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Analysis of Discontinuities in a Rectangular Waveguide Using Dyadic Green's Function Approach in Conjunction With Method of Moments

M. D. Deshpande
ViGYAN Inc., Hampton, VA. 23681

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National Aeronautics and
Space Administration
Langley Research Center
Hampton, Virginia 23681-0001

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a,b	x-, y-dimensions of rectangular waveguide
$\vec{A}$	magnetic vector potential
A_x, A_y, A_z	x-, y-, and z-components of $\vec{A}$, respectively
$\vec{E}$	Electric field inside rectangular waveguide
$\vec{E_s}$	scattered electric field
E_x, E_y, E_z	x-, y-, and z-components of $\vec{E}$, respectively
$\vec{E_i}$	incident electric field vector
f	frequency in cycles per second
$\overline{G}$	dyadic Green's function for magnetic potential
$\overline{G}_e$	dyadic Green's function of electric-type
$\overline{G}_{e0}$	dyadic Green's function of electric-type for source free region
G_{xx}, G_{yy}, G_{zz}	xx-, yy-, and zz-components of $\vec{G}$
g_{xx}, g_{yy}, g_{zz}	functions associated with G_{xx}, G_{yy}, G_{zz}, respectively
$\tilde{g}_{xx}, \tilde{g}_{yy}, \tilde{g}_{zz}$	Fourier transforms of g_{xx}, g_{yy}, g_{zz}, respectively
$\vec{H}$	Magnetic field inside rectangular waveguide
H_x, H_y, H_z	x-, y-, and z- components of magnetic field $\vec{H}$, respectively

$\bar{I}$	unit impulse current source
I_0	amplitude of current $\vec{J}$
$\vec{J}$	Electric current source
$\vec{J}_T$	test surface current density
J_z	z-component of $\vec{J}$
j	$= \sqrt{-1}$
k_0	free-space wave number
k	$= k_0\sqrt{\varepsilon_r\mu_r}$
k_z	propagation constants along z-direction inside rectangular waveguide
k_I	propagation constant along z-direction
m, n	integer associated with waveguide modes
T	transmission coefficient
V_y	reaction of test surface current density with the incident electric field
x, y, z	coordinates of field point in cartesian coordinate system
x', y', z'	coordinates of source point in cartesian coordinate system
x'', y'', z''	dummy variables of integration
$\hat{x}$, $\hat{y}$, $\hat{z}$	unit vector along the x-, y-, and z-axis, respectively
Z_{yy}	self impedance of the y-directed post current
Γ	reflection coefficient
∇	gradient operator
$\delta(.)$	delta function
ε_0, μ_0	permittivity and permeability of free-space
ε_m and ε_n	Neumann's numbers
η_0	free-space impedance
φ	variable of integration
ω	angular freqency equal to $2\pi f$
FEM	finite element method
EM	electromagnetic
EFIE	electric field integral equation
MoM	method of moment

Abstract

The dyadic Green's function for an electric current source placed in a rectangular waveguide is derived using a magnetic vector potential approach. A complete solution for the electric and magnetic fields including the source location is obtained by simple differentiation of the vector potential around the source location. The simple differentiation approach which gives electric and magentic fields identical to an earlier derivation is overlooked by the earlier workers in the derivation of the dyadic Green's function particularly around the source location. Numerical results obtained using the Green's function approach are compared with the results obtained using the Finite Element Method(FEM).

I. Introduction

Analysis and design of dipole, monopole, or aperture radiator to excite high intensity electromagnetic (EM) fields inside a reverberation chamber can be done using an integral equation approach. The EM fields inside a reverberation chamber due to a radiator can be determined by weighting an appropriate dyadic Green's function with an assumed antenna current. The Electric Field Integral Equation (EFIE) is then set up by forcing the total tangential electric field on the antenna surface to be zero. Using the Method of Moments (MoM), EFIE is then reduced to a matrix equation which can be solved for the antenna current. From the current, the EM field radiated by the antenna inside a reverberation chamber is determined. Also the input impedance of the antenna as a function of its location and frequency can be determined. This work is divided into two parts. In the first part we derive the appropriate dyadic Green's function for an electric current source located inside a rectangular waveguide and cavity. Detailed steps involved in this derivation are reported in this document. The second part of this work, which will be reported in subsequent documents, consists of an application of the dyadic Green's function to analyze a dipole antenna placed in a reverberation chamber.

Knowledge of a dyadic Green's function for cylindrical waveguides and cavities is essential for analyzing and designing antennas and arbitrarily shaped objects placed inside a cylindrical waveguide and cavity [1,2]. A detailed derivation of a dyadic Green's function for the rectangular waveguide was presented by Tai [3]. In deriving these dyadic Green's function valid for both source and source free regions, an additional term must be added to the classical representation of the field expressions [4]. To include the additional term in the classical representation, Tai [5] has presented an approach based upon the use of eigenvector functions. In [6], an electric-type dyadic Green's function is obtained through a magnetic-type dyadic Green's function obtained using the theory of distributions.

