ICASE

ON THE GIBBS PHENOMENON V: RECOVERING EXPONENTIAL ACCURACY FROM COLLOCATION POINT VALUES OF A PIECEWISE ANALYTIC FUNCTION

David Gottlieb
Chi-Wang Shu

ON THE GIBBS PHENOMENON V:

RECOVERING EXPONENTIAL ACCURACY FROM

COLLOCATION POINT VALUES OF A PIECEWISE ANALYTIC FUNCTION [1]

David Gottlieb and Chi-Wang Shu

Division of Applied Mathematics

Brown University

Providence, RI 02912

ABSTRACT

The paper presents a method to recover exponential accuracy at all points (including at the discontinuities themselves), from the knowledge of an approximation to the *interpolation polynomial* (or trigonometrical polynomial). We show that if we are given the collocation point values (or an highly accurate approximation) at the Gauss or Gauss-Lobatto points, we can reconstruct an *uniform exponentially convergent* approximation to the function $f(x)$ in any sub-interval of analyticity. The proof covers the cases of Fourier, Chebyshev, Legendre, and more general Gegenbauer collocation methods.

i

1   Introduction

We continue our investigation of overcoming the Gibbs phenomenon, i.e., recovering pointwise exponential accuracy *at all points* including at the discontinuities themselves, from the knowledge of a spectral partial sum of a discontinuous but piecewise analytic function. In previous papers [3], [4], [5] and [6] we have considered the cases of *Galerkin* approximation, which means that, given an orthogonal basis $\{\phi_k(x)\}$ over $[-1,1]$ with a weight function $w(x)$, we assume that the first N Galerkin coefficients

$$\hat{f}_k = \frac{1}{c_k} \int_{-1}^{1} w(x)f(x)\phi_k(x)dx, \qquad 1 \le k \le N \tag{1.1}$$

of a discontinuous but piecewise analytic function $f(x)$, are known. Here c_k are normalization constants. A procedure is given in our previous papers to recover uniform exponential accuracy, in any sub-interval of analyticity of $f(x)$. The Fourier Galerkin case ($\phi_k(x) = e^{ik\pi x}$, $w(x) = 1$) with a single discontinuity is analyzed in [3]; wave resolution issues are discussed in [4]; sub-interval reconstruction suitable for multiple discontinuities are analyzed in [5] for the Fourier case and the Legendre case ($\phi_k(x) = L_k(x)$, $w(x) = 1$), and in [6] for the general Gegenbauer case ($\phi_k(x) = C_k^\mu(x)$, $w(x) = (1 - x^2)^{\mu - \frac{1}{2}}$).

The reconstruction procedure analyzed in these papers consists of two steps:

1. Using the given spectral partial sum of the first N terms, to recover the first $m \sim N$ Gegenbauer expansion coefficients, based on a sub-interval $[a, b] \subset [-1, 1]$ in which the function is assumed analytic, with exponential accuracy. This can be achieved for any L_1 function, as long as we choose λ in the weight function of Gegenbauer polynomials to be proportional to N. The error incurred at this stage is called the *truncation error*.

2. For an analytic function in $[a, b]$, proving the exponential convergence of its Gegenbauer expansion, when the parameter λ in the weight function is proportional to the number of terms retained in the expansion. The error at this stage is labeled the *regularization error*.

In this paper, we will consider the case of collocation. We assume that the *point values* $f(x_i)$, where x_i are the Gauss or Gauss-Lobatto points of the orthogonal basis $\{\phi_k(x)\}$, of a discontinuous but piecewise analytic function $f(x)$, are given. Alternatively, we assume that we have an accurate approximation to the interpolation polynomial based on these points. The objective is to recover exponentially accurate point values over any subinterval $[a, b]$ of analyticity of $f(x)$. As before, we will separate the analysis of the error into two parts: truncation error and regularization error. Truncation error measures the difference between the exact Gegenbauer coefficients with $\lambda \sim N$, and those obtained by using the collocation point values. This will be investigated in Section 3. The regularization error measures the difference between the Gegenbauer expansion using the first $m \sim N$ Gegenbauer coefficients with $\lambda \sim N$, and the function itself, in a sub-interval $[a, b]$, in which the function is assumed analytic. This error is estimated in [5] and we will simply quote the result in Section 4. The results are summarized in Theorem 4.2. Section 5 contains a numerical example to illustrate our results. In Section 2 we shall give some useful properties and estimates, including a useful approximation result.

Throughout this paper, we will use A to denote a generic constant or at most a polynomial in the growing parameters, as will be indicated in the text. It may not be the same at different locations.

2 Preliminaries

We first collect some useful results about the Gegenbauer polynomials, to be used later. We rely heavily on the standardization in Bateman [1].

Definition 2.1. The Gegenbauer polynomial $C_n^\lambda(x)$, for $\lambda \geq 0$, is defined by

$$(1 - x^2)^{\lambda - \frac{1}{2}} C_n^\lambda(x) = G(\lambda, n) \frac{d^n}{dx^n} \left[(1 - x^2)^{n + \lambda - \frac{1}{2}} \right] \tag{2.1}$$

where $G(\lambda, n)$ is given by

$$G(\lambda, n) = \frac{(-1)^n \Gamma(\lambda + \frac{1}{2}) \Gamma(n + 2\lambda)}{2^n n! \Gamma(2\lambda) \Gamma(n + \lambda + \frac{1}{2})} \tag{2.2}$$

2

for $\lambda > 0$, by

$$G(0,n) = \frac{(-1)^n \sqrt{\pi}}{2^{n-1} n \Gamma(n + \frac{1}{2})} \tag{2.3}$$

for $\lambda = 0$ and $n \geq 1$, and by

$$G(0,0) = 1. \tag{2.4}$$

for $\lambda = 0$, $n = 0$. Notice that by this standardization, $C_n^0(x)$ is defined by (see [1]):

$$C_n^0(x) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} C_n^\lambda(x) = \frac{2}{n} T_n(x), \qquad n > 0; \qquad C_0^0(x) = 1, \tag{2.5}$$

where $T_n(x)$ are the Chebyshev polynomials.

