On the minimum of independent geometrically distributed random variables *

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Abstract

The expectations $E[X_{(1)}]$, $E[Z_{(1)}]$, and $E[Y_{(1)}]$ of the minimum of n independent geometric, modified geometric, or exponential random variables with matching expectations differ. We show how this is accounted for by stochastic variability and how $E[X_{(1)}]/E[Y_{(1)}]$ equals the expected number of ties at the minimum for the geometric random variables. We then introduce the "shifted geometric distribution", and show that there is a unique value of the shift for which the individual shifted geometric and exponential random variables match expectations both individually and in their minimums.

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1 Introduction

The purpose of this note is to compare the distributions of the minimums of two sets of random variables, respectively with geometric and exponential distributions, having pairwise matching means. The geometric distribution is the discrete analog of the exponential distribution and can be applied to a variety of performance models which can be analyzed by analytic or simulation methods. The following notation is used:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$, the natural numbers.
- $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$, the positive natural numbers.
- $N = \{1, 2, \ldots, n\}$, the first n natural numbers (n is a constant which will be clear from the context).
- $F_A(t) = \Pr\{A \leq a\}$, the cumulative distribution function (CDF) of a random variable A.
- $\overline{F}_A(t) = 1 - F_A(t)$, the complement of the CDF of A (its survivor function).

2 Geometric, modified geometric, and exponential distributions

Two random variables X and Z are said to have a geometric distribution with parameter $\alpha \in (0, 1)$, $X \sim \text{Geom}(\alpha)$, and a modified geometric distribution with parameter $\beta \in (0, 1)$, $Z \sim \text{ModGeom}(\beta)$, [4] if their probability mass functions (pmfs) are, respectively,

$$\forall k \in \mathbb{N}^+, \Pr\{X = k\} = \alpha(1-\alpha)^{k-1} \quad \text{and}$$

$$\forall k \in \mathbb{N}, \Pr\{Z = k\} = \beta(1-\beta)^k,$$

from which it follows that their CDFs at the mass values are

$$\forall k \in \mathbb{N}^+, \Pr\{X \leq k\} = \sum_{l=1}^{k} \alpha(1-\alpha)^{l-1} = 1 - (1-\alpha)^k \quad \text{and}$$

$$\forall k \in \mathbb{N}, \Pr\{Z \leq k\} = \sum_{l=0}^{k} \beta(1-\beta)^l = 1 - (1-\beta)^{k+1},$$

and that their expectations are

$$E[X] = \sum_{k=1}^{\infty} k\alpha(1-\alpha)^{k-1} = \frac{1}{\alpha} \quad \text{and}$$

$$E[Z] = \sum_{k=0}^{\infty} k\beta(1-\beta)^k = \frac{1-\beta}{\beta}.$$

Informally, the difference between a geometric and a modified geometric distribution with the same parameter is the way in which they count: the geometric distribution starts at one, the modified geometric distribution starts at zero. Hence, if $X \sim \text{Geom}(\alpha)$, $X - 1 \sim \text{ModGeom}(\alpha)$. Equivalently, the geometric distribution models the trial number of the first “success” in repeated independent Bernoulli trials, whereas the modified geometric distribution models the number of trials before the first success.

The above assumes that the "time-step" of the distribution is the same as the units in which time is measured. This restriction is removed by considering X and Z as random variables assuming values in $\{k\omega : k \in \mathbb{N}^+\}$ or $\{k\omega : k \in \mathbb{N}\}$, respectively, for some time-step $\omega > 0$:

$$X \sim \mathrm{Geom}(\alpha, \omega) \iff \forall t \in \mathbb{R}, \Pr\{X \leq t\} = \begin{cases} 1 - (1-\alpha)^{\lfloor \frac{t}{\omega} \rfloor} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$Z \sim \mathrm{ModGeom}(\beta, \omega) \iff \forall t \in \mathbb{R}, \Pr\{Z \leq t\} = \begin{cases} 1 - (1-\beta)^{\lfloor \frac{t}{\omega} \rfloor + 1} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

which imply

$$\forall k \in \mathbb{N}^+, \Pr\{X = k\omega\} = \alpha(1-\alpha)^{k-1} \quad \text{and}$$
$$\forall k \in \mathbb{N}, \Pr\{Z = k\omega\} = \beta(1-\beta)^{k}$$

and

$$E[X] = \frac{\omega}{\alpha} \quad \text{and}$$
$$E[Z] = \frac{\omega(1-\beta)}{\beta}.$$

It is well known that both the geometric and modified geometric distributions are discrete analogs of the exponential distribution. In particular, given an exponential random variable Y with rate $\lambda > 0$,

$$Y \sim \mathrm{Expo}(\lambda) \iff \forall t \geq 0, \Pr\{Y \leq t\} = 1 - e^{-\lambda t} \Rightarrow E[Y] = \lambda^{-1},$$

one can determine α and β so that X and Z match Y in expectation:

$$E[X] = \frac{\omega}{\alpha} = \lambda^{-1} = E[Y] \Rightarrow \alpha = \lambda\omega \quad \text{and}$$
$$E[Z] = \frac{\omega(1-\beta)}{\beta} = \lambda^{-1} = E[Y] \Rightarrow \beta = \frac{\lambda\omega}{1+\lambda\omega},$$

and then, using these values for α and β, the distributions of X and Z approximate that of Y arbitrarily well as the time-step ω is reduced:

$$\lim_{\omega \downarrow 0} \Pr\{X \leq t\} = \lim_{\omega \downarrow 0} 1 - (1-\lambda\omega)^{\lfloor \frac{t}{\omega} \rfloor} = 1 - e^{-\lambda t} = \Pr\{Y \leq t\} \quad \text{and}$$

$$\lim_{\omega \downarrow 0} \Pr\{Z \leq t\} = \lim_{\omega \downarrow 0} 1 - \left(1 - \frac{\lambda\omega}{1+\lambda\omega}\right)^{\lfloor \frac{t}{\omega} \rfloor + 1} = 1 - e^{-\lambda t} = \Pr\{Y \leq t\},$$

Note that $\alpha = \lambda\omega \in (0, 1)$ implies $\omega < \lambda^{-1}$, that is, it is not possible to match the mean of an exponential random variable $Y \sim \mathrm{Expo}(\lambda)$ with a geometric random variable having a time-step $\omega > \lambda^{-1}$. In the special case $\omega = \lambda^{-1}$, $\alpha = 1$ and the distribution of X degenerates to a constant: $X \sim \mathrm{Geom}(1, \omega) \equiv \mathrm{Const}(\omega)$. In the following, we allow this case and require $\omega \in (0, \lambda^{-1}]$. No such restriction exists in the case of the modified geometric distribution, where any $\omega > 0$ can be used.