The purpose of this communication is to present a simple method using the vector potential approach to determine the dyadic Green's function valid in the entire region of a cylindrical waveguide. For an arbitrarily oriented electric current source in a rectangular waveguide, expressions for the magnetic vector potential are obtained by solving the inhomogeneous Helmholtz equation. The electric fields and hence the dyadic Green's function of the electric-type is then obtained by taking the derivatives of the magnetic vector potential. In the process of finding the electric field, if the derivatives of the vector potential are carefully defined, the additional term discussed in [4-6] automatically follows. Reflection and transmission coefficients due to a y-directed cylindrical post placed in a rectangular waveguide and excited by a dominant mode are derived and numerical results are compared with the results obtained by the Finite Element Method [7].

II. Theory

Dyadic Green's Function for an Electric Current Source in a Rectangular Waveguide

(a) Solution of Inhomogeneous Helmholtz Equation:

Consider an infinite rectangular waveguide with electric current source $\vec{J}$ as shown in figure 1. The electromagnetic fields inside the waveguide due to $\vec{J}$ can be determined from

$$\vec{H}(x, y, z) = \frac{1}{\mu_0}\nabla\times\vec{A} \tag{1}$$

$$\vec{E}(x, y, z) = \frac{-j\omega}{k_0^2}\left[k_0^2\vec{A} + \nabla\left(\nabla\bullet\vec{A}\right)\right] \tag{2}$$

where the assumed time variation $e^{j\omega t}$ has been suppressed. The magnetic vector potential $\vec{A}(x, y, z)$ appearing in (1) and (2) statisfies the inhomogeneous wave equation

$$\nabla^2\vec{A}(x, y, z) + k_0^2\vec{A}(x, y, z) = -\mu_0\vec{J}(x', y', z') \tag{3}$$

If $\overline{G}(x, y, z, x', y', z')$ is the dyadic Green's function for the rectangular waveguide for a unit impulse current source $\bar{I}(x', y', z')$ inside the waveguide, then the magnetic vector potential $\vec{A}(x, y, z)$ can be written in the form

$$\vec{A}(x, y, z) = \int\int_{Source}\int\overline{G}(x, y, z, x', y', z') \bullet \vec{J}(x', y', z')\, dx'dy'dz' \tag{4}$$

Substituting (4) in (3) we get

$$\nabla^2\overline{G}(\,.\,) + k_0^2\overline{G}(\,.\,) = -\mu_0\bar{I}\delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{5}$$

where $\bar{I}$ is an unit dyadic, defined as $\bar{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$. Equation (5) may be written in component form as

$$\nabla^2 G_{xx}(\,.\,) + k_0^2 G_{xx}(\,.\,) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (6)$$

$$\nabla^2 G_{yy}(\,.\,) + k_0^2 G_{yy}(\,.\,) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (7)$$

$$\nabla^2 G_{zz}(\,.\,) + k_0^2 G_{zz}(\,.\,) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (8)$$

Because of the nature of the problem and the boundary conditions, the other components of the dyadic Green's function $\bar{G}(\,.\,)$ will not be excited and hence are not considered. The solutions of (6), (7), and (8) may be assumed in the following forms

$$G_{xx}(\,.\,) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{xx}(x', y', z', z)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad (9)$$

$$G_{yy}(\,.\,) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} g_{yy}(x', y', z', z)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad (10)$$

$$G_{zz}(\,.\,) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{zz}(x', y', z', z)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad (11)$$

Substituting (9) in (6), (10) in (7) and (11) in (8) we get

$$\left\{\frac{d^2}{dz^2}g_{xx}(\,.\,) + k_I^2 g_{xx}(\,.\,)\right\}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (12)$$

$$\left\{\frac{d^2}{dz^2}g_{yy}(\,.\,) + k_I^2 g_{yy}(\,.\,)\right\}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (13)$$

$$\left\{\frac{d^2}{dz^2}g_{zz}(\,.\,) + k_I^2 g_{zz}(\,.\,)\right\}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = -\mu_0 \delta(x-x')\,\delta(y-y')\,\delta(z-z') \qquad (14)$$

where $k_I^2 = k_0^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$. Multiply (12) by $\cos\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)$ and integrate over the cross section of waveguide we get

$$\left\{\frac{d^2}{dz^2}g_{xx}(\,.\,) + k_I^2 g_{xx}(\,.\,)\right\} = -\mu_0\frac{\varepsilon_m\varepsilon_n}{ab}\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\delta(z-z') \qquad (15)$$

