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A THEORY FOR INFLATED THIN-WALL CYLINDRICAL BEAMS

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SUMMARY

Nonlinear equilibrium equations are derived for the twisting, bending, and stretching of pressurized thin-wall cylindrical beams. Thin-wall beams, including those which depend on internal pressure for their load-carrying ability, appear to have application where lightweight structural members are required. Example problems involving column buckling and the bending of a beam column are solved with a set of linearized equations to illustrate the theory. The linearized equations reduce to the Timoshenko beam equations when internal pressure and axial force are set equal to zero.

INTRODUCTION

Lightweight inflatable plates and cylinders which depend on internal pressure for much of their load-carrying ability are receiving considerable attention in aerospace research. Packageability and automatic erectability make inflatable construction especially attractive for satellites and space stations. Inflatable construction may also provide a suitable means of building reentry vehicles with wing loadings low enough to alleviate significantly the problem of reentry heating. In addition, inflatable membrane cylinders may be quite useful as booms in a paraglider or similar vehicle.

For all these applications, information is needed on the load-carrying capabilities of inflatable structures. Simple methods of analysis have been used successfully to determine collapse loads for certain types of inflated structures (refs. 1 and 2). In addition, a linear theory for the structural analysis of inflatable plates has been presented in reference 3 and has been compared favorably with experiments in reference 4. Reference 5 contains a combined aerodynamic and structural analysis of a paraglider wing consisting of a flexible triangular sail between three equally spaced booms which are joined at the nose. In the analysis, stresses and deflections of the sail were calculated under the assumption that the booms were rigid. The results of the present paper may be of use in extending the analysis of the paraglider wing with rigid booms to the case of a wing with flexible booms.

The purpose of the present investigation is to develop for the long, inflated, circular-cylindrical beam a structural theory which allows for moderately large displacements, and which can be reduced in a consistent manner to a simple system of linear equations. The theory is applicable to pressurized membrane cylinders (that is, cylinders having negligible wall bending stiffness) when the internal pressure and applied forces are such that the cylinder wall remains in tension. The application, however, is not limited to membrane cylinders; the theory can be applied to pressurized or unpressurized thin-wall cylindrical beams in which wall bending stiffness is sufficient to prevent local buckling of the wall.

SYMBOLS

A	cross-sectional area of cylindrical shell, $2\pi rt$
B_1,B_2	arbitrary constants
C	constant of integration
E	Young's modulus
F	compressive axial force
G	shear modulus
I	area moment of inertia of cylindrical shell, $\pi r^3 t$
K_B	Euler buckling load, $\dfrac{EI\pi^2}{L^2}$
K_S	shear stiffness parameter, $G\pi rt$
L	length of cylindrical beam
M_1,M_2,M_3	moments about the X-, Y-, and Z-axes, respectively (see fig. 1)
N	total axial force on a beam cross section (see fig. 1)
$N_x,N_{x\theta}$	axial- and shear-stress resultants in shell
P	axial force due to internal pressure, $p\pi r^2$

p	internal pressure
Q	payload
q_1,q_2,q_3	applied distributed loads in the X-, Y-, and Z-directions, respectively
r	radius of circular cylindrical shell
T_1,T_2,T_3	applied distributed moments about X-, Y-, and Z-axes, respectively
t	cylinder wall thickness
U_1,U_2,U_3	translations of a beam cross section in X-, Y-, and Z-directions, respectively
u,v,w	axial, tangential, and radial displacements of a point in surface of shell (see fig. 1)
V_2,V_3	shear forces in Y- and Z-directions, respectively (see fig. 1)
ΔV	change in enclosed volume of cylindrical shell due to deformation
W	work done by externally applied loads during deformation
X,Y,Z	rectangular Cartesian coordinates (see fig. 1)
x,y,z	distances along X-, Y-, and Z-axes, respectively
β	parameter defined in problem 1, $\beta^2 = \frac{F}{E\pi r^3 t}\left(\frac{P + G\pi rt}{P + G\pi rt - F}\right)$
$\gamma_{x\theta}$	local shear strain in membrane
γ_1	angle of twist of beam cross section per unit length
γ_2,γ_3	shear strain of a beam cross section in the Y- and Z-directions, respectively
ϵ_x	local extensional strain in shell in X-direction

ϵ_1 extensional strain of cylindrical beam in X-direction

ξ, η, ζ coordinates of a point in deformed cylindrical shell, referred to the X-, Y-, and Z-axes, respectively

θ circumferential angular coordinate

κ_2, κ_3 curvature of cross-section center line in XZ- and XY-planes, respectively

λ parameter defined in problem 2, $\lambda^2 = \dfrac{\bar{N}(P + K_S)}{EI(P + K_S - \bar{N})}$