3 The minimum of a set of random variables

Consider now three sets of $n \geq 2$ independent random variables, $\{X_i : i \in N\}$, $\{Z_i : i \in N\}$, and $\{Y_i : i \in N\}$ with matching means:

$$\forall i \in N, X_i \sim \text{Geom}(\alpha_i, \omega), E[X_i] = \lambda_i^{-1} \Rightarrow \alpha_i = \lambda_i \omega,$$

$$\forall i \in N, Z_i \sim \text{ModGeom}(\beta_i, \omega), E[Z_i] = \lambda_i^{-1} \Rightarrow \beta_i = \frac{\lambda_i \omega}{1 + \lambda_i \omega}, \quad \text{and}$$

$$\forall i \in N, Y_i \sim \text{Expo}(\lambda_i), E[Y_i] = \lambda_i^{-1}.$$

Since $\omega \in \bigcap_{i \in N}(0, \lambda_i^{-1}]$, we obtain $\omega \in (0, \lambda_{MAX}^{-1}]$, where $\lambda_{MAX} = \max\{\lambda_i : i \in N\}$.

It is well known that the minimum of each of these sets has the same type of distribution as the elements of the set [5, 2]:

$$X_{(1)} = \min\{X_i : i \in N\} \sim \text{Geom}\left(1 - \prod_{i \in N}(1 - \alpha_i), \omega\right),$$

$$Z_{(1)} = \min\{Z_i : i \in N\} \sim \text{ModGeom}\left(1 - \prod_{i \in N}(1 - \beta_i), \omega\right), \quad \text{and}$$

$$Y_{(1)} = \min\{Y_i : i \in N\} \sim \text{Expo}\left(\sum_{i \in N} \lambda_i\right).$$

Hence, $X_{(1)}$, $Z_{(1)}$, and $Y_{(1)}$ have different expectations:

$$E[X_{(1)}] = \frac{\omega}{1 - \prod_{i \in N}(1 - \alpha_i)} = \frac{\omega}{1 - \prod_{i \in N}(1 - \lambda_i \omega)} \neq \left(\sum_{i \in N} \lambda_i\right)^{-1} = E[Y_{(1)}] \quad \text{and} \tag{1}$$

$$E[Z_{(1)}] = \frac{\omega \prod_{i \in N}(1 - \beta_i)}{1 - \prod_{i \in N}(1 - \beta_i)} = \frac{\omega}{-1 + \prod_{i \in N}(1 + \lambda_i \omega)} \neq \left(\sum_{i \in N} \lambda_i\right)^{-1} = E[Y_{(1)}]. \tag{2}$$

Theorem 1. For $n \geq 2$, $E[X_{(1)}] > E[Y_{(1)}] > E[Z_{(1)}]$.

Proof. We prove that $E[X_{(1)}] > E[Y_{(1)}]$ by induction on n, hence we make the index n explicit by writing $E[X_{(1,n)}]$ and $E[Y_{(1,n)}]$.

Base step: For $n = 2$,

$$E[X_{(1,2)}] = \frac{\omega}{1 - (1 - \lambda_1 \omega)(1 - \lambda_2 \omega)} = \frac{1}{\lambda_1 + \lambda_2 - \lambda_1 \lambda_2 \omega} > \frac{1}{\lambda_1 + \lambda_2} = E[Y_{(1,2)}].$$

Inductive Hypothesis: Assume that, for a given n, $E[X_{(1,n)}] > E[Y_{(1,n)}]$. Then,

$$\frac{\omega}{1 - \prod_{i \in N}(1 - \lambda_i \omega)} > \left(\sum_{i \in N} \lambda_i\right)^{-1} \Rightarrow \prod_{i \in N}(1 - \lambda_i \omega) > 1 - \sum_{i \in N} \lambda_i \omega$$

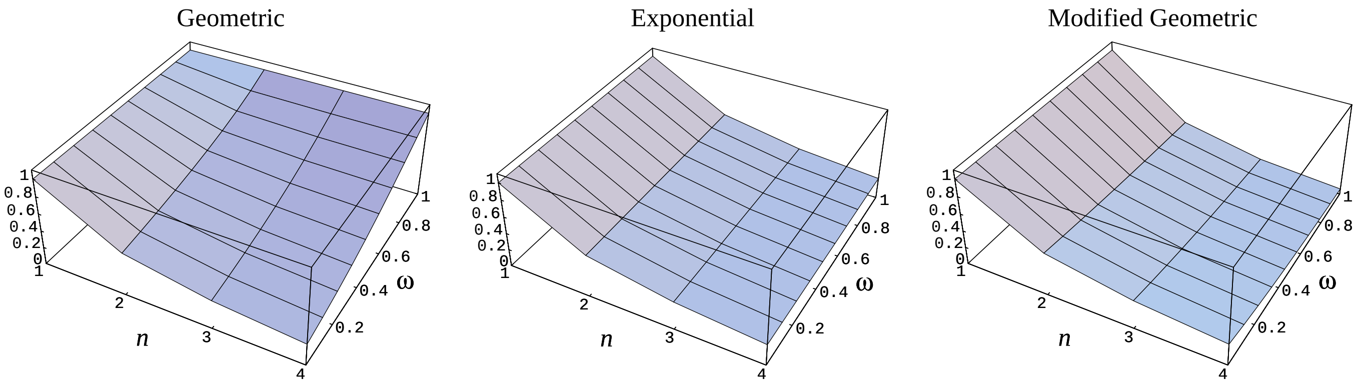

Figure 1: $E[X_{(1)}]$, $E[Y_{(1)}]$, and $E[Z_{(1)}]$ ($n = 1, \cdots, 5, \forall i, \lambda_i = 1$) as a function of ω.