Likewise the equations (13) and (14) yield

$$\left\{\frac{d^2}{dz^2}g_{yy}(\,.\,) + k_I^2 g_{yy}(\,.\,)\right\} = -\mu_0\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)\delta(z-z') \qquad (16)$$

$$\left\{\frac{d^2}{dz^2}g_{zz}(\,.\,) + k_I^2 g_{zz}(\,.\,)\right\} = -\mu_0\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\delta(z-z') \qquad (17)$$

where ε_m and ε_n are Neumann's numbers [7] and equal to 1 for $m = 0$ and 2 for $m \neq 0$. In order to determine the solution of the inhomogeneous differential equation (15) let us

assume
$$g_{xx}(\,.\,) = \int_{-\infty}^{\infty} \tilde{g}_{xx}(k_z)\, e^{jk_z z} dk_z \qquad (18)$$

Substitution of (18) in (15), multiplying by $e^{-jk_z' z}$ and integrating over z leads to

$$\tilde{g}_{xx}(k_z) = -\mu_0\frac{\varepsilon_m\varepsilon_n}{2\pi ab}\frac{\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)}{\left(-k_z^2 + k_I^2\right)}e^{-jk_z z'} \qquad (19)$$

Substitution of (19) in (18) yields

$$g_{xx}(\,.\,) = \mu_0\frac{\varepsilon_m\varepsilon_n}{2\pi ab}\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\int_{-\infty}^{\infty}\frac{1}{\left(k_z^2 - k_I^2\right)}e^{-jk_z z'}e^{jk_z z}dk_z \qquad (20)$$

The integrand in equation (20) has poles at $k_z = \pm k_I$, therefore the integral in (20) can be evaluated using contour integration in the complex domain [8]. Hence

$$g_{xx}(\ .\ ) = \frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')} \tag{21}$$

where + sign in the exponential is taken when $(z-z') < 0$ and - sign in the exponential is taken when $(z-z') > 0$. Likewise, $g_{yy}(\ .\ )$ and $g_{zz}(z)$ are obtained as

$$g_{yy}(\ .\ ) = \frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')} \tag{22}$$

$$g_{zz}(\ .\ ) = \frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')} \tag{23}$$

Substituting (21), (22), and (23) in (9), (10), and (11), respectively, the x-, y-, and z-components of the dyadic Green's function are obtained as

$$G_{xx}(\ .\ ) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\pm jk_I(z-z')} \tag{24}$$

$$G_{yy}(\ .\ ) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{\pm jk_I(z-z')} \tag{25}$$

$$G_{zz}(\ .\ ) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{\pm jk_I(z-z')} \tag{26}$$

The x-, y- and z-components of the magnetic vector potential due to the x-,y-, and z-directed currents are then obtained as

$$A_x(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int\!\!\!\int\!\!\!\int_{Source} J_x(x',y',z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{27}$$

$$A_y(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_y(x',y',z')\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (28)$$

$$A_z(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x',y',z')\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (29)$$

The expressions in (27)-(29) are the required solution of inhomogeneous Helmoltz equation given in (3).

(b) Electromagnetic Fields Due to Transverse Currents:

The electric and magnetic fields due to $A_x(x,y,z)$ are obtained from (2) as

$$E_x(A_x) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(\left(k_0^2-\left(\frac{m\pi}{a}\right)^2\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right)$$

$$\int_{Source}\iint J_x(x',y',z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (30)$$

$$E_y(A_x) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-\frac{m\pi}{a}\right)\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_x(x',y',z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (31)$$

$$E_z(A_x) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\left(-\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int\limits_{Source}\iint J_x(x', y', z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \qquad (32)$$

$$H_x(A_x) = 0 \qquad (33)$$

$$H_y(A_x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int\limits_{Source}\iint J_x(x', y', z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \qquad (34)$$

$$H_z(A_x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(\frac{n\pi y}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int\limits_{Source}\iint J_x(x', y', z')\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \qquad (35)$$

Similarly, the electric and magnetic fields due to $A_y(x, y, z)$ are obtained as

$$E_x(A_y) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-\frac{n\pi}{b}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int\limits_{Source}\iint J_y(x', y', z')\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \qquad (36)$$

$$E_y(A_y) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(k_0^2-\left(\frac{n\pi}{b}\right)^2\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int\limits_{Source}\iint J_y(x', y', z')\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \qquad (37)$$

$$E_z(A_y) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\left(-\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_y(x', y', z')\,\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{38}$$

$$H_x(A_y) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\,\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_y(x', y', z')\,\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{39}$$

$$H_y(A_y) = 0 \tag{40}$$

$$H_z(A_y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_y(x', y', z')\,\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{41}$$

(c) Electromagnetic Fields Due to Longitudinal Current:

The transverse electric fields due to $A_z(x, y, z)$ are obtained from (2) as

$$E_x(A_z) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x', y', z')\,\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{42}$$

$$E_y(A_z) = \frac{-j\omega}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x', y', z')\,\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}\,dv \tag{43}$$

In obtaining the longitudinal electric field representation due to $A_z(x, y, z)$, special attention is required in performing the differentiation with respect to z on $A_z(x, y, z)$. Since

$A_z(x, y, z)$ is continuous as a function of z, the first derivative of $A_z(x, y, z)$ is straightforward, and therefore causes no difficulty. Hence

$$\frac{\partial}{\partial z}A_z(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}(\pm jk_I)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x'', y'', z'')\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)e^{\pm jk_I(z - z'')}dv'' \quad (44)$$

where double prime quantities are the dummy variables of integration. Clearly $\frac{\partial}{\partial z}A_z(x, y, z)$ is discontinuous at $z = z''$, so in performing the derivative of $\frac{\partial}{\partial z}A_z(x, y, z)$ with respect to z around $z = z''$, care must be exercised to account for the jump in $\frac{\partial}{\partial z}A_z(x, y, z)$ as one crosses the $z = z''$ point. The behavior at $z = z''$ is properly accounted for by an impulse function at the point whereas the differentiation throughout the rest of region poses no problem, therefore,

$$\frac{\partial^2}{\partial z^2}A_z(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-k_I^2\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x'', y'', z'')\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)e^{\pm jk_I(z - z'')}dv''$$

$$+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{-j\mu_0}{2}\frac{\varepsilon_m\varepsilon_n}{ab}(-2j)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x'', y'', z'')\,\delta(z - z'')\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)e^{\pm jk_I(z - z'')}dv'' \quad (45)$$

Integrating on z'' in the second term of equation (45) yields

$$\frac{\partial^2}{\partial z^2}A_z(x,y,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-k_I^2\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\int\int J_z(x'',y'',z'')\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)e^{\pm jk_I(z-z'')}dv''$$

$$\left(-\mu_0\int_{source}\int J_z(x'',y'',z)\frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)\right)dx''dy''$$

(46)

Expanding $\delta(x-x'')\,\delta(y-y'')$ in the Fourier sine series over the domains $0 \le x \le a$ and $0 \le y \le b$ where $0 \le x'' \le a$ and $0 \le y'' \le b$ [9], it can be shown that

$$\delta(x-x'')\,\delta(y-y'') = \frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right) \quad (47)$$

Using (47), (46) can be written as

$$\frac{\partial^2}{\partial z^2}A_z(x,y,z) = -\mu_0 J_z(x,y,z) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(-k_I^2\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\int\int J_z(x'',y'',z'')\sin\left(\frac{m\pi x''}{a}\right)\sin\left(\frac{n\pi y''}{b}\right)e^{\pm jk_I(z-z'')}dv'' \quad (48)$$

The longitudinal component of the electric field is then obtained using (2) as

$$E_z(A_z) = \frac{j\omega}{k_0^2}\mu_0 J_z(x,y,z) + \frac{-j\omega}{k_0^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{-j\mu_0}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\left(k_0^2-k_I^2\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\int\int J_z(x',y',z')\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \quad (49)$$

The magnetic field components due to A_z are obtained as

$$H_x(A_z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x', y', z')\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (50)$$

$$H_y(A_z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$\int_{Source}\iint J_z(x', y', z')\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)e^{\pm jk_I(z-z')}dv \qquad (51)$$

$$H_z(A_z) = 0 \qquad (52)$$

The total electric and magnetic fields inside the waveguide due to $\vec{J}$ is then obtained by superpostion of the electromagnetic fields due to A_x, A_y, and A_z.

(d) Dyadic Green's Function for Electric Field:

It is instructive at this point to defined the dyadic Green's function for the electri field formulation. To this end, we write the vector wave equation for the electric field as

$$\nabla\times\nabla\times\vec{E} - k_0^2\vec{E} = -j\omega\mu_0\vec{J} \qquad (53)$$

If the electric field in terms of the dyadic Green's function $\overline{G_e}(x', y', z'/x, y, z)$ is given as

$$\vec{E}(x, y, z) = -j\omega\mu_0\iiint\overline{G_e}(x', y', z'/x, y, z)\bullet\vec{J}(x', y', z')\,dv' \qquad (54)$$

Substituting (54) in (53) the wave equation for the dyadic Green's function of electric-type is obtained as

$$\nabla\times\nabla\times\overline{G_e}(\,.\,) - k_0^2\overline{G_e}(\,.\,) = \bar{I}\delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{55}$$

