timescale architecture. In this case, the X, Y, and U state variables utilize a standard discretization, while the V state variable utilizes a finer discretization. Specifically, four segments are employed for the state V for every one segment for states X, Y, and U. Note that for this test case V is chosen arbitrarily to act as the “high-frequency” state for the two-timescale discretization. There is no significant difference in the rate at which V changes in comparison to the other states. The intention is simply to show how well the solution of the problem can be accomplished using two timescales rather than just one. For an equivalent comparison to the two-timescale solution, the standard one-timescale collocation architecture utilizes 40 segments. Table 2 summarizes the results.

Table 2
COMPARISON OF THE STANDARD AND FOUR-TO-ONE
TWO-TIMESCALE ARCHITECTURES

	# of Segments	# of NLP Parameters	# of Defects Equations	# of Iterations	Max Difference
Standard	40	486	320	199	— N.A. —
Case 1	10/40	186	140	64	0.08%, 0.3%

As seen, the number of NLP parameters is significantly reduced from 486 for the standard one-timescale collocation architecture to 186 for the two-to-one two-timescale discretization collocation architecture. The number of algebraic constraint equations is reduced considerably as well from 320 to 140. Consequently, the resulting number of iterations required for solution convergence is significantly reduced from 199 to 64. Furthermore, nearly identical results are obtained with the two-timescale collocation architecture. The maximum difference observed in any of the state variables is less than 0.08 percent between the two methods. The difference in the control variable is only slightly higher (less than 0.3 percent). These results again illustrate that a significant reduction in the number of NLP parameters, and hence solution convergence, can be achieved without sacrificing solution accuracy.

CONCLUSIONS

The development of a two-timescale discretization scheme for collocation is presented. This scheme allows a larger discretization to be utilized for smoothly-varying state variables and a second finer discretization to be utilized for state variables having higher frequency dynamics. As such, the finer discretization timescale can be tailored to only those state variables that have higher frequency dynamics in the governing equations, while the rest of the state variables can utilize a much larger discretization timescale. In so doing, the size of the overall NLP problem is significantly reduced as compared to the conventional single-timescale collocation architecture.

Two two-timescale architecture discretization schemes are described: 1) using two segments to represent the higher frequency state variables for every one segment of the low frequency state variables, and 2) using four segments to represent the higher frequency state variables for every one segment of the low frequency state variables. Results from two test cases are presented to validate the two-timescale collocation architecture, and compared to the solution obtained from the conventional collocation method. Comparison shows a very good agreement between the two methods with a maximum difference of less than 0.08 percent for the state variables and less than 0.5 percent for the control variables. The number of NLP parameters and iterations required for convergence for the two-timescale scheme can be reduced by two-thirds. Consequently, a significant reduction in the number of NLP parameters, and hence solution convergence, can be achieved without sacrificing solution accuracy.

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