Inductive Step: Then $E[X_{(1,n+1)}] > E[Y_{(1n+1)}]$, since

$$\begin{aligned} E[X_{(1,n+1)}] &= \frac{\omega}{1 - \left(\prod_{i \in N} (1-\lambda_i \omega)\right)(1-\lambda_{n+1}\omega)} \\ &> \frac{\omega}{1 - \left(1 - \sum_{i \in N} \lambda_i \omega\right)(1-\lambda_{n+1}\omega)} \\ &= \frac{1}{\sum_{i \in N} \lambda_i + \lambda_{n+1} - \left(\sum_{i \in N} \lambda_i\right)\lambda_{n+1}\omega} > \left(\sum_{i \in N \cup \{n+1\}} \lambda_i\right)^{-1} = E[Y_{(1,n+1)}] \end{aligned}$$

The proof that $E[Z_{(1)}] < E[Y_{(1)}]$ is analogous and is omitted. QED.

In other words, the minimum of n independent exponential random variables is always strictly bounded in expectation by the minimums of n independent geometric and modified geometric random variables with matching means. For example, if $n = 2$, and $\lambda_1 = \lambda_2 = \lambda$,

$$E[X_{(1)}] = (2\lambda(1 - \lambda\omega/2))^{-1} > E[Y_{(1)}] = (2\lambda)^{-1} > E[Z_{(1)}] = (2\lambda(1 + \lambda\omega/2))^{-1}.$$

$E[X_{(1)}]$ and $E[Z_{(1)}]$ coincide with $E[Y_{(1)}]$ only in the limit, as $\omega \downarrow 0$ (see figure 1):

$$\lim_{\omega \downarrow 0} E[X_{(1)}] = \lim_{\omega \downarrow 0} \frac{\omega}{1 - \prod_{i \in N}(1 - \lambda_i \omega)} = \lim_{\omega \downarrow 0} \frac{\omega}{\sum_{i \in N} \lambda_i \omega + o(\omega)} = \left(\sum_{i \in N} \lambda_i\right)^{-1} = E[Y_{(1)}] \quad \text{and}$$

$$\lim_{\omega \downarrow 0} E[Z_{(1)}] = \lim_{\omega \downarrow 0} \frac{\omega}{-1 + \prod_{i \in N}(1 + \lambda_i \omega)} = \lim_{\omega \downarrow 0} \frac{\omega}{\sum_{i \in N} \lambda_i \omega + o(\omega)} = \left(\sum_{i \in N} \lambda_i\right)^{-1} = E[Y_{(1)}].$$

The convergence of $E[X_{(1)}]$ and $E[Z_{(1)}]$ to $E[Y_{(1)}]$ as $\omega \downarrow 0$ can also be derived observing that

$$E[X_{(1)}] - \omega < E[Z_{(1)}] < E[Y_{(1)}] < E[X_{(1)}] < E[Z_{(1)}] + \omega,$$

which follows from the fact that $(X_i - \omega) \sim \text{ModGeom}(\alpha_i, \omega)$ and $(Z_i + \omega) \sim \text{Geom}(\beta_i, \omega)$, and from $\forall i \in N, \alpha_i < \beta_i$, which imply that $E[X_{(1)} - \omega] < E[Z_{(1)}]$ and $E[Z_{(1)} + \omega] > E[X_{(1)}]$.

The next section contains an explanation for these inequalities.

4 Stochastic variability

Random variables with the same mean can be compared using the notion of stochastic variability, described in Ross [3], for which there are two equivalent definitions. Y is said to be stochastically more variable than X, $X \leq_{\mathrm{v}} Y$, if

$$\forall \text{ increasing convex function } g, E[g(X)] \leq E[g(Y)]$$

or, equivalently, if

$$\forall a \geq 0, \int_a^\infty \overline{F}_X(t)\, dt \quad \leq \int_a^\infty \overline{F}_Y(t)\, dt.$$

An additional useful notion codifies the idea that the remaining lifetime of a random variable conditioned on exceeding some value a has never greater expectation (NBUE: New Better Than Used in Expectation), or never smaller expectation (NWUE: New Worse Than Used in Expectation), than the original lifetime. Formally, a nonnegative random variable A is NBUE if

$$\forall a \geq 0, E[A - a \mid A > a] \leq E[A]$$

and is NWUE if

$$\forall a \geq 0, E[A - a \mid A > a] \geq E[A].$$

Ross lists some important consequences of these definitions:

- If X and Y are nonnegative, $X \leq_{\mathrm{v}} Y$, and $E[X] = E[Y]$, then $-X \leq_{\mathrm{v}} -Y$.
- If $g : \mathbb{R}^n \to \mathbb{R}$ is an increasing convex function, if $\forall i \in N, X_i \leq_{\mathrm{v}} Y_i$, $\{X_i : i \in N\}$ are independent, and $\{Y_i : i \in N\}$ are independent, then

$$g(X_1, X_2, \ldots, X_n) \leq_{\mathrm{v}} g(Y_1, Y_2, \ldots, Y_n).$$

- If X is NBUE, and Y is exponential with the same mean as X, then $X \leq_{\mathrm{v}} Y$.
- If Z if NWUE and Y is exponential with the same mean as Z then $Y \leq_{\mathrm{v}} Z$.

These last two facts are used to relate $X \sim \mathrm{Geom}(\alpha, \omega)$, $Z \sim \mathrm{ModGeom}(\beta, \omega)$, and $Y \sim \mathrm{Expo}(\lambda)$ with the same mean, by showing that the geometric distribution is NBUE and that the modified geometric distribution is NWUE. Let $X \sim \mathrm{Geom}(\alpha, \omega)$, $Z \sim \mathrm{ModGeom}(\beta, \omega)$, and choose any $a \geq 0$. Using the memoryless property of the geometric distribution, we can derive:

$$\begin{aligned} E[X - a \mid X > a] &= E[X \mid X > a] - a = \left\lfloor \frac{a}{\omega} \right\rfloor \omega + E[X] - a \leq E[X] \text{ and} \\ E[Z - a \mid Z > a] &= E[Z \mid Z > a] - a = \left(\left\lfloor \frac{a}{\omega} \right\rfloor + 1 \right) \omega + E[Z] - a > E[Z]. \end{aligned}$$

Therefore, $X \leq_{\mathrm{v}} Y \leq_{\mathrm{v}} Z$.

Considering again the three sets of independent random variables with matching means $\{X_i : i \in N\}$, $\{Z_i : i \in N\}$, and $\{Y_i : i \in N\}$ observe that

$$\min\{a_i : i \in N\} = -\max\{-a_i : i \in N\}.$$

Since max is an increasing convex function and $\forall i \in N, -X_i \leq_{\mathrm{v}} -Y_i \leq_{\mathrm{v}} -Z_i$,

$$\max\{-X_i : i \in N\} \leq_{\mathrm{v}} \max\{-Y_i : i \in N\} \leq_{\mathrm{v}} \max\{-Z_i : i \in N\},$$

implying that

$$-E[\max\{-X_i : i \in N\}] \geq -E[\max\{-Y_i : i \in N\}] \geq -E[\max\{-Z_i : i \in N\}],$$

and thus that

$$E[\min\{X_i : i \in N\}] = E[X_{(1)}] \geq E[\min\{Y_i : i \in N\}] = E[Y_{(1)}] \geq E[\min\{Z_i : i \in N\}] = E[Z_{(1)}].$$

5 Matching the minimums by changing the time-step

This section presents an explanation for the existence of the strict Inequality (1), and its quantification, based on the possibility of a tie for the minimum in the set $\{X_i : i \in N\}$. A confirmation of this intuition is found by defining a new random variable, $W_{(1)}$, obtained dividing $X_{(1)}$ by the expected number of random variables tied for the minimum: the expectation of this "weighted minimum" $W_{(1)}$ is indeed the same as that of $Y_{(1)}$.

The discrete nature of the geometric distribution implies that several random variables in $\{X_i : i \in N\}$ might coincide with $X_{(1)}$. Define $I_{[1]}$ to be the the set of indices among $\{1, \ldots n\}$ corresponding to such random variables ($I_{[1]}$ is itself random):

$$I_{[1]} = \{i \in N : X_i = X_{(1)}\} \subseteq N, I_{[1]} \neq \emptyset.$$

The pmf of $I_{[1]}$ is

$$\begin{aligned}
\forall s \subseteq N, s \neq \emptyset, \Pr\{I_{[1]} = s\} &= \Pr\{\forall i \in s, X_i = X_{(1)} \wedge \forall j \in N \setminus s, X_j > X_{(1)}\} \\
&= \sum_{k=1}^{\infty} \Pr\{\forall i \in s, X_i = k\omega \wedge \forall j \in N \setminus s, X_j > k\omega\} \\
&= \sum_{k=1}^{\infty} \left(\prod_{i \in s} \alpha_i (1-\alpha_i)^{k-1}\right) \left(\prod_{j \in N \setminus s} (1-\alpha_j)^k\right) \\
&= \sum_{k=1}^{\infty} \left(\prod_{i \in s} \alpha_i\right) \left(\prod_{j \in N \setminus s} (1-\alpha_j)\right) \left(\prod_{l \in N} (1-\alpha_l)^{k-1}\right) \\
&= \frac{\left(\prod_{i \in s} \alpha_i\right) \left(\prod_{j \in N \setminus s} (1-\alpha_j)\right)}{1 - \prod_{l \in N} (1-\alpha_l)}.
\end{aligned}$$

This result is more easily obtained observing that, because of the absence of memory of the geometric distribution, $I_{[1]}$ and $X_{(1)}$ are independent, hence $\Pr\{I_{[1]} = s\}$ is simply the product of the one-step probability of success for the elements of s and of the one-step probability of failure for the elements not in s, normalized by the probability that at least one success occurs.

For example, if $n = 2$, the three possible values for $I_{[1]}$ and their probabilities are:

$$\Pr\{I_{[1]} = \{1\}\} = \Pr\{X_1 < X_2\} = \frac{\alpha_1(1-\alpha_2)}{1-(1-\alpha_1)(1-\alpha_2)} = \frac{\lambda_1 - \lambda_1\lambda_2\omega}{\lambda_1+\lambda_2-\lambda_1\lambda_2\omega}$$
$$\Pr\{I_{[1]} = \{2\}\} = \Pr\{X_1 < X_2\} = \frac{\alpha_2(1-\alpha_1)}{1-(1-\alpha_1)(1-\alpha_2)} = \frac{\lambda_2 - \lambda_1\lambda_2\omega}{\lambda_1+\lambda_2-\lambda_1\lambda_2\omega}$$
$$\Pr\{I_{[1]} = \{1,2\}\} = \Pr\{X_1 = X_2\} = \frac{\alpha_1\alpha_2}{1-(1-\alpha_1)(1-\alpha_2)} = \frac{\lambda_1\lambda_2\omega}{\lambda_1+\lambda_2-\lambda_1\lambda_2\omega}$$

In general, the probability that a particular X_i is equal $X_{(1)}$, or that $i \in I_{[1]}$, is

$$\begin{aligned}
\Pr\{X_i = X_{(1)}\} &= \sum_{k=1}^{\infty} \Pr\{X_i = k\omega \wedge \forall j \in N, j \neq i, X_j \geq k\omega\} \\
&= \sum_{k=1}^{\infty} \alpha_i(1-\alpha_i)^{k-1} \prod_{j\in N, j\neq i} (1-\alpha_j)^{k-1} \\
&= \sum_{k=1}^{\infty} \alpha_i \prod_{j\in N} (1-\alpha_j)^{k-1} \\
&= \frac{\alpha_i}{1-\prod_{j\in N}(1-\alpha_j)},
\end{aligned}$$

hence, the expected number of completions at time $X_{(1)}$ among $\{X_i : i \in N\}$ is

$$E[|I_{[1]}|] = \sum_{i\in N} \Pr\{X_i = X_{(1)}\} = \frac{\sum_{i\in N}\alpha_i}{1-\prod_{j\in N}(1-\alpha_j)}.$$

We can define the "weighted" random variables $\{W_i : i \in N\}$, where

$$\forall i \in N, W_i = \frac{X_i}{E[|I_{[1]}|]} \sim \text{Geom}\left(\alpha_i, \frac{\omega}{E[|I_{[1]}|]}\right) \equiv \text{Geom}\left(\alpha_i, \omega \cdot \frac{1-\prod_{j\in N}(1-\alpha_j)}{\sum_{i\in N}\alpha_i}\right).$$

which are still geometrically distributed random variables with the same success probabilities as their original counterparts $\{X_i : i \in N\}$, but with a reduced time-step. Then,

$$W_{(1)} = \min\{W_i : i \in N\} = \min\left\{\frac{X_i}{E[|I_{[1]}|]} : i \in N\right\} = \frac{X_{(1)}}{E[|I_{[1]}|]}$$

takes into account simultaneous completions by dividing the minimum completion time by the expected number of completions (the corresponding quantity for the continuous case is still simply $Y_{(1)}$, since the probability of simultaneous completions is zero in this case). The expected value of the weighted minimum for the geometric case coincides with the expected minimum for the exponential case:

$$E[W_{(1)}] = E\left[\frac{X_{(1)}}{E[|I_{[1]}|]}\right] = \frac{E[X_{(1)}]}{E[|I_{[1]}|]} = \frac{\dfrac{\omega}{1-\prod_{i\in N}(1-\alpha_i)}}{\dfrac{\sum_{i\in N}\alpha_i}{1-\prod_{j\in N}(1-\alpha_j)}} = \frac{\omega}{\sum_{i\in N}\alpha_i} = \left(\sum_{i\in N}\lambda_i\right)^{-1} = E[Y_{(1)}]$$