From equations (30)-(32), (36)-(38), (42), (43), and (49), the dyadic Green's function can be written as

$$\overline{G_e}(\,.\,) = \overline{G_{e0}}(\,.\,) - \frac{\delta(x-x')\,\delta(y-y')\,\delta(z-z')}{k_0^2}\hat{z}\hat{z} \tag{56}$$

where $\overline{G_{e0}}(\,.\,)$ is given by

$$\overline{G_{e0}}(\,.\,) = \frac{1}{k_0^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{-j}{2k_I}\frac{\varepsilon_m\varepsilon_n}{ab}e^{\pm jk_I(z-z')}$$

$$\left(\left[k_0^2-\left(\frac{m\pi}{a}\right)^2\right]\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{m\pi x'}{a}\right)\sin\frac{n\pi y'}{b}\hat{x}\hat{x}\right.$$

$$+\left(-\frac{m\pi}{a}\right)\frac{n\pi}{b}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{y}\hat{x}$$

$$+(\pm jk_I)\left(-\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{z}\hat{x}$$

$$+\left(-\frac{n\pi}{b}\right)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)\hat{x}\hat{y}$$

$$+\left[k_0^2-\left(\frac{n\pi}{b}\right)^2\right]\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\cos\left(\frac{n\pi y'}{b}\right)\hat{y}\hat{y}$$

$$+(\pm jk_I)\left(-\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{z}\hat{y}$$

$$+(\pm jk_I)\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{x}\hat{z}$$

$$+(\pm jk_I)\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{y}\hat{z}$$

$$\left.+\left[k_0^2-k_z^2\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m\pi x'}{a}\right)\sin\left(\frac{n\pi y'}{b}\right)\hat{z}\hat{z}\right) \tag{57}$$

The expression in (57) is identical to the Green's function reported in reference [1].

III. Application

Analysis of Cylindrical Post in a Rectangular Waveguide:

Consider a rectangular waveguide with a cylindrical post as shown in figure 2. It is assumed that the waveguide is excited by the dominant mode from the right. For simplicity it is assumed that the surface current density on the post as

$$\vec{J} = \hat{y} I_0 \delta\left(x' - \frac{a}{2}\right) \delta(z - z') \tag{58}$$

Let $\vec{E}_s(\vec{J})$ be the scattered electric field due to the current $\vec{J}$ and $\vec{E}_i$ be the incident electric field due to TE_{10} mode. The total electric field inside the waveguide is then given by $\vec{E}_s(\vec{J}) + \vec{E}_i$. Subjecting the total tangential electric field on the surface of the post to zero, we get following electric field integral equation:

$$\left(\vec{E}_s(\vec{J}) + \vec{E}_i\right)_t = 0 \tag{59}$$

where the subscript t is for the tangential component. Selecting a testing surface current density as $\overrightarrow{J_T}$ which resides on the cylindrical surface, Galerkin's procedure reduces equation (59) to

$$\langle \vec{E}_s(\vec{J}) \bullet \overrightarrow{J_T} \rangle + \langle \vec{E}_i \bullet \overrightarrow{J_T} \rangle = 0 \tag{60}$$

Equation (60) can be written in a algebric form as

$$Z_{yy} I_0 + V_y = 0 \tag{61}$$

where $Z_{yy}I_0 = \langle \vec{E_s}\left(\vec{J}\right) \bullet \vec{J_T} \rangle$, $V_y = \langle \vec{E_i} \bullet \vec{J_T} \rangle$, with the indicated integration performed in cylindrical coordinates. Using (54) and (56), the expression for Z_{yy} is obtained as

$$Z_{yy} = -(\omega\mu_0 2b/a) \sum_{m = 1, 3, 5, ..} \frac{1}{k_I\pi}\int_0^{\pi} e^{-jk_I r_0 \sin(\varphi)} \cos\left(\frac{m\pi r_0}{a}\cos(\varphi)\right) d\varphi \tag{62}$$

Assuming an unit amplitude dominant mode $\vec{E_i}$ can be written as

$$\vec{E_i} = \hat{y}\sqrt{\frac{2}{ab}}\sin\left(\frac{\pi x}{a}\right) e^{-j\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}z} \tag{63}$$

Using (63) the quantity V_y can be written as

$$V_y = \left(\sqrt{\frac{2b}{a}}\frac{2}{\pi}\right) e^{-j\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}z_1} \int_0^{\pi} e^{-j\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}r_0 \sin(\varphi)} \cos\left(\frac{\pi r_0}{a}\cos(\varphi)\right) d\varphi \tag{64}$$

The algebric equation (61) can be solved for I_0. The reflected amplitude of the dominant mode field at a reference plane $z = 0$ is then determined from

$$\Gamma = \frac{-k_0\eta_0 I_0}{\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}}\sqrt{\frac{b}{2a}}\, e^{-j\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}2z_1} \tag{65}$$