$\square$

Under this definition we have, for $\lambda > 0$,

$$C_n^\lambda(1) = \frac{\Gamma(n + 2\lambda)}{n! \Gamma(2\lambda)} \tag{2.6}$$

and

$$|C_n^\lambda(x)| \leq C_n^\lambda(1), \qquad -1 \leq x \leq 1 \tag{2.7}$$

The Gegenbauer polynomials are orthogonal under their weight function $(1 - x^2)^{\lambda - \frac{1}{2}}$:

$$\int_{-1}^1 (1 - x^2)^{\lambda - \frac{1}{2}} C_k^\lambda(x) C_n^\lambda(x) dx = \delta_{k,n} h_n^\lambda \tag{2.8}$$

where, for $\lambda > 0$,

$$h_n^\lambda = \pi^{\frac{1}{2}} C_n^\lambda(1) \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)(n + \lambda)} \tag{2.9}$$

We will need to use heavily the asymptotics of the Gegenbauer polynomials for large n and λ. For this we need the well-known Stirling's formula:

$$(2\pi)^{\frac{1}{2}} x^{x + \frac{1}{2}} e^{-x} \leq \Gamma(x + 1) \leq (2\pi)^{\frac{1}{2}} x^{x + \frac{1}{2}} e^{-x} e^{\frac{1}{12x}}, \qquad x \geq 1 \tag{2.10}$$

Lemma 2.2. There exists a constant A independent of λ and n such that

$$A^{-1} \frac{\lambda^{\frac{1}{2}}}{(n + \lambda)} C_n^\lambda(1) \leq h_n^\lambda \leq A \frac{\lambda^{\frac{1}{2}}}{(n + \lambda)} C_n^\lambda(1) \tag{2.11}$$

The proof follows from (2.9) and the Stirling's formula (2.10).

3

Let us assume that the function $f(x)$ is analytic in $[-1, 1]$ and satisfies the following

Assumption 2.3. There exists a constant $\rho \geq 1$ and $C(\rho)$ such that, for every $k \geq 0$,

$$\max_{-1 \leq x \leq 1} \left| \frac{d^k f}{dx^k}(x) \right| \leq C(\rho) \frac{k!}{\rho^k} \tag{2.12}$$

This is a standard assumption for analytic functions. ρ is the distance from $[-1, 1]$ to the nearest singularity of $f(x)$ in the complex plane (see for example [7]). For such function $f(x)$ we have the following

Lemma 2.4. If $f(x)$ is an analytic function satisfying the Assumption 2.3, then for any integer m such that $0 \leq m \leq \lambda - \frac{1}{2}$, we have

$$\max_{-1 \leq x \leq 1} \left| \frac{d^m}{dx^m} \left[(1 - x^2)^{\lambda - \frac{1}{2}} f(x) \right] \right| \leq C(\rho) \Gamma \left(\lambda + \frac{1}{2} \right) \left(2 + \frac{1}{\rho} \right)^m. \tag{2.13}$$

Proof:

$$\left| \frac{d^m}{dx^m} \left[(1 - x^2)^{\lambda - \frac{1}{2}} f(x) \right] \right| = \left| \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left(\frac{d^{m-k}}{dx^{m-k}} f(x) \right) \left(\frac{d^k}{dx^k} (1 - x^2)^{\lambda - \frac{1}{2}} \right) \right|$$

$$\leq C(\rho) \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{(m-k)!}{\rho^{m-k}} \frac{2^k k! \, \Gamma(2(\lambda - k)) \, \Gamma \left(\lambda + \frac{1}{2} \right)}{\Gamma \left(\lambda - k + \frac{1}{2} \right) \Gamma(2\lambda - k)} (1 - x^2)^{\lambda - k - \frac{1}{2}} \left| C_k^{\lambda - k}(x) \right|$$

$$\leq C(\rho) \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{(m-k)!}{\rho^{m-k}} \frac{2^k k! \, \Gamma(2(\lambda - k)) \, \Gamma \left(\lambda + \frac{1}{2} \right)}{\Gamma \left(\lambda - k + \frac{1}{2} \right) \Gamma(2\lambda - k)} \frac{\Gamma(2\lambda - k)}{k! \, \Gamma(2(\lambda - k))}$$

$$\leq C(\rho) 2^m \Gamma \left(\lambda + \frac{1}{2} \right) \sum_{k=0}^{m} \frac{m!}{k! \, \Gamma \left(\lambda - k + \frac{1}{2} \right)} \left(\frac{1}{2\rho} \right)^{m-k}$$

$$\leq C(\rho) 2^m \Gamma \left(\lambda + \frac{1}{2} \right) \sum_{k=0}^{m} \frac{m!}{k! \, (m-k)!} \left(\frac{1}{2\rho} \right)^{m-k}$$

$$= C(\rho) \Gamma \left(\lambda + \frac{1}{2} \right) \left(2 + \frac{1}{\rho} \right)^m$$

where we have used the analyticity assumption (2.12), and the formulas (2.1)-(2.2), for the second inequality. We have also used (2.6)-(2.7), and the fact that $(1 - x^2)^{\lambda - k - \frac{1}{2}} \leq 1$ because $\lambda - k - \frac{1}{2} \geq 0$, for the third inequality. In the fourth inequality we have used the fact that $\Gamma \left(\lambda - k + \frac{1}{2} \right) \geq (m-k)!$ because $\lambda - k + \frac{1}{2} \geq m - k + 1$.