We conclude this section by observing that, while the result $E[W_{(1)}] = E[Y_{(1)}]$ seems to imply that exact ties are the cause of Inequality (1), this is not correct, since the inequality holds even when ties are not possible. This can be shown by considering a set of geometric random variables $\{X_i^* : i \in N\}$, where

$$\forall i \in N, X_i^* \sim \text{Geom}(\alpha_i^*, \omega_i), E[X^*] = \frac{\omega_i}{\alpha_i^*} = \lambda_i^{-1} = E[Y_i]$$

and, $\forall i \in N, \forall j \in N, i \neq j$, the ratio ω_i/ω_j is not a rational number, hence, it is not possible to find two integers k_i and k_j that would results in a potential tie at time $k_i\omega_i = k_j\omega_j$.

6 Matching the minimums by time-shifting

In the previous section, we forced the expectation of the minimums of $\{X_i : i \in N\}$ and $\{Y_i : i \in N\}$ to coincide by reducing the time-step of the geometric distributions, that is, transforming $\{X_i : i \in N\}$ into $\{W_i : i \in N\}$. While the result $E[W_{(1)}] = E[Y_{(1)}]$ is appealing, the weighted random variables $\{W_i : i \in N\}$ do not match the original $\{Y_i : i \in N\}$ in expectation. A more interesting result would be to modify our initial set of random variables $\{X_i : i \in N\}$ so that both the individual random variables and the minimum match the corresponding exponential quantities in expectation.

In this section, we accomplish exactly this by introducing the "shifted geometric" distribution, a generalization of both the geometric and modified geometric distribution. Given $0 < \alpha \leq 1$, $\omega > 0$, and $\sigma \in \mathbb{R}$, we say that S has a shifted geometric distribution with parameters α, ω, and σ, $S \sim \text{ShiftGeom}(\alpha, \omega, \sigma)$, if its pmf is

$$\forall k \in \mathbb{N}, \Pr\{S = k\omega + \sigma\} = \alpha(1-\alpha)^k$$

which implies that its CDF is

$$\forall t \in \mathbb{R},\ \Pr\{S \leq t\} = \begin{cases} 1 - (1-\alpha)^{\lfloor \frac{t-\sigma}{\omega} \rfloor + 1} & \text{if } t \geq \sigma \\ 0 & \text{otherwise} \end{cases}$$

and that its expectation is

$$E[S] = \frac{1-\alpha}{\alpha}\omega + \sigma.$$

In other words, given a random variable $A \sim \text{ModGeom}(\alpha)$, $\omega > 0$, and $\sigma \in \mathbb{R}$, $S = A\omega + \sigma \sim \text{ShiftGeom}(\alpha, \omega, \sigma)$. Figure 2 shows the relationships between the geometric, modified geometric, shifted geometric, and exponential distributions.

Given $Y \sim \text{Expo}(\lambda)$, we can again consider the condition under which S and Y have the same expectation:

$$E[S] = \frac{1-\alpha}{\alpha}\omega + \sigma = \lambda^{-1} = E[Y] \Rightarrow \alpha = \frac{\omega\lambda}{1 - \sigma\lambda + \omega\lambda}. \tag{3}$$

Since α is a probability, it can only have values in $[0, 1]$. Furthermore, $E[S] = \infty$ when $\alpha = 0$, so we exclude this case. Then, S and Y have the same expectation for any choice of ω and σ, as long as

$$0 < \alpha = \frac{\omega\lambda}{1 - \sigma\lambda + \omega\lambda} \leq 1 \Rightarrow \sigma \leq \lambda^{-1}$$

and α is set according to Equation (3). A few observations are of particular interest:

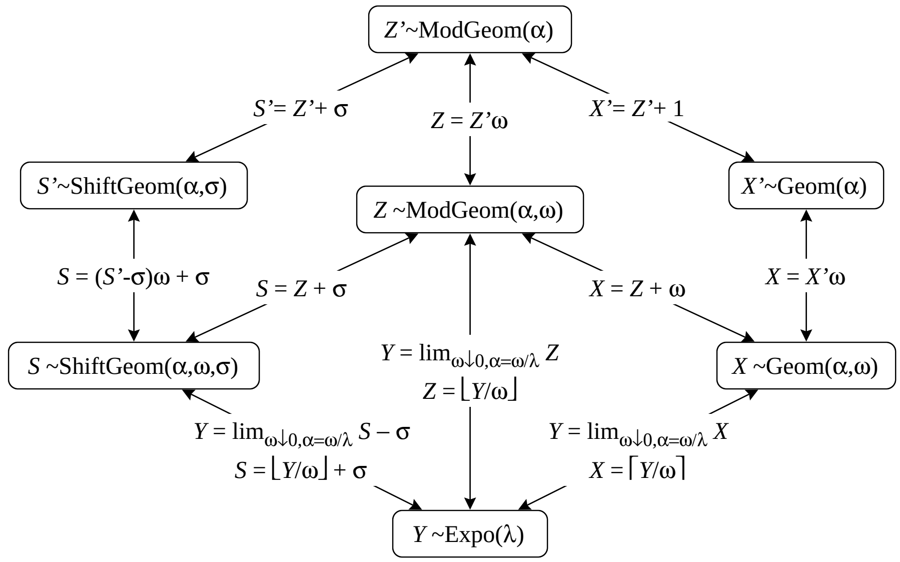

Figure 2: Relationships between the distributions discussed in this paper.

- Once the value of $E[S]$ is fixed at λ^{-1}, decreasing the time-shift σ by δ, possibly below zero, causes a decrease in α, so that $E[A]$ increases by δ/ω and $E[S] = E[A]\omega - \sigma$ remains constant. Since $E[A]$ can be arbitrarily large, this explains why there is no lower bound for σ.
- If $\sigma = 0$, $S \sim \text{ModGeom}(\alpha, \omega)$.
- If $\omega < \lambda^{-1}$ and $\sigma = \omega$, $S \sim \text{Geom}(\alpha, \omega)$.
- If $\sigma = \lambda^{-1}$, $\alpha = 1$, hence $S \sim \text{Const}(\sigma) \equiv \text{Const}(\lambda^{-1})$.