Π_1 strain energy of deformed cylindrical shell

Π_2 change in potential energy of pressurizing gas due to deformation

ρ_0, ρ_L variables of integration (see eq. (7))

$\omega_1, \omega_2, \omega_3$ rotation of a beam cross section in the YZ-, XZ-, XY-planes, respectively

Subscripts:

cr critical

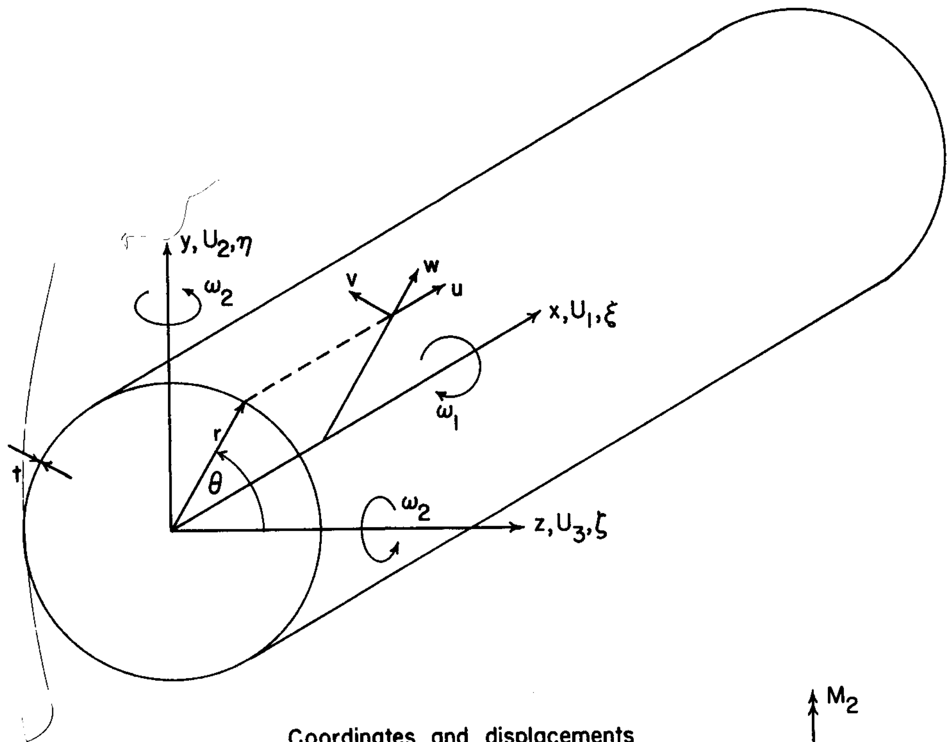
i index

Barred quantities represent the applied forces or moments at the ends of the beam.

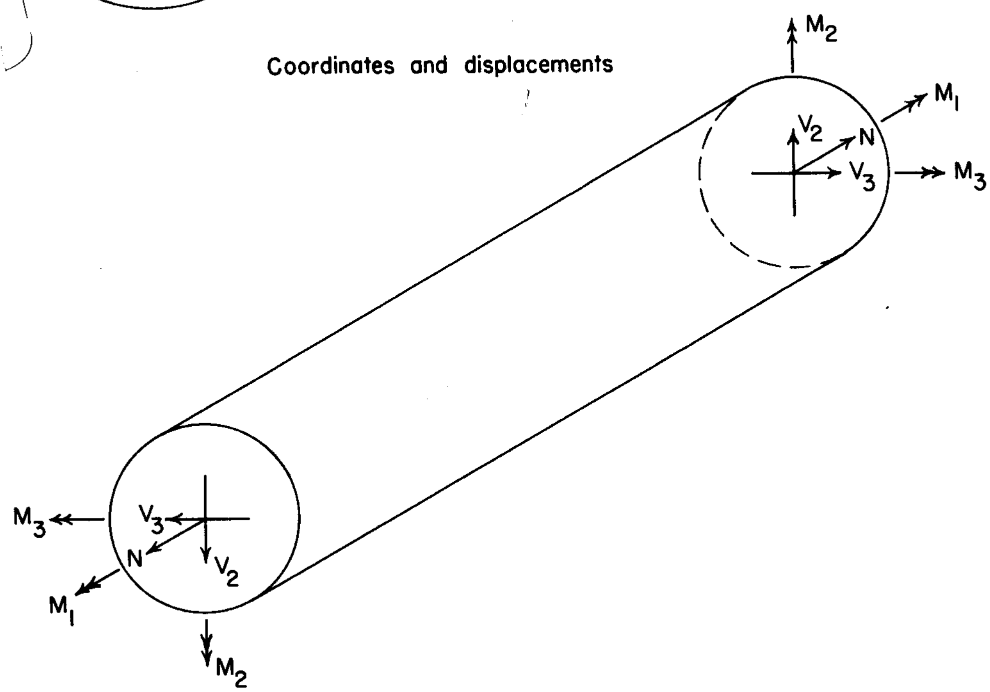
ANALYSIS

A sketch of the inflatable circular-cylindrical beam under investigation, showing the forces, moments, displacements, and coordinate systems, is presented in figure 1. The ends of the beam are considered to serve no function other than to contain the internal pressure.

Two important simplifying assumptions are made in the analysis: first, any cross section of the pressurized cylindrical beam remains undeformed under a system of loads, so that deformations of the beam are restricted to translations and rotations of a beam cross section; and second, the translations and rotations of a cross section are small enough that the displacements of a point in the surface of the beam can be represented



Coordinates and displacements



Forces and moments

Figure 1.- Sketch of cylindrical beam.

with little error by the vector sums of the displacements due to translations and rotations of a cross section. A further simplification attendant to these assumptions is that the circumferential strain is negligible and may be disregarded in the analysis. Under the present assumptions, the theory holds for long cylindrical beams having walls with either significant or negligible bending stiffness, provided that local buckling or wrinkling does not occur.

Derivation of Governing Nonlinear Equations

In order to derive a system of equations, use is made of the variational equation

$$\delta\left(\Pi_1 + \Pi_2 - W\right) = 0 \tag{1}$$

where Π_1 is the strain energy of the cylinder, Π_2 is the change in the potential energy of the pressurizing gas due to deformation, and W is the work done by externally applied loads during deformation. If the circumferential strain is neglected, then

$$\Pi_1 = \frac{1}{2} \int_0^L \int_0^{2\pi} \left(N_x \epsilon_x + N_{x\theta} \gamma_{x\theta}\right) r \, d\theta \, dx \tag{2}$$

Also,

$$\Pi_2 = -p\Delta V \tag{3}$$

where p is the internal pressure (assumed constant during deformation) and ΔV is the change in enclosed volume due to deformation.

From reference 6, the nonlinear strain-displacement relations which include all second-degree terms are found, after some manipulation, to be

$$\epsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{4a}$$

and

$$\gamma_{x\theta} = \frac{\partial v}{\partial x}\left(1 - \frac{\partial u}{\partial x}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\left(1 - \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{w}{r}\right) + \frac{1}{r}\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial \theta} - v\right)$$

However, with the assumption that $\frac{\partial u}{\partial x}$ and $\frac{1}{r}\left(\frac{\partial v}{\partial \theta} + w\right)$ are small compared with 1, the shear strain reduces to

$$\gamma_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial \theta} - v\right) \tag{4b}$$

In terms of the translations and small rotations of a cross section, the local displacements are given by:

$$u = U_1 + r \sin \omega_2 \cos \theta - r \sin \omega_3 \sin \theta$$

$$v = U_2 \cos \theta - U_3 \sin \theta - r \sin \omega_1 + \frac{r}{2}\left(\cos \omega_3 - \cos \omega_2\right)\sin 2\theta$$

$$w = U_2 \sin \theta + U_3 \cos \theta - r\left[1 - \cos \omega_1 + \left(1 - \cos \omega_2\right)\cos^2\theta + \left(1 - \cos \omega_3\right)\sin^2\theta\right]$$

For small rotations

$$\sin \omega_i \approx \omega_i$$

$$\cos \omega_i \approx 1 - \frac{\omega_i^2}{2}$$

and the local displacements reduce to

$$\left.\begin{aligned} u &= U_1 + r\omega_2 \cos \theta - r\omega_3 \sin \theta \\ v &= U_2 \cos \theta - U_3 \sin \theta - r\omega_1 + \frac{r}{4}\left(\omega_2^2 - \omega_3^2\right)\sin 2\theta \\ w &= U_2 \sin \theta + U_3 \cos \theta - \frac{r}{2}\left(\omega_1^2 + \omega_2^2\cos^2\theta + \omega_3^2\sin^2\theta\right) \end{aligned}\right\} \tag{5}$$

The change in enclosed volume due to these displacements may now be computed by expressing the coordinates of a point in the deformed cylindrical beam in terms of the local displacements and the coordinates of the undeformed beam. Let

$$\left.\begin{aligned} \xi &= u + x \\ \eta &= (r + w)\sin \theta + v \cos \theta \\ \zeta &= (r + w)\cos \theta - v \sin \theta \end{aligned}\right\} \tag{6}$$

Then by the use of Gauss' theorem (ref. 7) the change in volume is

$$\Delta V = \int\int_S \zeta(\xi,\eta) \frac{\partial(\xi,\eta)}{\partial(x,\theta)} \, dx \, d\theta$$

where the integration is over the total deformed surface S. By a transformation of coordinates,

$$\Delta V = -\pi r^2 L + \int_0^{2\pi}\int_0^L \zeta(\xi,\eta)\frac{\partial(\xi,\eta)}{\partial(x,\theta)}\,dx\,d\theta + \int_0^{2\pi}\int_0^r \zeta\left(\xi_L,\eta_L\right)\frac{\partial\left(\xi_L,\eta_L\right)}{\partial\left(\rho_L,\theta_L\right)}\,d\rho_L\,d\theta_L$$

$$- \int_0^{2\pi}\int_0^r \zeta_0\left(\xi_0,\eta_0\right)\frac{\partial\left(\xi_0,\eta_0\right)}{\partial\left(\rho_0,\theta_0\right)}\,d\rho_0\,d\theta_0 \tag{7}$$

where the subscripts 0 and L refer to the ends $x = 0$ and $x = L$, respectively.

Inasmuch as the cross-section displacements and rotations are functions only of x, integration with respect to θ and substitution from equations (5) and (6) yield

$$\Delta V = \pi r^2 \int_0^L \left[U_1' + U_2'\omega_3 - U_3'\omega_2 - \frac{1}{2}\left(\omega_2^2 + \omega_3^2\right)\right] dx \tag{8}$$

where primes denote total differentiation with respect to x, and where terms of the third or higher degree in the displacements and rotations and their derivatives are neglected.

To the same degree of approximation, the local strains expressed in terms of cross-section rotations and translations are obtained by substituting equations (5) into equations (4):

$$\left.\begin{aligned}
\epsilon_x &= U_1' + r\omega_2'\cos\theta - r\omega_3'\sin\theta + \frac{1}{2}\Big[U_2'^2 + U_3'^2 + r^2\omega_1'^2 \\
&\quad - 2r\left(\omega_1'U_2'\cos\theta - \omega_1'U_3'\sin\theta\right)\Big] \\
\gamma_{x\theta} &= -r\omega_1' + \left(U_2' - \omega_3 + \omega_1 U_3'\right)\cos\theta - \left(U_3' + \omega_2 - \omega_1 U_2'\right)\sin\theta \\
&\quad + \frac{r}{2}\left(\omega_2'\omega_2 - \omega_3'\omega_3\right)\sin 2\theta
\end{aligned}\right\} \tag{9}$$

Substitution of equations (9) into equation (2) gives

$$\begin{aligned}
\Pi_1 = \frac{r}{2}\int_0^{2\pi}\int_0^L\Bigg\{ & N_x\Big[U_1' + r\left(\omega_2' - \omega_1'U_2'\right)\cos\theta - r\left(\omega_3' - \omega_1'U_3'\right)\sin\theta \\
& + \frac{1}{2}\left(r^2\omega_1'^2 + U_2'^2 + U_3'^2\right)\Big] + N_{x\theta}\Big[-r\omega_1' + \left(U_2' - \omega_3 + \omega_1 U_3'\right)\cos\theta \\
& - \left(U_3' + \omega_2 - \omega_1 U_2'\right)\sin\theta + \frac{r}{2}\left(\omega_2'\omega_2 - \omega_3'\omega_3\right)\sin 2\theta\Big]\Bigg\}dx\,d\theta
\end{aligned} \tag{10}$$

In terms of the forces and moments on a beam cross section (see fig. 1), the local stress resultants are defined by

$$\left.\begin{aligned} N_x &= \frac{N}{2\pi r} + \frac{M_2}{\pi r^2}\cos\theta - \frac{M_3}{\pi r^2}\sin\theta \\ N_{x\theta} &= -\frac{M_1}{2\pi r^2} + \frac{V_2}{\pi r}\cos\theta - \frac{V_3}{\pi r}\sin\theta \end{aligned}\right\} \tag{11}$$

Substitution of equations (11) into equation (10) and integration over θ yield

$$\begin{aligned} \Pi_1 = \frac{1}{2}\int_0^L \Bigg\{ & N\left[U_1' + \frac{1}{2}\left(r^2\omega_1'^2 + U_2'^2 + U_3'^2\right)\right] + M_2\left(\omega_2' - \omega_1'U_2'\right) + M_3\left(\omega_3' - \omega_1'U_3'\right) \\ & + M_1\omega_1' + V_2\left(U_2' - \omega_3 + \omega_1U_3'\right) + V_3\left(U_3' + \omega_2 - \omega_1U_2'\right)\Bigg\} dx \end{aligned} \tag{12}$$