4

Finally, we would need to quote some approximation results for the interpolation polynomials. If the point values $f(x_i)$ of the function $f(x)$ are known at the N Gauss or Gauss-Lobatto points $\{x_i\}$ of the basis functions $\{\phi_k(x)\}$, we will denote the unique interpolation polynomial by

$$I_N f(x) = \sum_{k=1}^{N} \hat{f}_k \phi_k(x) \tag{2.14}$$

which satisfies

$$I_N f(x_i) = f(x_i), \qquad i = 1, 2, \ldots, N \tag{2.15}$$

We then have the following error estimates for the interpolation polynomial $I_N f(x)$:

Lemma 2.5. If $f(x)$ has m continuous derivatives in $[-1, 1]$, then the interpolation polynomial $I_N f(x)$ defined by (2.14), (2.15), where the basis functions $\{\phi_k(x)\}$ are either trigonometric polynomials $e^{ik\pi x}$ or the Gegenbauer polynomials C_k^μ with $\mu > -\frac{1}{2}$ (which includes, for example, the Chebyshev case $\mu = 0$ and the Legendre case $\mu = \frac{1}{2}$), satisfies the following estimate:

$$||f - I_N f||_{L_w^2} \leq \frac{A}{N^m} ||f^{(m)}||_{L^\infty}, \tag{2.16}$$

where the weighted L^2 norm is defined by

$$||f||_{L_w^2}^2 = \int_{-1}^{1} w(x)|f(x)|^2 dx \tag{2.17}$$

and A is a constant independent of N and m.

Proof: The proof of this Lemma can be found in [8] for the trigonometric polynomial case, and in [2] (Theorem 6.6.1, first line of formula (6.6.3), for which the only restriction about the weight corresponds to our $\mu > -\frac{1}{2}$) for the Gegenbauer polynomial case. Notice that in both situations the constant A is actually independent of m, although the authors did not explicitly point this out.

□

3 Truncation Error

Consider an L_1 function $f(x)$ defined in $[-1, 1]$. We assume that the point values $f(x_i)$ at the N Gauss or Gauss-Lobatto points (i.e., the zeros of $\phi_{N+1}(x)$, or the zeros of $\frac{d}{dx}\phi_N(x)$ plus the two

boundary points ± 1), are given. Here, the orthogonal basis $\{\phi_k(x)\}$ is either the trigonometric polynomials $e^{ik\pi x}$, or the Gegenbauer polynomials $C_k^\mu(x)$ with $\mu > -\frac{1}{2}$.

We are interested in approximating the Gegenbauer expansion of $f(x)$, with $\lambda \sim N$, based on a sub-interval $[a, b] \subset [-1, 1]$ of analyticity. We start by introducing the local variable ξ:

Definition 3.1. The local variable ξ is defined by

$$\xi = \xi(x) = \frac{x - \delta}{\epsilon}, \qquad x = x(\xi) = \epsilon\xi + \delta \tag{3.1}$$

where

$$\epsilon = \frac{b - a}{2}, \qquad \delta = \frac{b + a}{2} \tag{3.2}$$

Thus when $a \leq x \leq b$, $-1 \leq \xi \leq 1$.

$\square$

We assume that the function $f(x)$ is analytic in this sub-interval $[a, b]$ satisfying the Assumption 2.3:

$$\max_{a \leq x \leq b} \left| \frac{d^k f}{dx^k}(x) \right| \leq C(\rho) \frac{k!}{\rho^k} \tag{3.3}$$

for any $k \geq 0$, with some $\rho \geq 1$.

The function $f(x)$ has a Gegenbauer expansion in the sub-interval $[a, b]$, with $\lambda \sim N$. With ξ, ϵ and δ defined in (3.1)-(3.2), we have

$$f(\epsilon\xi + \delta) = \sum_{l=0}^{\infty} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi), \qquad -1 \leq \xi \leq 1 \tag{3.4}$$

where the Gegenbauer coefficients $\hat{f}_\epsilon^\lambda(l)$ are defined by

$$\hat{f}_\epsilon^\lambda(l) = \frac{1}{h_l^\lambda} \int_{-1}^{1} (1 - \xi^2)^{\lambda - \frac{1}{2}} C_l^\lambda(\xi) f(\epsilon\xi + \delta) d\xi \tag{3.5}$$

Our goal in this section is to find a good approximation to the first $m \sim N$ Gegenbauer coefficients $\hat{f}_\epsilon^\lambda(l)$ in (3.5).

Based on the known point values $f(x_i)$, we define not the usual interpolation polynomial $I_N f(x)$ in (2.14)-(2.15), but

$$g_N(x) = I_N (\alpha \cdot f)(x), \tag{3.6}$$

6

where the function $\alpha(x)$ is defined by

$$\alpha(x) = \begin{cases} (1 - \xi(x)^2)^{\lambda - \frac{1}{2}}, & a \leq x \leq b, \\ 0, & otherwise. \end{cases} \tag{3.7}$$

with $\xi(x)$ defined by (3.1). This works for the case $\int_{-1}^{1} \frac{1}{w(x)} dx < \infty$, where $w(x)$ is the weight function of the basis $\{\phi_k\}$. Most commonly used bases belong to this class, for example the Fourier case where $w(x) = 1$ and the Gegenbauer case where $w(x) = (1 - x^2)^{\mu - \frac{1}{2}}$, when $\mu < \frac{3}{2}$. Otherwise, if the basis $\{\phi_k\}$ consists of the Gegenbauer polynomials $C_k^{\mu}(x)$ with $\mu \geq \frac{3}{2}$, we should choose

$$\alpha(x) = \begin{cases} (1 - \xi(x)^2)^{\lambda - \frac{1}{2}} / (1 - x^2)^{\frac{\mu}{2} - \frac{1}{4}}, & a \leq x \leq b, \\ 0, & otherwise. \end{cases} \tag{3.8}$$

Intuitively, the function $g_N(x)$ has about $\lambda \sim N$ continuous derivatives, hence would produce nice error estimates.