Consider now a set of modified geometric random variables with time-step one, $\{A_i : i \in N\}$ and the set of shifted geometric random variables $\{S_i : i \in N\}$ obtained from them by changing the time-step to ω and applying a time-shift σ:

$$\forall i \in N,\ A_i \sim \text{ModGeom}(\alpha_i), S_i = A_i\omega + \sigma \Rightarrow S_i \sim \text{ShiftGeom}(\alpha_i, \omega, \sigma)$$

and set the parameters $\{\alpha_i : i \in N\}$ so that:

$$\forall i \in N,\ E[S_i] = \frac{1-\alpha_i}{\alpha_i}\omega + \sigma = \lambda_i^{-1} = E[Y_i] \Rightarrow \alpha_i = \frac{\omega\lambda_i}{1 - \sigma\lambda_i + \omega\lambda_i}.$$

Since $\forall i \in N, 0 < \alpha_i \leq 1$, the maximum value of σ is

$$\sigma \leq \min_{i \in N}\left\{\lambda_i^{-1}\right\} = \left(\max_{i \in N}\{\lambda_i\}\right)^{-1} = \lambda_{MAX}^{-1}.$$

The expectation of $S_{(1)} = \min\{S_i : i \in N\} = A_{(1)}\omega + \sigma$ is then

$$E[S_{(1)}] \quad = \quad \frac{\prod_{i \in N}(1-\alpha_i)}{1 - \prod_{i \in N}(1-\alpha_i)}\omega + \sigma$$

$$= \frac{\prod_{i \in N}\left(1 - \frac{\omega\lambda_i}{1-\sigma\lambda_i+\omega\lambda_i}\right)}{1 - \prod_{i \in N}\left(1 - \frac{\omega\lambda_i}{1-\sigma\lambda_i+\omega\lambda_i}\right)}\omega + \sigma$$

$$= \frac{\prod_{i \in N}(1-\sigma\lambda_i)}{\prod_{i \in N}(1-\sigma\lambda_i+\omega\lambda_i) - \prod_{i \in N}(1-\sigma\lambda_i)}\omega + \sigma = \frac{p_n}{q_n - p_n}\omega + \sigma$$

where

$$p_n = \prod_{i \in N}(1-\sigma\lambda_i) \quad \text{and} \quad q_n = \prod_{i \in N}(1-\sigma\lambda_i+\omega\lambda_i)$$

satisfy

- $\forall \sigma \le \lambda_{MAX}^{-1}, p_n < q_n$.
- $p_n|_{\sigma=0} = 1,\ q_n|_{\sigma=0} = \prod_{i \in N}(1+\omega\lambda_i) > 1$.
- $p_n|_{\sigma=\lambda_{MAX}^{-1}} = 0$.
- If $\omega < \lambda_{MAX}^{-1},\ q_n|_{\sigma=\omega} = 1$.

Theorem 2. There exists a unique value $\sigma^* \le \lambda_{MAX}^{-1}$ for which $E[S_{(1)}] = E[Y_{(1)}]$.

Proof. To show the existence of σ^*, it is sufficient to observe that $E[S_{(1)}]$ is a continuous function of σ over $(-\infty, \lambda_{MAX}^{-1}]$, that

$$E[S_{(1)}]\Big|_{\sigma=0} = \frac{\omega}{\prod_{i \in N}(1+\omega\lambda_i) - 1} < \left(\sum_{i \in N}\lambda_i\right)^{-1} = E[Y_{(1)}]$$

(this is Inequality (2)), and that

$$E[S_{(1)}]\Big|_{\sigma=\lambda_{MAX}^{-1}} = \lambda_{MAX}^{-1} > \left(\sum_{i \in N}\lambda_i\right)^{-1} = E[Y_{(1)}].$$

Hence, by continuity, there must exist a value $\sigma^* \in (0, \lambda_{MAX}^{-1})$ satisfying

$$E[S_{(1)}]\Big|_{\sigma=\sigma^*} = E[Y_{(1)}].$$

Furthermore, if $\omega < \lambda_{MAX}^{-1}$,

$$E[S_{(1)}]\Big|_{\sigma=\omega} = \frac{\prod_{i \in N}(1-\omega\lambda_i)}{1 - \prod_{i \in N}(1-\omega\lambda_i)}\omega + \omega = \frac{\omega}{1 - \prod_{i \in N}(1-\omega\lambda_i)} > \left(\sum_{i \in N}\lambda_i\right)^{-1} = E[Y_{(1)}]$$

(this is Inequality (1)), hence, in general, $\sigma^* \in (0, \min\{\omega, \lambda_{MAX}^{-1}\})$.

We prove the uniqueness of σ^* by induction on n, showing that $E[S_{(1)}]$ is a strictly increasing function of σ over $(-\infty, \lambda_{MAX}^{-1}]$, hence we make the index n explicit in $E[S_{(1)}]$ by writing

$$E[S_{(1,n)}] = \min\{S_i : i \in N\}.$$

Base step: For $n = 2$,

$$\begin{aligned} E[S_{(1,2)}] &= \frac{(1-\sigma\lambda_1)(1-\sigma\lambda_2)}{(1-\sigma\lambda_1+\omega\lambda_1)(1-\sigma\lambda_2+\omega\lambda_2)-(1-\sigma\lambda_1)(1-\sigma\lambda_2)}\omega+\sigma \\ &= \frac{1+\sigma\lambda_1\lambda_2(\omega-\sigma)}{\lambda_1+\lambda_2+\lambda_1\lambda_2(\omega-2\sigma)} \end{aligned}$$

and

$$\begin{aligned} \frac{dE[S_{(1,2)}]}{d\sigma} &= \frac{\lambda_1\lambda_2(\omega(\lambda_1+\lambda_2-2\lambda_1\lambda_2\sigma)+2(1-\sigma\lambda_1-\sigma\lambda_2+\sigma^2\lambda_1\lambda_2)+\omega^2\lambda_1\lambda_2)}{(\lambda_1+\lambda_2+\lambda_1\lambda_2(\omega-2\sigma))^2} \\ &= \frac{\lambda_1\lambda_2(\overbrace{\omega(\lambda_1(1-\sigma\lambda_2)+\lambda_2(1-\sigma\lambda_1)+2(1-\sigma\lambda_1)(1-\sigma\lambda_2)}^{\geq 0 \text{ for } \sigma\leq\lambda_M^{-1}AX}+\omega^2\lambda_1\lambda_2)}{(\lambda_1+\lambda_2+\lambda_1\lambda_2(\omega-2\sigma))^2} > 0. \end{aligned}$$

In particular,

$$\lim_{\sigma\to-\infty} E[S_{(1,2)}] = -\infty \quad \text{and} \quad \lim_{\sigma\to-\infty}\frac{dE[S_{(1,2)}]}{d\sigma} = \frac{1}{2}$$

and

$$E[S_{(1,2)}]\Big|_{\sigma=\lambda_{MAX}^{-1}} = \lambda_{MAX}^{-1} \quad \text{and} \quad \frac{dE[S_{(1,2)}]}{d\sigma}\bigg|_{\sigma=\lambda_{MAX}^{-1}} = \frac{\omega}{\omega+\lambda_{MIN}^{-1}-\lambda_{MAX}^{-1}} \in \left(\frac{\omega}{\omega+\lambda_{MIN}^{-1}}, 1\right]$$

where $\lambda_{MIN} = \min\{\lambda_1, \lambda_2\}$.

Inductive Hypothesis: Assume that, for a given n,

$$E[S_{(1,n)}] = \frac{p_n}{q_n - p_n}\omega + \sigma$$

is a strictly increasing function of σ over $(-\infty, \lambda_{MAX}^{-1}]$, that is

$$\frac{dE[S_{(1,n)}]}{d\sigma} = \frac{p_n'(q_n-p_n)-p_n(q_n'-p_n'}{(q_n-p_n)^2}\omega+1 = \frac{p_n'q_n-p_nq_n'}{(q_n-p_n)^2}\omega+1>0$$

which implies

$$\forall\sigma\in(-\infty,\lambda_{MAX}^{-1}],\ \omega(p_n'q_n-p_nq_n') > -(q_n-p_n)^2$$

Inductive Step: Then the same holds for $n+1$, that is,

$$E[S_{(1,n+1)}] = \frac{p_{n+1}}{q_{n+1}-p_{n+1}}\omega+\sigma = \frac{p_n(1-\sigma\lambda_{n+1})}{q_n(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})-p_n(1-\sigma\lambda_{n+1})}\omega+\sigma,$$

where λ_{n+1} is the rate of the $(n+1)$-th exponential random variable, is a strictly increasing function of σ over $(-\infty, \max\{\lambda_{MAX}, \lambda_{n+1}\}^{-1}]$, that is

$$\begin{aligned} \frac{dE[S_{(1,n+1)}]}{d\sigma} &= \frac{(1-\sigma\lambda_{n+1})(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})\quad\overbrace{(p_n'q_n-p_nq_n')\omega}^{>-(q_n-p_n)^2 \text{ for } \sigma\leq\lambda_{MAX}^{-1}}\quad-\lambda_{n+1}^2\omega^2q_np_n}{(q_n(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})-p_n(1-\sigma\lambda_{n+1}))^2}+1 \\ &> \frac{(1-\sigma\lambda_{n+1})(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})(-(q_n-p_n)^2)-\lambda_{n+1}^2\omega^2q_np_n}{(q_n(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})-p_n(1-\sigma\lambda_{n+1}))^2}+1 \\ &= \frac{\lambda_{n+1}\omega(q_n-p_n)((q_n+p_n)(1-\sigma\lambda_{n+1})+\lambda_{n+1}\omega q_n}{(q_n(1-\sigma\lambda_{n+1}+\omega\lambda_{n+1})-p_n(1-\sigma\lambda_{n+1}))^2} > 0 \end{aligned}$$

since, $\sigma < \lambda_{n+1}^{-1}$ and $\forall \sigma \in (-\infty, \max\{\lambda_{MAX}, \lambda_{n+1}\}^{-1}], q_n > p_n$. QED.

We might now ask whether this value σ^* for which $E[S_{(1)}] = E[Y_{(1)}]$ is such that the other order statistics coincide as well, that is, whether

$$\forall i \in N, i > 1,\ E[S_{(i)}]|_{\sigma=\sigma^*} = E[Y_{(i)}].$$

This is indeed true for $n = 2$, since

$$E[S_{(1)}] + E[S_{(2)}] = E[S_1] + E[S_2] = E[Y_1] + E[Y_2] = E[Y_{(1)}] + E[Y_{(2)}]$$

implies that, whenever $E[S_{(1)}] = E[Y_{(1)}]$,

$$E[S_{(2)}] = E[Y_{(2)}]$$

Unfortunately, this is not true in general for $n \geq 3$, as it can be seen considering the homogeneous case. When $\forall i \in N, \lambda_i = \lambda$,

$$\begin{aligned}
\Pr\{S_{(2)} > k\omega + \sigma\} &= \Pr\{A_{(2)} > k\} \\
&= \Pr\left\{(\forall i \in N, A_i > k) \vee \left(\bigvee_{i\in N} (A_i \leq k \wedge \forall j \in N, j \neq i, A_j > k)\right)\right\} \\
&= \Pr\{\forall i \in N, A_i > k\} + \sum_{i\in N} \Pr\{A_i \leq k \wedge \forall j \in N, j \neq i, A_j > k\} \\
&= ((1-\alpha)^n)^{k+1} + n(1-(1-\alpha)^{k+1})((1-\alpha)^{n-1})^{k+1} \\
&= n(1-\alpha)^{(n-1)(k+1)} - (n-1)(1-\alpha)^{n(k+1)}
\end{aligned}$$

and

$$\begin{aligned}
E[A_{(2)}] &= \sum_{k=0}^{\infty} \Pr\{A_{(2)} > k\} \\
&= \sum_{k=0}^{\infty} n(1-\alpha)^{(n-1)(k+1)} - (n-1)(1-\alpha)^{n(k+1)} \\
&= \frac{n(1-\alpha)^{n-1}}{1-(1-\alpha)^{n-1}} - \frac{(n-1)(1-\alpha)^n}{1-(1-\alpha)^n}
\end{aligned}$$

Hence, considering $S^{(2)} = A_{(2)}\omega + \sigma$ and substituting α from (3),

$$E[S_{(2)}] = \left(\frac{n(1-\sigma\lambda)^{n-1}}{(1-\sigma\lambda-\omega\lambda)^{n-1} - (1-\sigma\lambda)^{n-1}} - \frac{(n-1)(1-\sigma\lambda)^n}{(1-\sigma\lambda-\omega\lambda)^n - (1-\sigma\lambda)^n}\right)\omega + \sigma,$$

while, due to the absence of memory of the exponential distribution,

$$E[Y_{(2)}] = (n\lambda)^{-1} + ((n-1)\lambda)^{-1}.$$

It can be easily verified numerically, for example when $n = 3$, $\omega = 1/2$, $\lambda = 1$, that the only real root of $E[S_{(1)}] = E[Y_{(1)}]$ is $\sigma \approx 0.173927$, while the only real root less than λ^{-1} of $E[S_{(2)}] = E[Y_{(2)}]$ is $\sigma \approx 0.346961$.