If now direct, bending, and shearing strains for the beam are defined by

$$\epsilon_1 = U_1' + \frac{1}{2}\left(\underline{r^2\omega_1'^2} + U_2'^2 + U_3'^2\right) \tag{13a}$$

$$\kappa_2 = \omega_2' - \underline{\omega_1'U_2'} \tag{13b}$$

$$\kappa_3 = \omega_3' - \underline{\omega_1'U_3'} \tag{13c}$$

$$\gamma_1 = \underline{\omega_1'} \tag{13d}$$

$$\gamma_2 = U_2' - \omega_3 + \underline{\omega_1U_3'} \tag{13e}$$

$$\gamma_3 = U_3' + \omega_2 - \underline{\omega_1U_2'} \tag{13f}$$

then

$$\Pi_1 = \frac{1}{2}\int_0^L \left(N\epsilon_1 + M_2\kappa_2 + M_3\kappa_3 + M_1\gamma_1 + V_2\gamma_2 + V_3\gamma_3\right)dx \tag{14}$$

In terms of the beam strains, the forces and moments may be defined by

$$N = 2E\pi rt\epsilon_1 \tag{15a}$$

$$V_2 = G\pi rt\gamma_2 \tag{15b}$$

$$V_3 = G\pi rt\gamma_3 \tag{15c}$$

$$M_1 = 2G\pi r^3 t\gamma_1 \tag{15d}$$

$$M_2 = E\pi r^3 t\kappa_2 \tag{15e}$$

$$M_3 = E\pi r^3 t\kappa_3 \tag{15f}$$

Because of the assumption that the circumferential strain is zero, the material constants E and G may be defined independently and, hence, the definitions of these constants hold for orthotropic materials. Therefore, the variation of the strain energy may be written as

$$\delta\Pi_1 = \int_0^L \left(N\delta\epsilon_1 + M_2\delta\kappa_2 + M_3\delta\kappa_3 + M_1\delta\gamma_1 + V_2\delta\gamma_2 + V_3\delta\gamma_3\right) dx \tag{16}$$

The variation of work due to externally applied loads may be defined by

$$\begin{aligned}\delta W = &\int_0^L \left(q_1\delta U_1 + q_2\delta U_2 + q_3\delta U_3 + T_1\delta\omega_1 + T_2\delta\omega_2 + T_3\delta\omega_3\right)dx \\ &+ \left(\bar{N}\delta U_1 + \bar{V}_2\delta U_2 + \bar{V}_3\delta U_3 + \bar{M}_1\delta\omega_1 + \bar{M}_2\delta\omega_2 + \bar{M}_3\delta\omega_3\right)\Big|_0^L\end{aligned} \tag{17}$$

Substitution of equations (3), (8), (16), and (17) into equation (1) yields the variational equation

$$\begin{aligned}\delta\left(\Pi_1 + \Pi_2 - W\right) = &\int_0^L \Bigg\{\bigg(N\delta\epsilon_1 + V_2\delta\gamma_2 + V_3\delta\gamma_3 + M_1\delta\gamma_1 + M_2\delta\kappa_2 \\ &+ M_3\delta\kappa_3\bigg) - p\pi r^2\delta\left[U_1{}' + U_2{}'\omega_3 - U_3{}'\omega_2 - \frac{1}{2}\left(\omega_2{}^2 + \omega_3{}^2\right)\right] \\ &- \left(q_1\delta U_1 + q_2\delta U_2 + q_3\delta U_3 + T_1\delta\omega_1 + T_2\delta\omega_2 + T_3\delta\omega_3\right)\Bigg\}dx \\ &- \left(\bar{N}\delta U_1 + \bar{V}_2\delta U_2 + \bar{V}_3\delta U_3 + \bar{M}_1\delta\omega_1 + \bar{M}_2\delta\omega_2 + \bar{M}_3\delta\omega_3\right)\Big|_0^L = 0\end{aligned} \tag{18}$$

Use of the fundamental lemma of the calculus of variations, and integration by parts where necessary, yield the equilibrium equations and boundary conditions:

$$\left.\begin{aligned}
&N' + q_1 = 0 \\
&\left(NU_2'\right)' - \left(M_2\omega_1'\right)' + V_2' - \left(V_3\omega_1'\right)' - P\omega_3' + q_2 = 0 \\
&\left(NU_3'\right)' - \left(M_3\omega_1'\right)' + V_3' + \left(V_2\omega_1'\right)' + P\omega_2' + q_3 = 0 \\
&r^2\left(N\omega_1'\right)' + M_1' - \left(M_2U_2' + M_3U_3'\right)' + V_3U_2' - V_2U_3' + T_1 = 0 \\
&M_2' - V_3 - P\left(\omega_2 + U_3'\right) + T_2 = 0 \\
&M_3' + V_2 + P\left(U_2' - \omega_3\right) + T_3 = 0
\end{aligned}\right\} \tag{19}$$

where $P = p\pi r^2$. The boundary conditions at the ends of the beam are either

$$\left.\begin{aligned}
&U_1 = 0 \quad \text{or} \quad N - P - \bar{N} = 0 \\
&U_2 = 0 \quad \text{or} \quad NU_2' - M_2\omega_1' + V_2 - V_3\omega_1 - P\omega_3 - \bar{V}_2 = 0 \\
&U_3 = 0 \quad \text{or} \quad NU_3' - M_3\omega_1' + V_3 + V_2\omega_1 + P\omega_2 - \bar{V}_3 = 0 \\
&\omega_1 = 0 \quad \text{or} \quad r^2N\omega_1' + M_1 - M_2U_2' - M_3U_3' - \bar{M}_1 = 0 \\
&\omega_2 = 0 \quad \text{or} \quad M_2 - \bar{M}_2 = 0 \\
&\omega_3 = 0 \quad \text{or} \quad M_3 - \bar{M}_3 = 0
\end{aligned}\right\} \tag{20}$$

Equations (19), along with equations (13) substituted into equations (15), constitute a system of 12 equations in the 12 unknown displacements, rotations, forces, and moments. As long as the nonlinear terms are retained, solutions usually will be obtained only by numerical integration or by one of several approximate-solution methods.

Linearized Equations

For many practical applications involving pressurized cylindrical beams, twist about the longitudinal axis is negligibly small. With the assumption of zero twist about the X-axis, equations (19) and (20) simplify to

$$N' + q_1 = 0 \tag{21a}$$

$$\left(NU_2'\right)' + V_2' - P\omega_3' + q_2 = 0 \tag{21b}$$

$$\left(NU_3'\right)' + V_3' + P\omega_2' + q_3 = 0 \tag{21c}$$

$$M_2' - V_3 - P\left(U_3' + \omega_2\right) + T_2 = 0 \tag{21d}$$

$$M_3' + V_2 + P\left(U_2' - \omega_3\right) + T_3 = 0 \tag{21e}$$

with boundary conditions at the ends of the beam being

$$\left.\begin{array}{lll} U_1 = 0 & \text{or} & N - P - \bar{N} = 0 \\ U_2 = 0 & \text{or} & NU_2' + V_2 - P\omega_3 - \bar{V}_2 = 0 \\ U_3 = 0 & \text{or} & NU_3' + V_3 + P\omega_2 - \bar{V}_3 = 0 \\ \omega_2 = 0 & \text{or} & M_2 - \bar{M}_2 = 0 \\ \omega_3 = 0 & \text{or} & M_3 - \bar{M}_3 = 0 \end{array}\right\} \tag{22}$$

The assumption of zero twist allows the underscored terms in equations (13) to be dropped. Since the first of equations (13) is still nonlinear, equations (21) remain, in one sense, nonlinear, because they retain their ability to handle problems involving large lateral displacements. However, whether or not the first of equations (13) is linearized, equations (21) are "formally" linear because equation (21a) may be solved without regard for the lateral loads and deformations. Equations (21), which will be referred to as linear, are equivalent to the well-known Timoshenko beam equations (see ref. 8), with the additional effects of axial force, internal pressure, and large lateral displacements accounted for.

ILLUSTRATIVE EXAMPLES

Two sample problems have been solved to illustrate the usefulness of the simplified system of equations. The first problem is that of the buckling of a pin-end column under a compressive axial force. The second problem is concerned with finding the displacements and rotations of a pin-end beam column under a compressive axial force and a linearly varying transverse load. For simplicity, displacement and rotation only in the XY-plane are considered.

Problem 1.- The configuration to be analyzed is shown in figure 2.

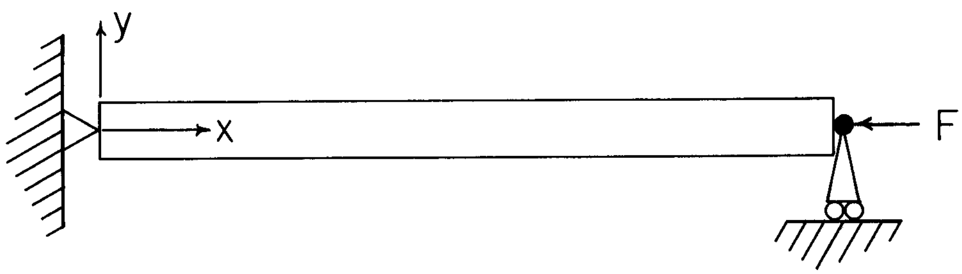


Figure 2.- Pin-end column under compressive axial load.

If, in equations (21),

$$q_1 = q_2 = q_3 = T_2 = T_3 = \bar{M}_2 = \bar{M}_3 = \omega_2 = U_3 = 0$$

and

$$\bar{N} = -F$$

then, from equation (21a),

$$N = P - F$$

and equations (21b) and (21e) become

$$(P + G\pi rt - F)U_2'' - (P + G\pi rt)\omega_3' = 0 \tag{23a}$$

$$E\pi r^3 t\omega_3'' - (P + G\pi rt)\left(U_2' - \omega_3\right) = 0 \tag{23b}$$

The boundary conditions are

$$U_2(0) = U_2(L) = \omega_3'(0) = \omega_3'(L) = 0 \tag{24}$$

Integration of equation (23a) and substitution into equation (23b) give

$$\omega_3'' + \beta^2\omega_3 = \frac{\beta^2 C}{F} \tag{25}$$

where

$$\beta^2 = \frac{F}{E\pi r^3 t}\left(\frac{P + G\pi rt}{P + G\pi rt - F}\right)$$

and C is a constant of integration. Then

$$\omega_3 = B_1 \sin \beta x + B_2 \cos \beta x + \frac{C}{F} \tag{26}$$

Use of the boundary condition $\omega_3'(0) = \omega_3'(L) = 0$ (eq. (24)) yields

$$B_1 = 0$$

and, for a nontrivial solution

$$\beta L = \pi \tag{27}$$

from which equation the lowest critical value of F is found. Substitution for β into equation (27) and substitution of I for $\pi r^3 t$ gives

$$F_{cr} = \frac{EI \frac{\pi^2}{L^2}(P + G\pi rt)}{EI \frac{\pi^2}{L^2} + P + G\pi rt} \tag{28}$$

This result is in agreement with the results presented in reference 9.

In equation (28) may be seen the relationship which obtains between the bending stiffness and the shear stiffness (including shear stiffness provided by internal pressure) in the expression for the critical axial compressive force. Denoting the Euler buckling load $EI \frac{\pi^2}{L^2}$ by K_B and the shear stiffness $P + G\pi rt$ by $P + K_S$ transforms equation (28) to

$$F_{cr} = \frac{K_B\left(P + K_S\right)}{K_B + P + K_S} = \frac{K_B}{1 + \frac{K_B}{P + K_S}} = \frac{P + K_S}{1 + \frac{P + K_S}{K_B}} \tag{29}$$

For a beam very stiff in shear $\left(\frac{K_B}{P + K_S} \to 0\right)$, there results

$$F_{cr} \to K_B$$

the Euler buckling load. Similarly, for a beam very weak in shear $\left(\frac{P + K_S}{K_B} \to 0\right)$,

$$F_{cr} \to P + K_S$$

the shear buckling load. Equation (29) shows that internal pressure increases the resistance to transverse shear deformation. In fact, for inflated bodies made of single-layer fabrics for which a low effective shear modulus may be expected, the pressure contribution may well predominate. (See ref. 10.)

Problem 2.- The second example is an application of the present results to a simplified paraglider problem. It was shown in reference 5 that the stresses normal to the leading edges of the paraglider sail vary linearly along the edges. Then if the leading-edge boom is considered to be a long inflatable cylindrical beam, and the payload Q is positioned directly under the centroid of the linearly varying load on the boom by two shroud lines of lengths l_1 and l_2 (see fig. 3), the problem may be defined in the following manner: The conditions of problem 1 still hold, except that

$$q_2 = \frac{2Qx}{L^2}$$

and

$$N = P - \bar{N}$$

where, from considerations of equilibrium and the geometry of the boom-shroud-payload configuration,

$$\bar{N} = \frac{Q}{3\left[\left(\frac{3l_1}{2L}\right)^2 - 1\right]^{1/2}}$$

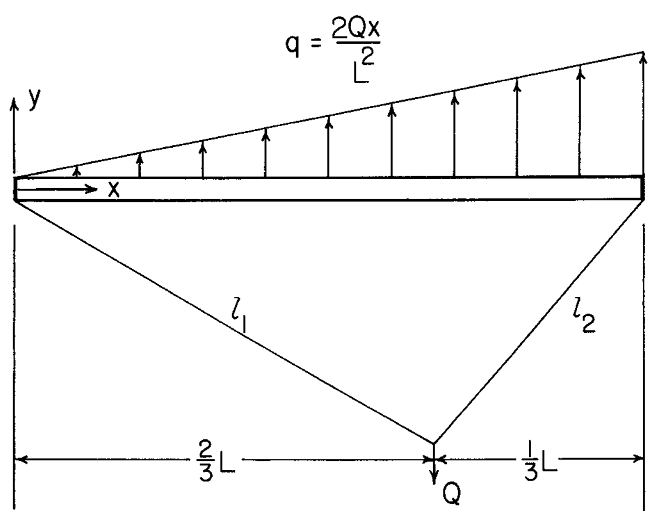


Figure 3.- Pin-end beam column under compressive axial force and transverse load.

and

$$l_2 = \left(l_1^2 - \frac{L^2}{3}\right)^{1/2}$$

with l_1 given $\left(l_1 > \frac{2}{3} L\right)$.