Our candidate for approximating the Gegenbauer coefficients $\hat{f}_{\epsilon}^{\lambda}(l)$ in (3.5) is:

$$\hat{g}_{\epsilon}^{\lambda}(l) = \frac{1}{h_l^{\lambda}} \int_{-1}^{1} g_N(\epsilon\xi + \delta) C_l^{\lambda}(\xi) d\xi \tag{3.9}$$

if (3.7) is used to define $\alpha(x)$, and is

$$\hat{g}_{\epsilon}^{\lambda}(l) = \frac{1}{h_l^{\lambda}} \int_{-1}^{1} (1 - x(\xi)^2)^{\frac{\mu}{2} - \frac{1}{4}} g_N(\epsilon\xi + \delta) C_l^{\lambda}(\xi) d\xi \tag{3.10}$$

if (3.8) is used to define $\alpha(x)$.

How well do $\hat{g}_{\epsilon}^{\lambda}(l)$ approximate $\hat{f}_{\epsilon}^{\lambda}(l)$? To answer this question we define

Definition 3.2. The truncation error is defined by

$$TE(\lambda, m, N, \epsilon) = \max_{-1 \leq \xi \leq 1} | \sum_{l=0}^{m} (\hat{f}_{\epsilon}^{\lambda}(l) - \hat{g}_{\epsilon}^{\lambda}(l)) C_l^{\lambda}(\xi)| \tag{3.11}$$

where $\hat{f}_{\epsilon}^{\lambda}(l)$ are defined by (3.5) and $\hat{g}_{\epsilon}^{\lambda}(l)$ are defined by (3.9).

$\square$

The truncation error is the measure of the distance between the true Gegenbauer expansion in the interval $[a, b]$ and its approximation based on the collocation point values of $f(x)$ in $[-1, 1]$.

Using the result of the previous section, we have the following estimate for the truncation error:

Lemma 3.3. The truncation error can be estimated by

$$TE(\lambda, m, N, \epsilon) \leq A \frac{m(m+\lambda)\Gamma\left(\lambda + \frac{1}{2}\right)\Gamma(m+2\lambda)}{\sqrt{\lambda}\, m!\, \Gamma(2\lambda) N^{\lambda - \frac{3}{2} - \frac{\mu}{2}}} \left[\frac{1}{\epsilon}\left(2 + \frac{1}{\rho}\right)\right]^{\lambda} \tag{3.12}$$

Proof: We shall just prove the case when (3.7) and (3.9) are used. For this case we have:

$$
\begin{aligned}
TE(\lambda, m, N, \epsilon) &\leq \sum_{l=0}^{m} |\hat{f}_{\epsilon}^{\lambda}(l) - \hat{g}_{\epsilon}^{\lambda}(l)| C_{l}^{\lambda}(1) \\
&\leq \sum_{l=0}^{m} \frac{C_{l}^{\lambda}(1)}{h_{l}^{\lambda}} \left| \int_{-1}^{1} \left[(1 - \xi^2)^{\lambda - \frac{1}{2}} f(\epsilon\xi + \delta) - g_N(\epsilon\xi + \delta)\right] C_{l}^{\lambda}(\xi) d\xi \right| \\
&\leq \sum_{l=0}^{m} \frac{\left(C_{l}^{\lambda}(1)\right)^2}{h_{l}^{\lambda}} \frac{1}{\epsilon} \int_{-1}^{1} |\alpha(x) f(x) - g_N(x)|\, dx \\
&\leq \sum_{l=0}^{m} \frac{\left(C_{l}^{\lambda}(1)\right)^2}{\epsilon h_{l}^{\lambda}} \left(\int_{-1}^{1} w(x) |\alpha(x) f(x) - g_N(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{-1}^{1} \frac{1}{w(x)} dx\right)^{\frac{1}{2}} \\
&\leq \sum_{l=0}^{m} \frac{\left(C_{l}^{\lambda}(1)\right)^2}{\epsilon h_{l}^{\lambda}} \frac{A}{N^{[\lambda - \frac{1}{2}]}} \|\alpha \cdot f\|_{H_w^{[\lambda - \frac{1}{2}]}} \\
&\leq \frac{A}{N^{[\lambda - \frac{1}{2}]}} \sum_{k=0}^{[\lambda - \frac{1}{2}]} \left[\frac{1}{\epsilon^k} \Gamma\left(\lambda + \frac{1}{2}\right)\left(2 + \frac{1}{\rho}\right)^k\right] \cdot \sum_{l=0}^{m} \frac{\left(C_{l}^{\lambda}(1)\right)^2}{\epsilon h_{l}^{\lambda}} \\
&\leq A \frac{m(m+\lambda)\Gamma\left(\lambda + \frac{1}{2}\right)\Gamma(m+2\lambda)}{\sqrt{\lambda}\, m!\, \Gamma(2\lambda) N^{\lambda - \frac{3}{2}}} \left[\frac{1}{\epsilon}\left(2 + \frac{1}{\rho}\right)\right]^{\lambda}
\end{aligned}
$$

where we have used (2.7) for the first inequality; the definitions of $\hat{f}_{\epsilon}^{\lambda}(l)$ and $\hat{g}_{\epsilon}^{\lambda}(l)$, in (3.5) and (3.9), for the second inequality; the definition of $\alpha(x)$ in (3.7) for the third inequality; Cauchy-Schwartz inequality for the fourth inequality; the approximation result (2.16) for the fifth inequality, with $[\lambda - \frac{1}{2}]$ denoting the greatest integer $\leq \lambda - \frac{1}{2}$; (2.13) for the sixth inequality; and (2.6) and (2.11) for the last inequality.

The case when (3.8) and (3.10) are used instead of (3.7) and (3.9) can be proven in a similar fashion. We only need to notice that the function $\left(\frac{1 - \xi(x)^2}{1 - x^2}\right)^{\frac{\mu}{2} - \frac{1}{4}}$ is always analytic in $a \leq x \leq b$, hence can be combined with $f(x)$ in the analysis. This accounts for the extra factor $N^{-\frac{\mu}{2}}$ in the denominator.