7 Variate generation application

The results of Section 3 can be used in variate generation for Monte Carlo simulation. For brevity, only the geometric distribution is considered. Results for the modified geometric and shifted geometric distributions are similar.

To generate a single Expo(λ) random variate Y by inversion [1]

$$Y \leftarrow -\frac{1}{\lambda}\ln(1-U),$$

where $U \sim$ Unif$(0,1)$. The random number $1-U$ can be replaced by U for increased speed although the direction of monotonicity is reversed. If $Y \sim$ Expo(λ) then $\lceil Y \rceil \sim$ Geom$(1-e^{-\lambda})$ since $\Pr\{\lceil Y \rceil = k\} = e^{-(k-1)\lambda}(1-e^{-\lambda})$ for $k \in \mathbb{N}^+$. Thus to generate a Geom(α) random variate X requires only a single line of code

$$X \leftarrow \left\lceil \frac{\ln(1-U)}{\ln(1-\alpha)} \right\rceil.$$

If the time-step is ω, then the appropriate modification to generate a Geom(α, ω) random variate is

$$X \leftarrow \left\lceil \frac{\ln(1-U)}{\ln(1-\alpha)} \right\rceil \omega.$$

The straightforward approach to generating the minimum $Y_{(1)}$ of n exponential random variables $\{Y_i \sim \text{Expo}(\lambda_i) : i \in N\}$ is to generate n exponential variates $Y_1, \ldots, Y_n$, then determine the minimum and the associated index (if required). This approach becomes time consuming as n increases. A much faster approach is to generate the minimum as

$$Y_{(1)} \leftarrow -\frac{\ln(1-U)}{\sum_{i \in N} \lambda_i},$$

where the denominator needs to be computed only once. This approach is both synchronized (one random variate from Unif$(0,1)$, U_i is needed to generate one random variate for $Y_{(1)_i}$) and monotone (given two random variates from Unif$(0,1)$, U_1 and U_2, $U_1 < U_2 \Rightarrow Y_{(1)_1} < Y_{(1)_2}$). To generate a variate corresponding to the index J of the minimum value, use the pmf

$$\Pr\{J = j\} = \frac{\lambda_j}{\sum_{i \in N} \lambda_i},$$

for $j \in N$.

There are two cases to be considered when generating the minimum of geometric random variables. The first is when the modeler wants the means of the individual random variables (but not of their minimums) to match. The second is when the modeler wants the means of the minimums (but not of the individual random variables) to match. Consider generating the minimum $X_{(1)}$ in the first case, where $\{X_i \sim \text{Geom}(\alpha_i, \omega) : i \in N\}$. First generate the minimum

$$X_{(1)} \leftarrow \left\lceil \frac{\ln(1-U)}{\sum_{i \in N} \ln(1-\alpha_i)} \right\rceil \omega.$$

To generate a random set of indices $I_{[1]}$ corresponding to completion at the minimum value, use the pmf

$$\Pr\{I_{[1]} = s\} = \frac{\left(\prod_{i \in s} \alpha_i\right)\left(\prod_{j \in N \setminus s} (1 - \alpha_j)\right)}{1 - \prod_{l \in N} (1 - \alpha_l)}.$$

for $s \subseteq N, s \neq \emptyset$.

There are two costs to consider when generating a set of indices corresponding to $X_{(1)}$. The first cost is the set-up cost incurred once at the beginning of a simulation. If all of the $2^n - 1$ subsets of indices are to be considered, the $(0, 1)$ interval must be partitioned into as many pieces prior to generating any variates. The second cost, often called the *marginal* cost to generate a variate, is incurred each time a random variate is generated. It involves generating a Unif$(0, 1)$ variate and searching the partition determined at the beginning of the simulation for the appropriate cell. This cell corresponds to a set of indices for the generated geometric random variable. The above scenario is worst-case, since time will be saved in both the set-up and marginal steps if, for example, the modeler is only interested in whether or not a tie occurred.

The generation of $W_{(1)}$, where the expected values of the minimums of the exponential and geometric random variables coincide, requires only a slight modification to the previous approach. At the beginning of a simulation, $E[|I_{[1]}|]$ should be calculated. Thus the reduced geometric is

$$W_{(1)} \leftarrow \frac{X_{(1)}}{E[|I_{[1]}|]}.$$

where $X_{(1)}$ is generated using the previous technique.

8 Conclusion

We have shown how, if the random variables $\{X_i : i \in N\}$, $\{Y_i : i \in N\}$, and $\{Z_i : i \in N\}$ model the same set of n concurrent activities using geometric, exponential, or modified geometric distributions, respectively, with given expectations $\{\lambda_i^{-1}$, the expected value of the minimums are different, $E[X_{(1)}] > E[Y_{(1)}] > E[Z_{(1)}]$. Stochastic variability is employed to justify the result.

We then consider two different ways to match the expectation of the minimums. First, by taking into account the possibility of ties in the geometric case, we define the "weighted minimum" $W_{(1)}$, and obtain $E[W_{(1)}] = E[Y_{(1)}]$, but this operation corresponds to decreasing the time-step of the individual geometric distributions, hence their expectation. Alternatively, we introduce the "shifted geometric distribution", which a generalizes both the geometric and the modified geometric. We can then define a set of shifted geometric random variables $\{S_i : i \in N\}$, which match in expectation the exponential random variables both individually, $E[S_i] = \lambda_i^{-1}$, and their minimum, $E[S_{(1)}] = E[Y_{(1)}]$. Generating variates is straightforward.

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