The transmitted amplitude of the dominant mode at the reference plane $z = 2z_1$ is obatined as

$$T = \left(1 + \frac{-k_0\eta_0 I_0}{\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}}\sqrt{\frac{b}{2a}}\right) e^{-j\sqrt{\left(k_0^2 - \left(\frac{\pi}{a}\right)^2\right)}2z_1} \tag{66}$$

IV. Numerical Results

To validate the Green's function derived in this report, a y-directed cylindrical post of radius $r_0 = 0.1cm$ placed at $\left(x = \frac{a}{2}, z = 0.5\right)$ in a rectangular waveguide with $a = 2.25cm$, $b = 1.02cm$ and excited by an unit amplitude dominant mode field is considered. The reflection coefficient at the $z = 0.0cm$ plane and the transmission coefficient at the plane $z = 1.0cm$ due to the presence of the probe are calculated using expressions (65) and (66) and presented in figures 3 and 4 along with the numerical results obtained using the FEM method [7,8]. The close agreement between the results obtained from two different numerical methods confirms the validity of the Green's functions derived here.

V. Conclusion

The complete dyadic Green's function for a electric current source located inside a rectangular waveguide is derived using the magnetic vector potential approach. The magnetic vector potential for an electric current source in a rectangular waveguide is obtained by solving the inhomogeneous Helmholtz's equation. The electric and magnetic fields are obtained from the magnetic vector potential through spatial differentiation. The fields which are valid over the source region are obtained by carefully differentiating the vector potential around the source location. The electric and magentic field expressions obtained by the present method are found to be identical with the expressions reported in the literature. Numerical results on the reflection and transmission coefficients using the Green's function approach are in a good agreement with the numerical results obtained using the FEM techniques.

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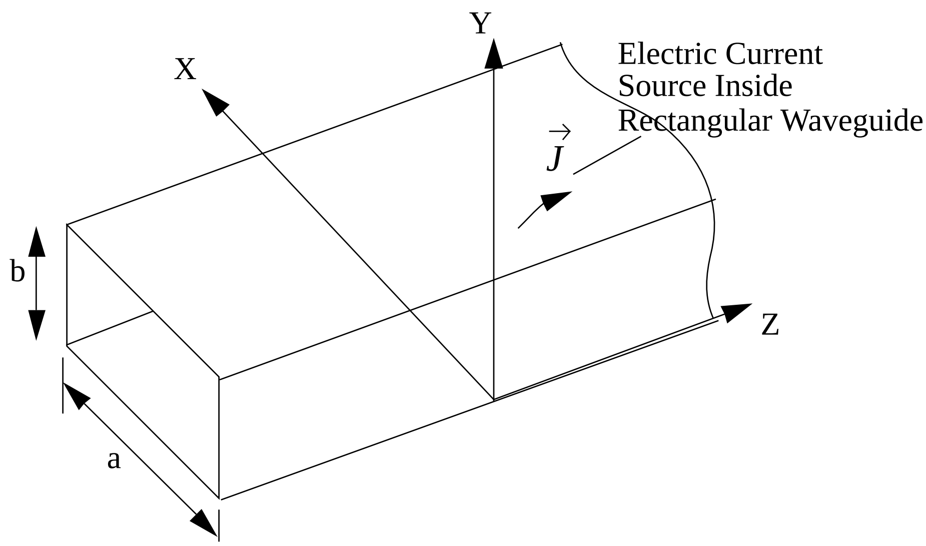

Figure 1 Electric current source inside a rectangular waveguide

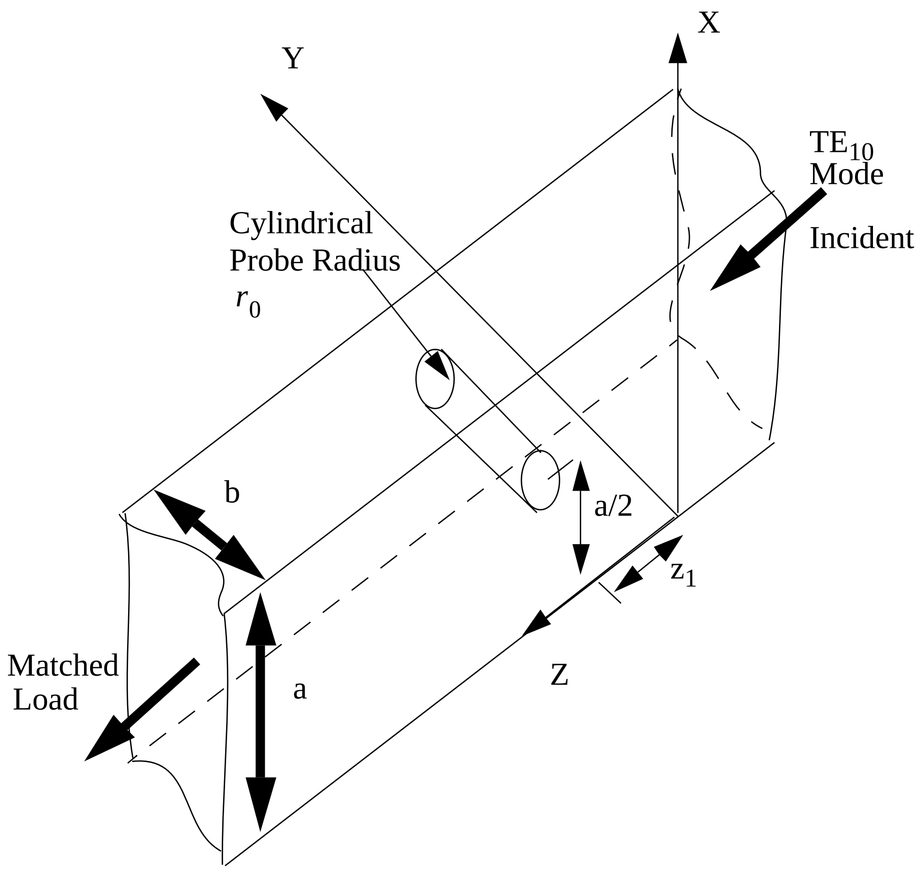

Figure 2 Rectangular waveguide with a cylindrical post parallel to y-axis placed at $x = a/2$, $z = z_1$.