$\square$

8

We are now ready for the main theorem of this section:

Theorem 3.4. Let the truncation error be defined in (3.11). Let $\lambda = \alpha\epsilon N$ and $m = \beta\epsilon N$ with $0 < \alpha, \beta < 1$, then

$$TE(\alpha\epsilon N, \beta\epsilon N, N, \epsilon) \leq A \left(\frac{(\beta + 2\alpha)^{\beta+2\alpha}}{e^\alpha \alpha^\alpha \beta^\beta} \cdot \left(\frac{1}{2} + \frac{1}{4\rho} \right)^\alpha \right)^{\epsilon N} \tag{3.13}$$

where A grows at most as $N^{\frac{5+\mu}{2}}$. In particular, if $\alpha = \beta < \frac{2e}{27\left(1+\frac{1}{2\rho}\right)}$, then

$$TE(\alpha\epsilon N, \alpha\epsilon N, N, \epsilon) \leq A q^{\epsilon N} \tag{3.14}$$

where

$$q = \left[\frac{27\alpha}{2e} \left(1 + \frac{1}{\rho} \right) \right]^\alpha < 1 \tag{3.15}$$

Proof: The theorem follows from Lemma 3.3 and the Stirling's formula (2.10).

$\square$

We now make two remarks:

Remark 3.5. Unlike in the Galerkin case [3], [5] and [6], where for the purpose of truncation error being exponentially small we only need to assume that the function $f(x)$ is in L^1, here for the collocation case we need to assume that the function $f(x)$ is actually analytic in the relevant sub-interval $[a, b]$.

$\square$

Remark 3.6. We would like to point out that the collocation procedure (3.6), where we collocate not the function $f(x)$ but its product with a weight $(1-x^2)^{\lambda-\frac{1}{2}}$, is somehow nonstandard. Another, probably more natural, procedure of collocation, is to interpolate $f(x)$ with $I_N f(x)$ and then define the approximate Gegenbauer coefficients $\hat{g}_\epsilon^\lambda(l)$ by

$$\hat{g}_\epsilon^\lambda(l) = \frac{1}{h_l^\lambda} \int_{-1}^1 (1 - \xi^2)^{\lambda-\frac{1}{2}} I_N f(\epsilon\xi + \delta) C_l^\lambda(\xi) d\xi, \tag{3.16}$$

instead of (3.9). Unfortunately, we have been unable to prove that the truncation error is exponentially small for (3.16). In fact it can be shown that for some cases the aliasing error reduces

9

the accuracy. However numerical experiments in Section 5 seem to indicate that both procedures produce similarly good results.

□

4 Regularization Error and the Main Theorem

The second part of the error, which is called the regularization error and is caused by using a finite Gegenbauer expansion based on a sub-interval $[a, b] \subset [-1, 1]$, to approximate a function $f(x)$ which is assumed analytic in this sub-interval, has been studied in [5]. We will thus just quote the result.

We again assume that $f(x)$ is an analytic function on $[a, b]$ satisfying the Assumption 2.3, see (3.3).

Let us consider the Gegenbauer partial sum of the first m terms for the function $f(\epsilon\xi + \delta)$:

$$f_m^{\lambda, \epsilon}(\xi) = \sum_{l=0}^{m} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi) \tag{4.1}$$

with ξ, ϵ and δ defined by (3.1) and (3.2), and the Gegenbauer coefficients $\hat{f}_\epsilon^\lambda(l)$, based on [a,b], defined by (3.5).

The regularization error in the maximum norm is defined by:

$$RE(\lambda, m, \epsilon) = \max_{-1 \leq \xi \leq 1} \left| f(\epsilon\xi + \delta) - \sum_{l=0}^{m} \hat{f}_\epsilon^\lambda(l) C_l^\lambda(\xi) \right| \tag{4.2}$$

We have the following result for the estimation of the regularization error, when $\lambda \sim m$ [5]:

Theorem 4.1. Assume $\lambda = \gamma m$ where γ is a positive constant. If $f(x)$ is analytic in $[a, b] \subset [-1, 1]$ satisfying the Assumption 2.3 in (3.3), then the regularization error defined in (4.2) can be bounded by

$$RE(\gamma m, m, \epsilon) \leq A q^m \tag{4.3}$$

where q is given by

$$q = \frac{\epsilon(1 + 2\gamma)^{1+2\gamma}}{\rho 2^{1+2\gamma} \gamma^\gamma (1 + \gamma)^{1+\gamma}} \tag{4.4}$$

10

which is always less than 1. In particular, if $\gamma = 1$ and $m = \beta \epsilon N$ where β is a positive constant, then

$$RE(\beta \epsilon N, \beta \epsilon N, \epsilon) \leq A q^{\epsilon N} \tag{4.5}$$

with

$$q = \left(\frac{27\epsilon}{32\rho} \right)^{\beta} \tag{4.6}$$

□

We can now combine the estimates for truncation errors and regularization errors to obtain the following main theorem of this paper:

Theorem 4.2. (Removal of the Gibbs Phenomenon for the collocation case).

Consider a L_1 function $f(x)$ on $[-1, 1]$, which is analytic in a sub-interval $[a, b] \subset [-1, 1]$ and satisfies the Assumption 2.3 in (3.3). Assume that the point values $f(x_i)$ at the N Gauss or Gauss-Lobatto points of $\{\phi_k(x)\}$, are given. Here $\{\phi_k(x)\}$ is either the trigonometric polynomials $e^{ik\pi x}$ or the Gegenbauer polynomials $C_k^\mu(x)$ for $\mu > -\frac{1}{2}$. Let $\hat{g}_\epsilon^\lambda(l)$, $0 \leq l \leq m$ be the Gegenbauer expansion coefficients, defined in (3.9) or (3.10), based on the sub-interval $[a, b]$, of the collocation polynomial $g_N(x)$ defined in (3.6). Then for $\lambda = m = \beta \epsilon N$ with $\beta < \frac{2e}{27\left(1+\frac{1}{2\rho}\right)}$, we have

$$\max_{-1 \leq \xi \leq 1} \left| f(\epsilon \xi + \delta) - \sum_{l=0}^{m} \hat{g}_\epsilon^\lambda(l) C_l^\lambda(\xi) \right| \leq A \left(q_T^{\epsilon N} + q_R^{\epsilon N} \right) \tag{4.7}$$

where

$$q_T = \left[\frac{27\beta}{2e} \left(1 + \frac{1}{2\rho} \right) \right]^{\beta} < 1, \qquad q_R = \left(\frac{27\epsilon}{32\rho} \right)^{\beta} < 1$$

and A grows at most as $N^{\frac{5+\mu}{2}}$

Proof: Just combine the results of Theorems 3.4 and 4.1.