Equations (21b) and (21e) become

$$\left(P + K_S - \bar{N}\right)U_2'' - \left(P + K_S\right)\omega_3' = -\frac{2Qx}{L^2} \tag{30}$$

and

$$EI\omega_3'' + \left(P + K_S\right)\left(U_2' - \omega_3\right) = 0 \tag{31}$$

with the boundary conditions

$$U_2(0) = U_2(L) = \omega_3'(0) = \omega_3'(L) = 0 \tag{32}$$

Solution of equations (30) and (31) subject to the boundary conditions yields

$$U_2 = \frac{2EIQ}{\bar{N}^2 L}\left(\frac{\sin \lambda x}{\sin \lambda L} - \frac{x}{L}\right) + \frac{Qx}{3\bar{N}}\left(\frac{x^2}{L^2} - 1\right) \tag{33}$$

and

$$\omega_3 = \frac{2Q}{\bar{N}}\left[\frac{\cos \lambda x}{\lambda L \sin \lambda L} + \frac{1}{2}\left(\frac{x^2}{L^2} - \frac{2}{\lambda^2 L^2} - \frac{1}{3}\right)\right] \tag{34}$$

where

$$\lambda^2 = \frac{\bar{N}\left(P + K_S\right)}{EI\left(P + K_S - \bar{N}\right)}$$

Expressions for the shear V_2 and the moment M_3 may be obtained by use of equations (33) and (34) along with equations (13) and (15):

$$V_2 = \frac{2QEI}{\bar{N}L^2\left(1 + \frac{P}{K_S}\right)}\left(\frac{\lambda L \cos \lambda x}{\sin \lambda L} - 1\right) \tag{35}$$

$$M_3 = \frac{2QEI}{\bar{N}L}\left(\frac{x}{L} - \frac{\sin \lambda x}{\sin \lambda L}\right) \tag{36}$$

Another quantity of interest is the relative axial displacement of the ends of the cylindrical beam. From equations (13a) and (15a), if $N = P - \bar{N}$,

$$U_1' = \frac{P - \bar{N}}{EA} - \frac{1}{2} U_2'^2$$

where $A = 2\pi rt$. With the condition that $U_1(0) = 0$, the axial displacement is

$$U_1 = \frac{P - \bar{N}}{EA} x - \frac{1}{2}\int_0^x U_2'^2 dx \tag{37}$$

Substitution for $U_2'(x)$ and integration yield for the relative axial displacement of the ends

$$U_1(L) = \frac{P - \bar{N}}{EA} L - L\left\{\frac{2Q^2}{45\bar{N}^2} + 2\left(\frac{EIQ}{\bar{N}^2L^2}\right)^2\left[\frac{\lambda^2L^2}{2 \sin^2\lambda L} + \frac{\lambda L}{2} \cot \lambda L - 1 + 2\left(1 - \frac{\bar{N}}{P + K_S}\right)\left(\lambda L \cot \lambda L + \frac{\lambda^2L^2}{3} - 1\right)\right]\right\} \tag{38}$$

If the transverse shear stiffness is taken to be infinite, these results can be shown to be in agreement with the results for the classical beam column.

CONCLUDING REMARKS

Under the assumption that the deformations of long pressurized cylindrical membrane beams can be accurately expressed in terms of translations and rotations of a cross section, nonlinear equilibrium equations have been derived for the bending, twisting, and stretching of such beams. It is found that for problems in which twist about the longitudinal axis can be neglected the equilibrium equations are linear. (Twisting will not occur in beams of circular cross section when they are loaded as a beam and/or column.) These linear equations are seen to be equivalent to the Timoshenko beam equations with the transverse shear stiffness augmented by the appropriate internal pressure contribution and with axial force accounted for.

Two illustrative problems are analyzed by using the linear equations. The first problem is that of the buckling of a simply supported pressurized column under axial compression. The results are seen to be in agreement with results obtained by another method.

The second problem, typical of those which could arise in the use of pressurized membrane beams as booms for a paraglider or similar vehicle, is concerned with the combined bending and axial compression of a simply supported inflated membrane column. Expressions are obtained for lateral deflection, cross-section rotation, shear force, bending moment, and column shortening. If the transverse shear stiffness of the inflated beam column is assumed to be infinite, the results can be shown to agree with the results for the classical beam column.

The analysis presented herein is valid for slender beams having significant wall bending stiffness, with or without internal pressure, provided local buckling of the wall does not occur. Since the theory contains no provision for the effects of local buckling, it is applicable to membrane cylinders only when the internal pressure and applied forces are such that the cylinder wall is in tension.

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