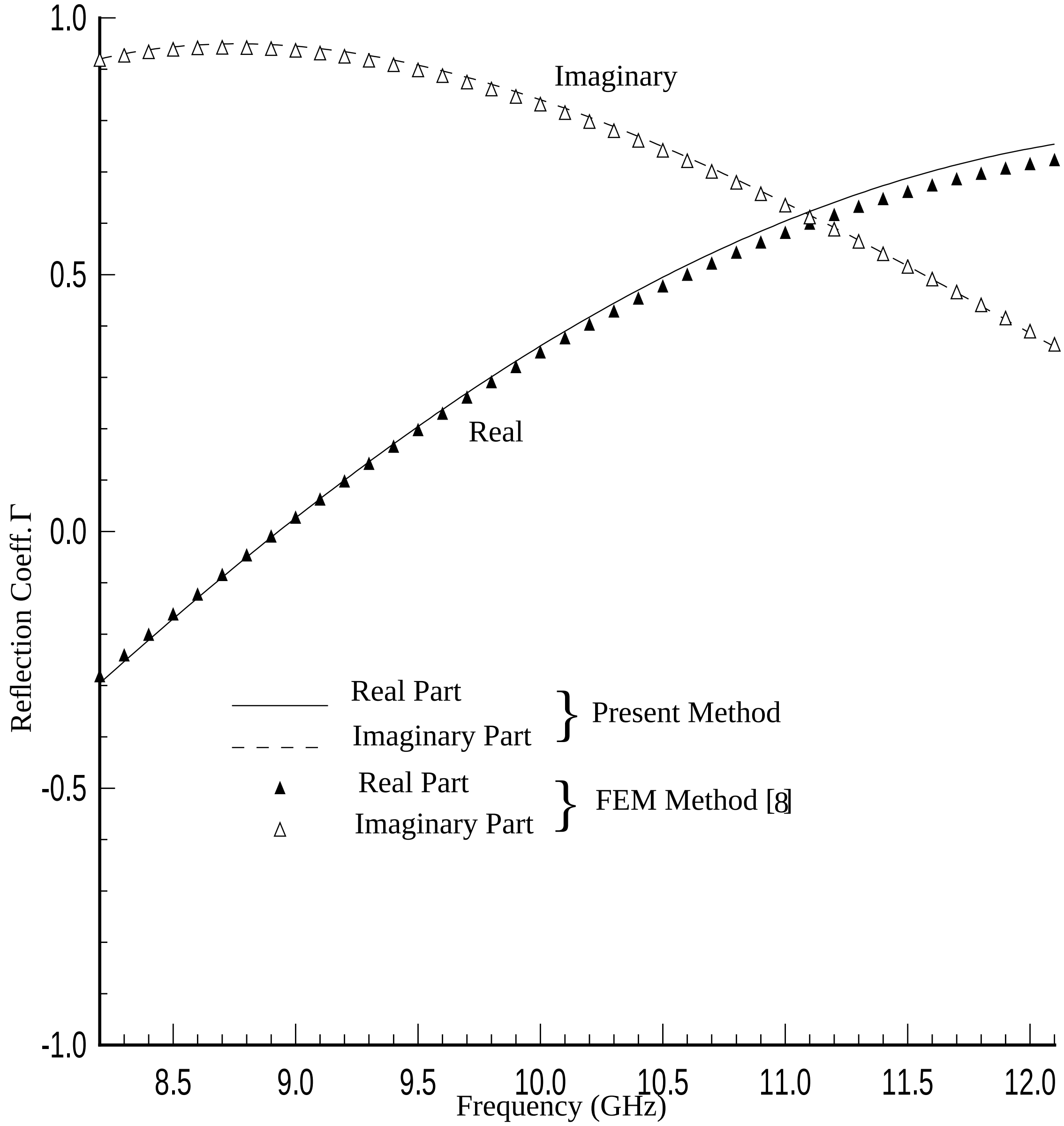

Figure 4 Reflection coefficient of a y-directed post in a rectangular waveguide as a function of frequency.

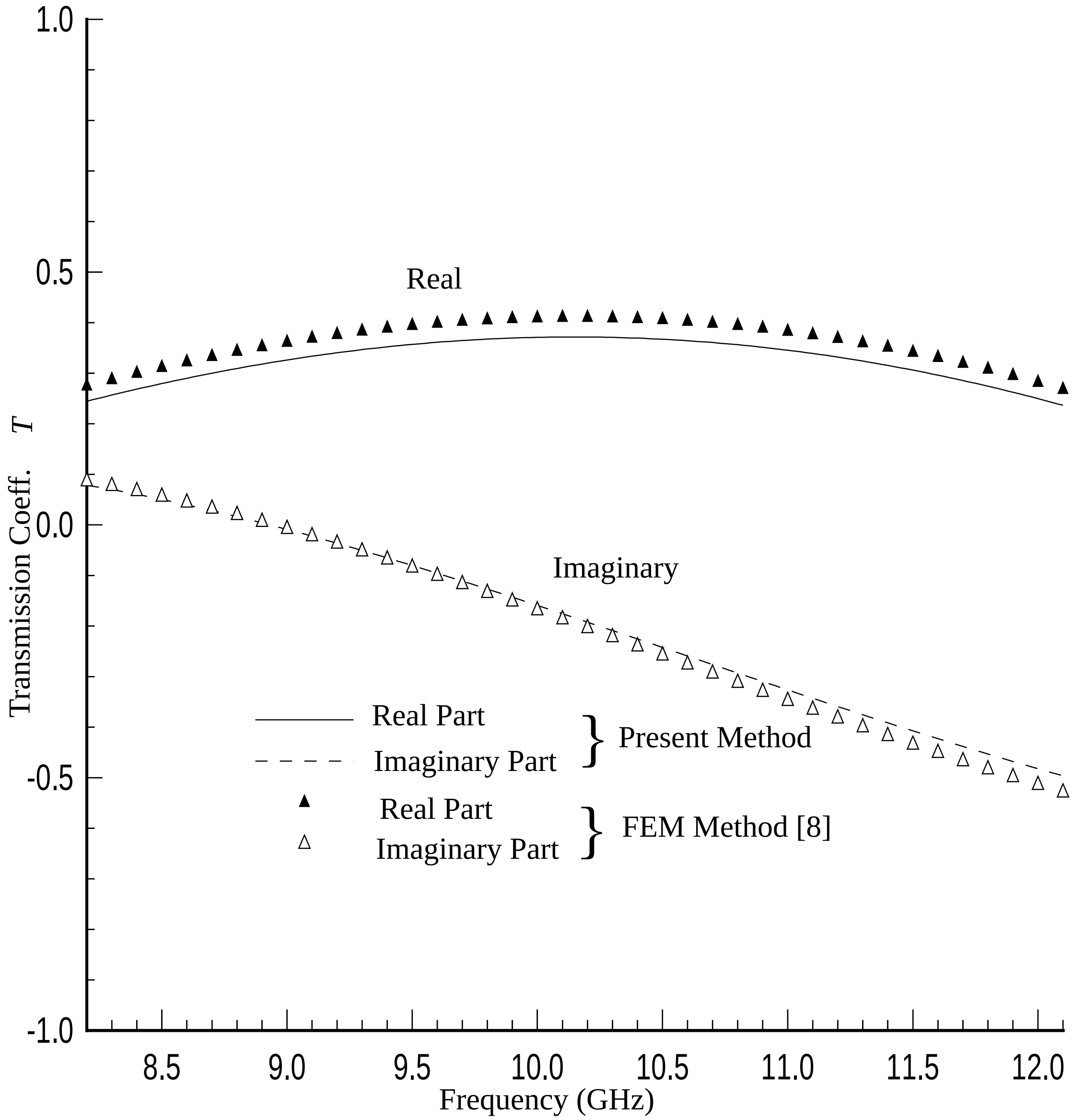

Figure 5 Transmission coefficient of a y-directed post in a rectangular waveguide as a function of frequency.