□

We remark that no attempt has been made to optimize the parameters.

5 Numerical Results

In this section we give a numerical example to illustrate our result. We will test both the Fourier and the Chebyshev cases.

Example 5.1. We take the following function

$$f(x) = \begin{cases} \sin(\cos(x)), & if \quad a \le x \le b \\ 0, & otherwise \end{cases} \tag{5.1}$$

and try to recover pointwise values of $f(x)$ for $a \le x \le b$. This function is also used in [6]. In order to see the effect of relative locations of the sub-interval, for the Chebyshev case we will consider two situations $[a, b] = [-0.5, 0.5]$ and $[a, b] = [0, 1]$. For the Fourier case we will only consider the case $[a, b] = [-0.5, 0.5]$ because of periodicity.

We first consider the Fourier case. We assume that the point values $f(x_i)$, for the $2N + 1$ equally spaced points:

$$x_i = \frac{2i}{2N + 1}, \qquad -N \le i \le N, \tag{5.2}$$

are given. Two collocation procedures are tested. Procedure A, which is the procedure proven in Section 3 to be exponentially accurate, is to interpolate $\alpha(x)f(x)$ as in (3.6) and (3.7), and then build the approximating Gegenbauer coefficients $\hat{g}_\epsilon^\lambda(l)$ by (3.9), finally use

$$g_m^{\lambda,\epsilon}(x) = \sum_{l=0}^{m} \hat{g}_\epsilon^\lambda(l) C_l^\lambda(\xi) \tag{5.3}$$

as our approximation to $f(x)$ in $[a, b]$. We also test the more natural collocation Procedure B, for which we interpolate $f(x)$ as in (2.14)-(2.15), and then build the approximating Gegenbauer coefficients $\hat{g}_\epsilon^\lambda(l)$ by using (3.16), finally using (5.3) to approximate $f(x)$ in $[a, b]$.

For the Fourier method, since the collocation points are equally spaced, we also have the choice to use all the collocation points in $[-1, 1]$ to build the approximation in $[a, b]$ (referred to as *global-local*); or use just the collocation points *inside* $[a, b]$, with suitable scaling to expand $[a, b]$ to $[-1, 1]$ (referred to as *local-local*).

We shall use the parameters

$$m = 0.2\epsilon N, \qquad \lambda = 0.4\epsilon N \tag{5.4}$$

12

where $2N + 1$ is the total number of collocation points in the full interval $[-1, 1]$. This choice is also used in [6] (with the right scaling). We will use (5.4) for all the test cases of Fourier collocation. No attempt has been made to optimize these parameters for each individual case.

In Fig. 1 we show the errors, in a logarithm scale, for $N = 10, 20, 40$ and 80, for the Procedure A (*global-local* at left, *local-local* at right). In Fig. 2 we show similar errors for the Procedure B. We can clearly see that both Procedure A and Procedure B produce similarly good results. Also, it seems that *global-local* and *local-local* are producing similar errors for the same choice of λ and m. Considering that the effective N (the number of collocation points) is smaller for the *local-local* case, hence round off errors should also be smaller, it seems that *local-local* should be used if possible (see also the discussions in [9]). However, there are situations (for example when we are solving a PDE and $I_N f$ is obtained from the time evolution) where the point values $f(x_i)$ may not be accurate, however the interpolation polynomial $I_N f$ (which may interpolates $f(x)$ at some other points than x_i) is accurate. In such situations *global-local* is more appropriate.
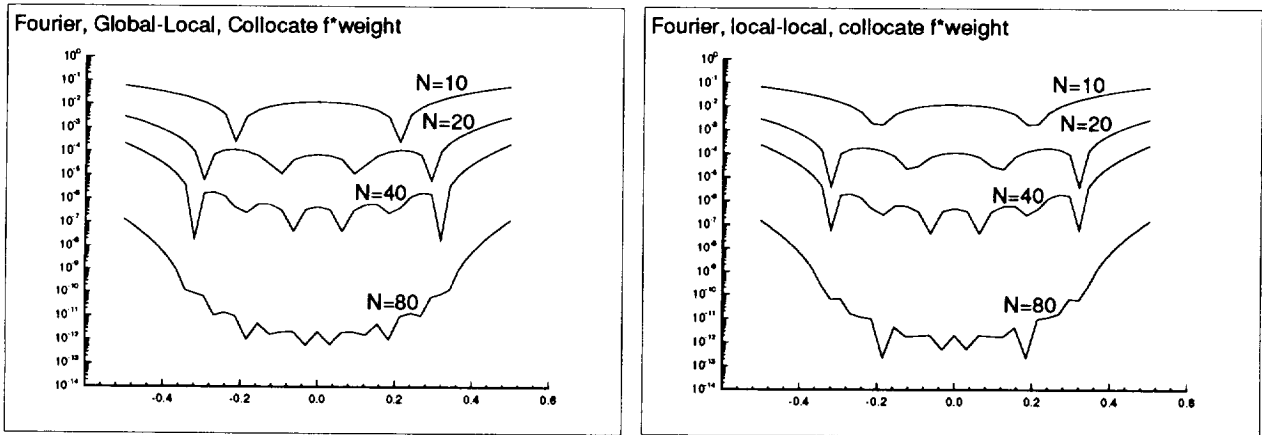

Fig. 1: Errors in log scale, $f(x)$ defined by (5.1). $[a, b] = [-0.5, 0.5]$. Fourier collocation with Procedure A, namely interpolating $f(x) * \alpha(x)$ with $\alpha(x)$ defined by (3.7). $\lambda = 0.4\epsilon N$ and $m = 0.2\epsilon N$, $N = 10, 20, 40, 80$. Left: *global-local*; Right: *local-local*.
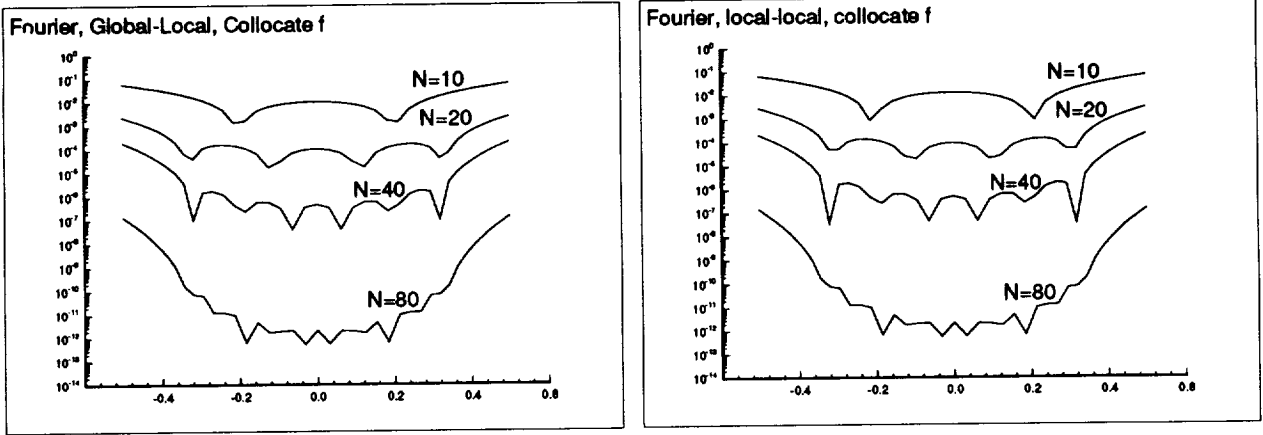
Fig. 2: Errors in log scale, $f(x)$ defined by (5.1). $[a,b] = [-0.5, 0.5]$. Fourier collocation with Procedure B, namely interpolating $f(x)$ only. $\lambda = 0.4\epsilon N$ and $m = 0.2\epsilon N$, $N = 10, 20, 40, 80$. Left: *global-local*; Right: *local-local*.

Next we consider the Chebyshev case. We assume that the point values $f(x_i)$, at the $N + 1$ Gauss-Lobatto points

$$x_i = \cos\left(\frac{i\pi}{N}\right), \qquad 0 \le i \le N, \tag{5.5}$$

are given. We again test both Procedure A and Procedure B, now for the middle interval $[a,b] = [-0.5, 0.5]$ as well as for the boundary interval $[a,b] = [0,1]$. The choice of λ and m are the same as in [6]:

$$m = 0.1\epsilon N, \qquad \lambda = 0.2\epsilon N \tag{5.6}$$

This is also the same as (5.4) for the Fourier case, considering that now the number of collocation points is $N + 1$ instead of $2N + 1$.

In Fig. 3 we show the errors, in a logarithm scale, for $N = 20$, 40, 80 and 160, for the Procedure A ($[a,b] = [-0.5, 0.5]$ at left, $[a,b] = [0,1]$ at right). In Fig. 4 we show similar errors for the Procedure B. We can again see that both procedure A and Procedure B produce similarly good results. Also, these errors are comparable with those obtained in [6] where Galerkin instead of collocation is used.
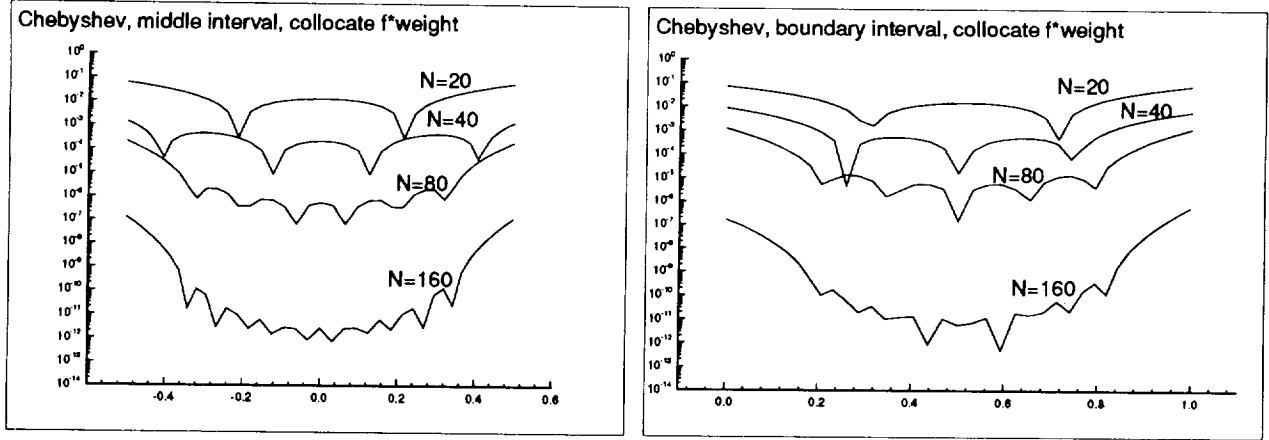
14

Fig. 3: Errors in log scale, $f(x)$ defined by (5.1). Chebyshev collocation with Procedure A, namely interpolating $f(x)*\alpha(x)$ with $\alpha(x)$ defined by (3.7). $\lambda = 0.2\epsilon N$ and $m = 0.1\epsilon N$, $N = 20, 40, 80, 160$. Left: $[a,b] = [-0.5, 0.5]$; Right: $[a,b] = [0,1]$.
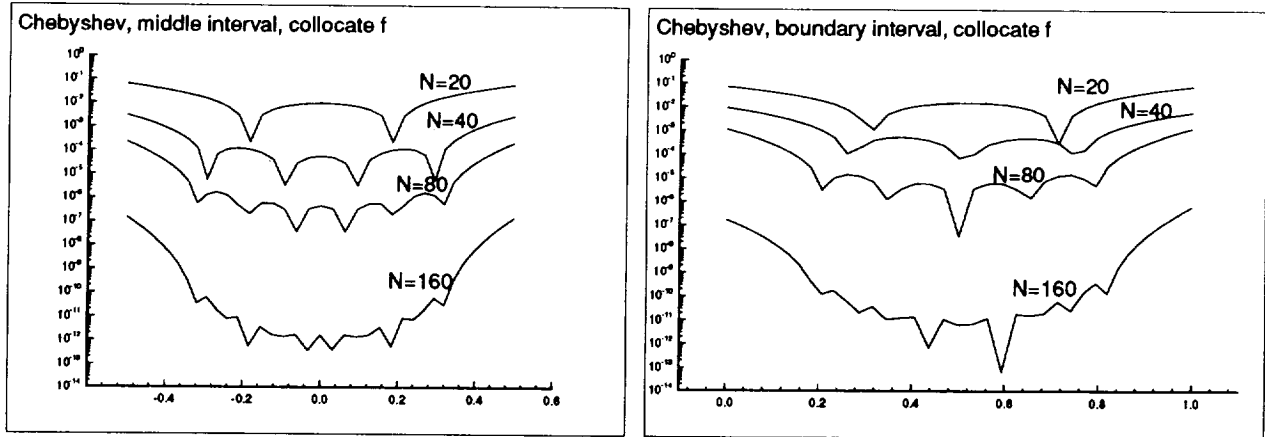

Fig. 4: Errors in log scale, $f(x)$ defined by (5.1). Chebyshev collocation with Procedure B, namely interpolating $f(x)$ only. $\lambda = 0.2\epsilon N$ and $m = 0.1\epsilon N$, $N = 20, 40, 80, 160$. Left: $[a,b] = [-0.5, 0.5]$; Right: $[a,b] = [0,1]$.

6 Concluding Remarks

We have proven the exponential convergence in the maximum norm, of a reconstruction procedure using Gegenbauer series based on $C_l^\lambda(x)$ with large λ, for any L_1 function in any sub-interval $[a,b]$ in which the function is analytic, if we are given the collocation point values of the function at the

Gauss or Gauss-Lobatto points. The proof covers Fourier, Chebyshev, Legendre and more general Gegenbauer collocation methods. Numerical examples are also given.

References

[1] H. Bateman, *Higher Transcendental Functions*, v2, McGraw-Hill, 1953.

[2] D. Funaro, *Polynomial Approximation of Differential Equations*, Lecture Notes in Physics, m8, Springer-Verlag, 1992.

[3] D. Gottlieb, C.-W. Shu, A. Solomonoff and H. Vandeven, *On The Gibbs Phenomenon I: recovering exponential accuracy from the Fourier partial sum of a non-periodic analytic function*, J. Comput. Appl. Math., v43, 1992, pp.81-92.

[4] D. Gottlieb and C.-W. Shu, *Resolution properties of the Fourier method for discontinuous waves*, ICASE Report No. 92-27, NASA Langley Research Center, 1992. Comput. Math. Appl. Mesh. Engin., to appear.

[5] D. Gottlieb and C.-W. Shu, *On The Gibbs Phenomenon III: recovering exponential accuracy in a sub-interval from the spectral partial sum of a piecewise analytic function*, ICASE Report No. 93-82, NASA Langley Research Center, 1993. SIAM J. Numer. Anal., to appear.

[6] D. Gottlieb and C.-W. Shu, *On The Gibbs Phenomenon IV: recovering exponential accuracy in a sub-interval from the Gegenbauer partial sum of a piecewise analytic function*, Math. Comp., to appear.

[7] F. John, *Partial Differential Equations*, Springer-Verlag, 1982.

[8] E. Tadmor, *The exponential accuracy of Fourier and Chebyshev differencing methods*, SIAM J. Numer. Anal., v23, 1986, pp.1-10.

[9] L. Vozovoi, M. Israeli and A. Averbuch, *Analysis and application of Fourier-Gegenbauer method to stiff differential equations*, to appear in J. of Scientific Computing.

REPORT DOCUMENTATION PAGE		Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY(Leave blank)	2. REPORT DATE July 1994	3. REPORT TYPE AND DATES COVERED Contractor Report	

4. TITLE AND SUBTITLE ON THE GIBBS PHENOMENON V: RECOVERING EXPONENTIAL ACCURACY FROM COLLOCATION POINT VALUES OF A PIECEWISE ANALYTIC FUNCTION	5. FUNDING NUMBERS C NAS1-19480 WU 505-90-52-01
6. AUTHOR(S) David Gottlieb and Chi-Wang Shu	

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23681-0001	8. PERFORMING ORGANIZATION REPORT NUMBER ICASE Report No. 94-61

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Langley Research Center Hampton, VA 23681-0001	10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA CR-194952 ICASE Report No. 94-61

12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified–Unlimited Subject Category 64	12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

The paper presents a method to recover exponential accuracy at all points (including at the discontinuities themselves), from the knowledge of an approximation to the *interpolation polynomial* (or trigonometrical polynomial). We show that if we are given the collocation point values (or an highly accurate approximation) at the Gauss or Gauss-Lobatto points, we can reconstruct an *uniform exponentially convergent* approximation to the function $f(x)$ in any sub-interval of analyticity. The proof covers the cases of Fourier, Chebyshev, Legendre, and more general Gegenbauer collocation methods.

14. SUBJECT TERMS Gibbs phenomenon, collocation, Fourier, Chebyshev, Legendre, Gegenbauer, exponential accuracy	15. NUMBER OF PAGES 18
	16. PRICE CODE A03

17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT