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EXPONENTIALLY ACCURATE APPROXIMATIONS TO PIECE-WISE SMOOTH PERIODIC FUNCTIONS

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ABSTRACT

A family of simple, periodic basis functions with "built-in" discontinuities are introduced, and their properties are analyzed and discussed. Some of their potential usefulness is illustrated in conjunction with the Fourier series representation of functions with discontinuities. In particular, it is demonstrated how they can be used to construct a sequence of approximations which converges *exponentially in the maximum norm* to a piece-wise smooth function. The theory is illustrated with several examples and the results are discussed in the context of other sequences of functions which can be used to approximate discontinuous functions.

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1. Introduction. Fourier series are used widely in many branches of applied mathematics. For example, they are often used together with separation of variables to construct analytical solutions to boundary value problems for differential equations, and with a variety of spectral methods to find approximate solutions to these problems numerically. For practical purposes, approximate solutions to these problems are often obtained using only a finite number of the terms in a Fourier series. This truncation procedure may lead to nonuniformly valid approximations. In particular, when the function being approximated has a point of discontinuity, the Gibbs phenomena is present. It is well known (see, e.g., [1] or [12]) that the magnitude of the "overshoot" associated with the Gibbs phenomena is *not eliminated* by increasing the number of terms in the approximation, but rather the overshoot approaches an asymptotic limit of about 18% as the number of terms approaches infinity. In addition, for any Fourier series partial sum, the "oscillations" caused by this phenomena typically propagate into regions away from the singularity, and, hence, degrade the quality of the partial sum approximation in these regions.

In a series of papers, Gottlieb and several of his co-investigators [3],[4],[5],[6],[7] have proposed a way of overcoming the Gibbs phenomena. Their technique involves the construction of a new series using the Gegenbauer polynomials. For a function f that is analytic on the interval $[-1,1]$, but is not periodic, they prove that their technique leads to a series which converges exponentially to f in the maximum norm. To do this, they require that the parameter λ, which appears in the weight factor $(1-x^2)^{\lambda-1/2}$, grows with the number of Fourier modes considered. Recently, Geer [2] introduced and studied a class of approximations $\{F_{N,M}\}$ to a periodic function f which uses the ideas of Padé, or rational function, approximations based on the Fourier series representation of f, rather than on the Taylor series representation of f. Each approximation $F_{N,M}$ is the quotient of a trigonometric polynomial of degree N and a trigonometric polynomial of degree M. The coefficients in these polynomials are determined by requiring that an appropriate number of the Fourier coefficients of $F_{N,M}$ agree with those of f. Explicit expressions were derived for these coefficients in terms of the Fourier coefficients of f and it was proven that these "Fourier-Padé" approximations converge *point-wise* to $(f(x^+)+f(x^-))/2$ more rapidly (in some cases by a factor of $1/k^{2M}$) than the Fourier series partial sums on which they were based. Although these approximations do *not* "eliminate" the Gibbs phenomena, they do mitigate its effect. In particular, the asymptotic value of the magnitude of the overshoot is reduced to about 6%, and, outside a "small" neighborhood of a point of discontinuity of f, the "unwanted" oscillations *can* (for practical purposes) essentially be eliminated.

To fix notation, we let f be a 2π-periodic function with enough smoothness and regularity properties so that its Fourier series exists and converges to $(f(x^+)+f(x^-))/2$ for all $-\pi \le x \le \pi$. Let $C^M[-\pi,\pi]$ denote the class of 2π-periodic functions that have at least M continuous derivatives on $[-\pi,\pi]$. Then it is well known (see, e.g., [1]) that, if $f \in C^M[-\pi,\pi]$, then the Fourier coefficients $\{a_j, b_j\}$ of f are $O(1/j^{M+2})$ as $j \to \infty$. Moreover, if f is $2\pi-$periodic and *analytic* on $[-\pi,\pi]$, then there exists a constant θ, with $0<\theta<1$, such that a_j and b_j are $O(\theta^j)$, as $j \to \infty$. In this case, it follows that

the Fourier series of f converges exponentially to f in the maximum norm.

In this paper, we introduce a new, simple class of periodic "basis" functions which have certain "built-in" singularities, and which can be used to construct a sequence of approximations which *converges exponentially* to f in the maximum norm. In particular, this implies that the Gibbs phenomena can be completely eliminated, even when f has several points of discontinuity in the interval $[-\pi, \pi]$. In section 2, the first two of these basis functions are introduced and it is shown how they can be used to improve the convergence properties of a function with one or more singularities in the interval $[-\pi, \pi]$. These ideas are illustrated by two simple examples in section 3. In section 4, the complete class of basis functions are introduced, and various convergence properties of approximations constructed using them are studied in section 5. In particular, it is proven that, under only mild restrictions on the original function f, they can be used to construct a sequence of approximations that converge exponentially to f in the maximum norm. The results of the theorems of section 5 are illustrated by three more examples in section 6. We discuss our results in the final section and compare our approximations with those used by other investigators.

2. Basis functions S_0 and S_1. Consider the 2π-periodic functions

$$S_0(x) \equiv \frac{1}{2\sqrt{2}} \frac{\sin(x)}{\sqrt{1-\cos(x)}} \quad \text{and} \quad S_1(x) \equiv \frac{1}{\sqrt{2}} \sqrt{1-\cos(x)}, \tag{1}$$

which are illustrated in Figure 1. Using the Taylor series expansions of $\sin(x)$ and $\cos(x)$ about $x = 0$, it is straightforward to show that

$$S_0(x) = \frac{x}{|x|} \left\{ \frac{1}{2} + O(x^2) \right\}, \quad S_1(x) = |x| \left\{ \frac{1}{2} + O(x^2) \right\}, \quad \text{as } x \to 0. \tag{2}$$

From these relations it follows that S_0 and S_1 satisfy the following jump conditions at $x = 0$:

$$[S_0(0)] = 1, \ \left[\frac{dS_0}{dx}(0) \right] = 0, \ [S_1(0)] = 0, \ \left[\frac{dS_1}{dx}(0) \right] = 1. \tag{3}$$

Here we have used the notation $[g(x)] \equiv g(x^+) - g(x^-)$. (In the second and fourth equations in (3), the derivatives of S_j are understood to be derivatives "from the right" and "from the left", respectively.) We also note that both S_0 and S_1 have continuous derivatives of all orders for $-\pi \leq x < 0$ and $0 < x \leq \pi$.

Consider now the problem of approximating the piece-wise continuous (smooth) 2π-periodic function $f(x)$, which can be represented at points of continuity of f by its Fourier series, i.e.,

$$f(x) = \lim_{N \to \infty} F^{(N)}(x), \quad F^{(N)}(x) \equiv \frac{a_0}{2} + \sum_{j=1}^{N} a_j \cos(jx) + b_j \sin(jx),$$

$$\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \begin{pmatrix} \cos(jx) \\ \sin(jx) \end{pmatrix} dx, \ j = 0, 1, \ldots . \tag{4}$$

We shall assume that $f(x)$ and/or $f'(x)$ may have discontinuities at a finite number of points, say at $x = x_j$, $j = 1, 2, ..., n$, in the interval $(-\pi, \pi]$, and that the one-sided limits of f and f' exist and are finite at each of these points.

To construct a more rapidly converging Fourier series type representation of f, we first define the function

$$R_1(x) \equiv \sum_{j=1}^{n} \{A_{0,j} S_0(x - x_j) + A_{1,j} S_1(x - x_j)\}, \tag{5}$$

where the constants $\{A_{0,j}\}$ and $\{A_{1,j}\}$ are determined so that the "jumps" in $R_1(x)$ and its derivative coincide with the corresponding jumps in $f(x)$. Using the relations (3), this requirement leads to the expressions

$$A_{0,j} = [f(x_j)], \quad A_{1,j} = [f'(x_j)], \quad j = 1, 2, ...n. \tag{6}$$

Then we write

$$f(x) = R_1(x) + f_1(x), \quad \text{where} \quad f_1(x) \equiv f(x) - R_1(x). \tag{7}$$

By the way it has been defined, the function $f_1(x)$ is $C^1[-\pi, \pi]$ (at least) and, hence, its Fourier series will converge at a faster rate than the Fourier series of f. To construct its Fourier series, we first note that S_0 and S_1 can be expanded in a Fourier series as

$$S_0(x) \equiv \frac{1}{2\sqrt{2}} \frac{\sin(x)}{\sqrt{1 - \cos(x)}} = \sum_{j=1}^{\infty} b_{0,j} \sin(jx), \quad b_{0,j} = \frac{4}{\pi} \frac{j}{4j^2 - 1}, \tag{8}$$

and

$$S_1(x) \equiv \frac{1}{\sqrt{2}} \sqrt{1 - \cos(x)} = \frac{a_{1,0}}{2} + \sum_{j=1}^{\infty} a_{1,j} \cos(jx), \quad a_{1,j} = \frac{4}{\pi(1 - 4j^2)}. \tag{9}$$

Then, denoting the Fourier coefficients of f_1 by $\{a_j^{(1)}, b_j^{(1)}\}$, we find, using equations (4)-(9),

$$a_j^{(1)} = a_j - \sum_{s=1}^{n} \{A_{1,s} a_{1,j} \cos(jx_s) - A_{0,s} b_{0,j} \sin(jx_s)\},$$

$$b_j^{(1)} = b_j - \sum_{s=1}^{n} \{A_{1,s} a_{1,j} \sin(jx_s) + A_{0,s} b_{0,j} \cos(jx_s)\}, \quad j = 1, 2, \ldots . \tag{10}$$

As we shall demonstrate explicitly in the next sections, the coefficients $\{a_j^{(1)}, b_j^{(1)}\}$ decay faster (by at least a factor of $1/j^2$) as $j \to \infty$ than the original Fourier coefficients $\{a_j, b_j\}$. Using these definitions, we define the new approximations

$$f^{(1,N)}(x) \equiv R_1(x) + \frac{a_0^{(1)}}{2} + \sum_{j=1}^{N} a_j^{(1)} \cos(jx) + b_j^{(1)} \sin(jx), \; N \geq 0. \tag{11}$$

These functions are the first in a sequence of "better" approximations to f, which we shall illustrate, generalize, and then analyze and discuss in the following sections. (We note that the general idea of "subtracting" from the Fourier series of f the Fourier series of another "simple" function with an appropriate singularity has been considered by other investigators, for example, by Tolstov [13], pp. 144-147. However, this idea does not seem to have been pursued in any systematic or exhaustive manner.)

3. Examples. Before we generalize the family of approximations $\left\{f^{(1,N)}\right\}$ and discuss some of their convergence properties, we illustrate these approximations by two examples.

Example 1: We consider first the function $f(x)$ defined by $f(x) = x + \pi$, for $-\pi \le x \le 0$, $f(x) = x - \pi$, for $0 < x \le \pi$, and $f(x + 2\pi) = f(x)$, for all x. Then $x = 0$ is the only point of discontinuity of $f(x)$ in the interval $-\pi < x \le \pi$. Thus, we set $n = 1$ with $x_1 = 0$ in equation (5) above and use the facts that $[f(0)] = -2\pi$ and $[f'(0)] = 0$ in equations (6) to define

$$R_1(x) = A_{0,1}S_0(x) + A_{1,1}S_1(x) = -\frac{\pi}{\sqrt{2}}\frac{\sin(x)}{\sqrt{1-\cos(x)}}. \tag{12}$$

Thus, $R_1(x)$ has the same jump at $x = 0$ as the function $f(x)$, which is clearly evident in Figure 2, where we have plotted both $f(x)$ and $R_1(x)$. Hence, we now consider the function $f_1(x) \equiv f(x) - R_1(x)$, which is $C^1[-\pi, \pi]$, and is also shown in Figure 2. Using equations (10), we expand $f_1(x)$ in a Fourier series as

$$f_1(x) = \sum_{j=1}^{\infty} b_j^{(1)} \sin(jx), \quad b_j^{(1)} = \frac{2}{j(4j^2-1)}. \tag{13}$$

We note that the coefficients $\{b_j^{(1)}\}$ are $O(1/j^3)$ as $j \to \infty$, while the Fourier coefficients $b_j = -2/j$ of f are only $O(1/j)$ as $j \to \infty$. Then, using equation (13) in equation (11), we define the approximations

$$f^{(1,N)}(x) = R_1(x) + \sum_{j=1}^{N} b_j^{(1)} \sin(jx). \tag{14}$$

In Figure 3 we have plotted both $f(x)$ and $f^{(1,3)}(x)$, where the good agreement is evident.

Example 2: As a somewhat more complicated example, we consider the function f defined by

$$f(x) = \begin{cases} -(x+\pi/3), & -\pi < x \le -\pi/3, \\ 2x, & -\pi/3 < x \le 0, \\ 1, & 0 < x \le \pi/2, \\ (x-\pi/2)^2 + 1/2, & \pi/2 < x \le \pi, \end{cases} \tag{15}$$

and $f(x + 2\pi) = f(x)$ for all x. This function is illustrated in Figure 4, along with the Fourier series partial sums $F^{(N)}(x)$ of $f(x)$, defined by equations (4), for $N = 10$, and the error (difference) $f(x) - F^{(10)}(x)$.

The function f has discontinuities at the $n = 4$ points $x_1 = -\pi/3$, $x_2 = 0$, $x_3 = \pi/2$, and $x_4 = \pi$. Using the definition (15) and the relations (6), we find that the coefficients $\{A_{0,j}\}$ and $\{A_{1,j}\}$ in the definition (5) of $R_1(x)$ are given in Table I.

Table I: Coefficients $A_{k,j}$ for Example 1.

j	$A_{0,j}$	$A_{1,j}$
1	$-2\pi/3$	3
2	1	-2
3	$-1/2$	0
4	$(8\pi - 6 - 3\pi^2)/12$	$-(1+\pi)$

The functions $f(x)$, $R_1(x)$, as well as the difference $f_1(x) \equiv f(x) - R_1(x)$, are shown in Figure 5. Obviously, $f_1(x)$ is much smoother than $f(x)$, and in fact, the three-term Fourier series partial sum of $f_1(x)$ is virtually "identical" to $f_1(x)$, to within plotting accuracy. Thus, the approximation $f^{(1,3)}(x)$, which is shown in Figure 6 along with $f(x)$, is a much better approximation to $f(x)$ than the original Fourier series representation of f, even when many more terms are included in $F^{(N)}(x)$ (cp. Figure 4).

4. Generalizations. The basis functions introduced in the section 2 can be generalized and extended to allow us to construct functions which have appropriate discontinuities at, say, the nth derivative. To this end, we define the functions $S_n(x)$ by

$$S_{2k}(x) \equiv \frac{2^{k-3/2}}{(2k)!} \sin(x) \left(1 - \cos(x)\right)^{k-1/2},$$

$$S_{2k+1}(x) \equiv \frac{2^{k-1/2}}{(2k+1)!} \left(1 - \cos(x)\right)^{k+1/2}, \quad k = 0, 1, 2, \ldots . \tag{16}$$

Then it is straightforward to show that $S_n(x)$ is $C^{n-1}[-\pi, \pi]$, while the jump in its nth derivative at $x = 0$ is 1. More generally, each S_n has continuous derivatives of all orders for $-\pi \le x < 0$ and $0 < x \le \pi$, and, at $x = 0$, satisfies the jump conditions

$$\left[S_n^{(q)}(0)\right] \equiv \left[\left(\frac{d}{dx}\right)^q S_n(x)\right]_{x=0} = 0, \text{ if } q < n; \quad \left[S_n^{(n)}(0)\right] = 1;$$

$$\left[S_1^{(2p+1)}(0)\right] = \left[S_0^{(2p)}(0)\right] = (-1/4)^p, \; p \ge 0;$$

$$\left[S_{2k+1}^{(2p)}(0)\right] = \left[S_{2k}^{(2p+1)}(0)\right] = 0, \; p \ge k \ge 1,$$

$$\begin{aligned} \left[S_{2k+1}^{(2p+1)}(0)\right] &= \left[S_{2k}^{(2p)}(0)\right] \\ &= \frac{(-1)^{p+1}}{k(2k-1)} \sum_{j=k-1}^{p-1} \binom{2p}{2j} (-1)^j \left[S_{2k-2}^{(2j)}(0)\right], \; p \ge k \ge 1. \end{aligned} \tag{17}$$

In particular, specific values for the jumps of $\left[S_{2k}^{(2p)}(0)\right]$ are summarized in Table II for a few small values of k and p.

Table II: The jumps $\left[S_{2k+1}^{(2p+1)}(0)\right] = \left[S_{2k}^{(2p)}(0)\right]$

	$p=0$	1	2	3	4	5	6
$k=0$	1	$-1/4$	$1/16$	$-1/64$	$1/256$	$-1/1024$	$1/4096$
1	0	1	$-5/2$	$91/16$	$-205/16$	$7381/256$	$-33215/512$
2	0	0	1	$-35/4$	$483/8$	$-12485/32$	$631631/256$
3	0	0	0	1	-21	$2541/8$	$-68497/16$
4	0	0	0	0	1	$-165/4$	$18447/16$
5	0	0	0	0	0	1	$-143/2$
6	0	0	0	0	0	0	1

Using the definitions (16), we find that the Fourier series of S_n can be expressed as

$$S_{2k+1}(x) = \frac{a_{2k+1,0}}{2} + \sum_{j=1}^{\infty} a_{2k+1,j}\cos(jx), \quad S_{2k}(x) = \sum_{j=1}^{\infty} b_{2k,j}\sin(jx), \tag{18}$$

$$a_{2k+1,j} = (-1)^{k+1}\frac{4^{k+1}}{\pi}\frac{1}{\prod_{i=0}^{k}\left(4j^2-(2i+1)^2\right)},$$

$$b_{2k,j} = -ja_{2k+1,j}, \quad j = 0,1,2,\ldots, \tag{19}$$

for $k = 0,1,2,\ldots$. (For convenience in certain formulas which follow, we also define $a_{2k,j} = b_{2k+1,j} = 0$, for $k \geq 0$.)

Now let the 2π-periodic function f have possible discontinuities in $f^{(k)}$ for $0 \leq k \leq M$ at $x = x_j$, $j = 1,2,\ldots,n$, where $-\pi < x_j \leq \pi$. (Here we assume that all of the one-sided limits $f^{(k)}(x_j^+)$ and $f^{(k)}(x_j^-)$ exist and are finite.) We define the function

$$R_M(x) \equiv \sum_{k=0}^{M}\sum_{j=1}^{n} A_{k,j}S_k(x-x_j), \tag{20}$$

where the constants $\{A_{k,j}\}$ are determined so that $f_M(x) \equiv f(x) - R_M(x)$ is $C^M[-\pi,\pi]$. Using the relations (17), we find that these constants can be determined recursively from the relations

$$A_{k,j} = \left[f^{(k)}(x_j)\right] - \sum_{i=0}^{k-1} A_{i,j}\left[S_i^{(k)}(0)\right], \quad j = 1,2,\ldots n, \quad k = 0,1,\ldots,M. \tag{21}$$

Thus, the function $f_M(x)$ will be $C^M[-\pi,\pi]$, at least, and hence its Fourier series will converge at a faster rate than the Fourier series of f.

Once $R_M(x)$ has been defined, we define the family of approximations $f^{(M,N)}$ to f by

$$f^{(M,N)}(x) \equiv R_M(x) + \frac{a_0^{(M)}}{2} + \sum_{j=1}^{N} a_j^{(M)} \cos(jx) + b_j^{(M)} \sin(jx),$$

$$a_j^{(M)} = a_j - \sum_{k=0}^{M} \sum_{s=1}^{n} A_{k,s} \left\{ a_{k,j} \cos(jx_s) - b_{k,j} \sin(jx_s) \right\},$$

$$b_j^{(M)} = b_j - \sum_{k=0}^{M} \sum_{s=1}^{n} A_{k,s} \left\{ a_{k,j} \sin(jx_s) + b_{k,j} \cos(jx_s) \right\}, \quad j = 0, 1, 2, \ldots . \tag{22}$$

These approximations have several nice convergence properties, which we shall discuss in the next section.

Returning to Example 1, since the function $f_1(x)$ (see equation (7)) is $C^1[-\pi, \pi]$ and satisfies the jump conditions $[f_1''(0)] = -\pi/2$ and $[f_1'''(0)] = 0$, we use the definition of $R_M(x)$ in equation (20) with $M = 3$ to define

$$R_3(x) \equiv -2\pi S_0(x) - \frac{\pi}{2} S_2(x)$$

$$= -\frac{\pi}{\sqrt{2}} \frac{\sin(x)}{\sqrt{1 - \cos(x)}} - \frac{\pi}{4\sqrt{2}} \sin(x) \sqrt{1 - \cos(x)}. \tag{23}$$

Then $f_3(x) \equiv f(x) - R_3(x)$ is $C^3[-\pi, \pi]$, and the Fourier series coefficients in the definition (22) of $f^{(3,N)}(x)$ are given by

$$a_j^{(3)} = 0, \ j \geq 0; \quad b_j^{(3)} = -\frac{18}{j(4j^2 - 1)(4j^2 - 9)}, \ j \geq 1. \tag{24}$$

We note that $b_j^{(3)} = O(1/j^5)$ as $j \to \infty$, while the original Fourier coefficients of f are only $O(1/j)$ as $j \to \infty$.

5. Convergence properties. We now investigate some of the convergence properties of the function $\left\{ f^{(M,N)} \right\}$ defined in the previous section, as $M, N \to \infty$. For this purpose, we first define the errors

$$E^{(M,N)}(x) \equiv f(x) - f^{(M,N)}(x)$$

$$= \sum_{j=N+1}^{\infty} a_j^{(M)} \cos(jx) + b_j^{(M)} \sin(jx), \quad M \geq 0, \ N \geq 0, \tag{25}$$

where the coefficients $\{a_j^{(M)}, b_j^{(M)}\}$ are defined in equations (22). Since the function $f_M(x) \equiv f(x) - R_M(x)$ is $C^M[-\pi, \pi]$, it follows that both $a_j^{(M)}$ and $b_j^{(M)}$ are $O(j^{-(M+2)})$,

as $j \to \infty$ (see, e.g., [1]). Thus, there exists a constant $K = K(M, N)$, independent of j, such that

$$|a_j^{(M)}| \leq \frac{K}{j^{M+2}}, \quad |b_j^{(M)}| \leq \frac{K}{j^{M+2}}, \quad \text{for all } j \geq N+1. \tag{26}$$

Using the bounds (26), we find from equation (25) that

$$\left|E^{(M,N)}(x)\right| \leq 2K \sum_{j=N+1}^{\infty} \frac{1}{j^{M+2}} < \frac{2K}{M+1} \frac{1}{N^{M+1}}, \tag{27}$$

which follows easily from the integral comparison test.

Suppose now that K satisfies a bound of the form

$$K \leq \tilde{K}\, \theta^M M^p\, M!, \quad \text{as } M, N \to \infty, \tag{28}$$

where $\tilde{K}$ and θ are constants independent of M and N. Then, if we let N be proportional to M, say, $N = \lambda M$, where $\lambda > \theta/e$, it follows that the errors $E^{(M,N)}(x)$ decay exponentially to zero as $M \to \infty$. To see this, we substitute (28) into (27) and use Stirling's approximation to the factorial to write

$$\begin{aligned} \left|E^{(M,N)}(x)\right| &< \frac{2K}{M+1} \frac{1}{N^{M+1}} \leq 2\tilde{K} \frac{M^p \theta^M}{M+1} \frac{\sqrt{2\pi M}\,(M/e)^M}{(\lambda M)^{M+1}} \\ &\leq \hat{K} M^{p-3/2} \left(\frac{\theta}{\lambda e}\right)^M, \end{aligned} \tag{29}$$

where $\hat{K}$ is another constant, independent of M and N, and $(\theta/\lambda e) < 1$. Thus, we have proved the following theorem.

THEOREM 1. *Let $f(x)$ be a piece-wise smooth, 2π-periodic function with possible discontinuities in f and/or some of its derivatives at a finite number of points in the interval $-\pi < x \leq \pi$. Suppose that the Fourier coefficients $\{a_j^{(M)}, b_j^{(M)}\}$ defined by equations (21)-(22) satisfy the inequalities (26), where K satisfies the bound (28). Then the approximations $\{f^{(M,\lambda M)}\}$ defined by equations (20)-(22) converge exponentially to $f(x)$ in the L_∞-norm as $M \to \infty$, if λ is chosen so that $\lambda > \theta/e$, i.e.*

$$\max_{-\pi \leq x \leq \pi} |f(x) - f^{(M,\lambda M)}(x)| \leq \hat{K} M^{p-3/2} \left(\frac{\theta}{\lambda e}\right)^M,$$

where $\hat{K}$ is a constant independent of M.

To illustrate this theorem, the coefficients $\{b_j^{(M)}\}$ for *Example 1* for $M = 2m+1$ are easily shown to be given by

$$b_j^{(2m+1)} = 2(-1)^m \frac{(1 \cdot 3 \cdot 5 \cdots (2m+1))^2}{j \prod_{i=0}^{m} (4j^2 - (2i+1)^2)}, \quad m = 0, 1, 2, \ldots . \tag{30}$$

Noting that $1 \cdot 3 \cdot 5 \cdots (2m+1) = (2m+1)!/(2^m m!)$, we use Stirling's approximation to the factorial in equation (30) to write

$$\left|b_j^{(2m+1)}\right| \leq \tilde{K}\frac{\sqrt{M}M!}{j^{M+2}}\tilde{\theta}^M, \quad \tilde{\theta}^M \equiv \frac{1}{2^M \prod_{i=0}^{m}(1-((2i+1)/2j)^2)}, \tag{31}$$

where $\tilde{K}$ is a constant independent of j and M. In the Appendix we show that there is another positive constant, $\hat{K}$, such that

$$\tilde{\theta}^M \leq \hat{K}\theta_0^M, \quad \theta_0 = \theta_0(\lambda) \equiv 1/\left(2\sqrt{1-1/(4\lambda^2)}\right), \tag{32}$$

for all $\lambda \geq \tilde{\lambda}$, where $\tilde{\lambda}$ is any arbitrary number greater than $1/2$, and for all $j \geq N+1 = \lambda M + 1$. Thus we see that $b_j^{(2m+1)}$ satisfies the bound (26), where K has the form of the right side of equation (28) with $p = 1/2$ and $\theta = \theta_0(\lambda)$. (See the Appendix, also, for a derivation a sharper bound for $\tilde{\theta}$.) Then the condition that $\lambda > \theta_0/e$ leads to the requirement that $\lambda > \lambda_0 \equiv \sqrt{1+e^2}/2e \doteq 0.533$. In Figure 7 we have plotted the normalized errors $M(\lambda e/\theta_0)^M E^{(M,N)}(x)$ as a function of x, with $\lambda = 1$ (and hence $N = M$) for $M = 5$, 9, and 13. We note that the error, when regarded as a function of x, is much more uniformly distributed over the interval $[-\pi, \pi]$ than is the corresponding error for the original Fourier series, which is highly concentrated around $x = 0$. In Figure 8 we have plotted the normalized L_∞−errors $M(\lambda e/\theta_0)^M \max|E^{(M,N)}(x)|$, with $N = \lambda M$ as a function of $1/M$ for several choices of $\lambda > \lambda_0$. The figure clearly illustrates that these quantities are bounded as $1/M \to 0$ and, hence, that the errors $E^{(M,N)}(x)$ decay uniformly and exponentially to zero as M increases.

To investigate the convergence properties of the approximations $\{f^{(M,N)}\}$ for a general function f, we first consider the following lemma.

LEMMA 1: *Let $f(x)$ be a 2π−periodic function, n a positive integer, and the points $x_0 = -\pi < x_1 < \ldots \leq x_n = \pi$ be defined such that $f(x) = p_k(x)$ for $x_{k-1} < x < x_k$, $k = 1, 2, \ldots, n$, where p_k is a polynomial of degree at most m. Then the Fourier coefficients of f defined by equations (4) can be expressed as*

$$a_j = \frac{1}{\pi}\sum_{s=1}^{n}\left\{\sin(jx_s)\left(\sum_{k=0}^{[m/2]}\frac{(-1)^{k+1}}{j^{2k+1}}\left[f^{(2k)}(x_s)\right]\right)\right.$$

$$\left. + \cos(jx_s)\left(\sum_{k=0}^{[(m-1)/2]}\frac{(-1)^{k+1}}{j^{2k+2}}\left[f^{(2k+1)}(x_s)\right]\right)\right\},$$

$$b_j = \frac{1}{\pi}\sum_{s=1}^{n}\left\{\cos(jx_s)\left(\sum_{k=0}^{[m/2]}\frac{(-1)^{k}}{j^{2k+1}}\left[f^{(2k)}(x_s)\right]\right)\right.$$

$$\left. + \sin(jx_s)\left(\sum_{k=0}^{[(m-1)/2]}\frac{(-1)^{k+1}}{j^{2k+2}}\left[f^{(2k+1)}(x_s)\right]\right)\right\},$$

where, in the upper limits of the inner summations, we have let $[q]$ *denote the greatest integer not exceeding* q.

The proof of this lemma follows easily by using the formulas (4) and integrating each of these expressions by parts m times (see, e.g., [10], pp. 489-493, for more details).

Inserting these expressions into equations (4) and rearranging the resulting series, we can express the Fourier series of f as

$$\frac{1}{2}\left(f(x^+)+f(x^-)\right)=\frac{a_0}{2}+\sum_{s=1}^{n}\sum_{k=0}^{m}\left[f^{(k)}(x_s)\right]h_k(x-x_s),$$

$$h_{2k}(x)=\frac{(-1)^k}{\pi}\sum_{j=1}^{\infty}\frac{\sin(jx)}{j^{2k+1}},\quad h_{2k+1}(x)=\frac{(-1)^{k+1}}{\pi}\sum_{j=1}^{\infty}\frac{\cos(jx)}{j^{2k+2}}, \tag{33}$$

for $k=0,1,\ldots$. Hence, to establish the exponential convergence of our approximations for a "piece-wise polynomial" function f, as defined in LEMMA 1, it is sufficient to establish that our approximations converge exponentially to each of the functions $h_k(x)$ defined in equations (33). In fact, we shall show that

$$\left|h_i(x)-h_i^{(M,\lambda M)}(x)\right|\le\frac{K_0}{M}\frac{|x|^i}{i!}\left(\frac{\theta_0}{e\lambda}\right)^M,\quad i=0,1,\ldots, \tag{34}$$

and hence

$$\max_{-\pi\le x\le\pi}\left|h_i(x)-h_i^{(M,\lambda M)}(x)\right|\le\frac{K_0}{M}\frac{\pi^i}{i!}\left(\frac{\theta_0}{e\lambda}\right)^M,\quad i=0,1,\ldots, \tag{35}$$

where K_0 is a constant, independent of M, $\theta_0\equiv 1/\left(2\sqrt{1-1/(4\lambda^2)}\right)$, with $\lambda\equiv N/M$, and

$$h_{2p}^{(M,N)}(x)\equiv\sum_{\substack{k=2p\\ k\ even}}^{M}A_k^{(2p)}S_k(x)+\sum_{j=1}^{N}b_j^{(2p,M)}\sin(jx), \tag{36}$$

$$h_{2p+1}^{(M,N)}(x)\equiv\sum_{\substack{k=2p+1\\ k\ odd}}^{M}A_k^{(2p+1)}S_k(x)+\frac{a_0^{(2p+1,M)}}{2}+\sum_{j=1}^{N}a_j^{(2p+1,M)}\cos(jx), \tag{37}$$

for $p=0,1,\ldots$. Here the constants $A_k^{(2p)}$ and $A_k^{(2p+1)}$, as well as the coefficients $\{b_j^{(2p,M)}\}$ and $\{a_j^{(2p+1,M)}\}$, are determined as described in section 4.

To show that equations (34)-(37) hold, we shall use induction. The function $h_0=(-1/2\pi)f(x)$, where $f(x)$ is the function of *Example 1.* Hence, using equation (33) with $k=0$ and equation (36) with $p=0$, we can write

$$h_0(x)-h_0^{(M,N)}(x)=-\frac{1}{2\pi}\sum_{j=N+1}^{\infty}b_j^{(M)}\sin(jx), \tag{38}$$

where $b_j^{(M)}$, with $M = 2m+1$, is given by the right side of equation (30). Using the bounds (26) and (28), with $\theta = \theta_0$, $p = 1/2$, and $N = \lambda M$, as well as the remarks below equation (30) and Stirling's approximation to the factorial, we can write

$$\left|h_0(x) - h_0^{(M,N)}(x)\right| \leq \frac{K_0}{M}\left(\frac{\theta_0}{\lambda e}\right)^M,$$

where K_0 is another constant, independent of M. Hence, equation (34) holds with $i = 0$ and the exponential convergence of our approximations to h_0 is established.

Suppose now that equations (35)-(37) hold for $i = 0, 1, ..., q-1$. To investigate the behavior of $h_q^{(M,N)}$, we observe first that

$$h'_{i+1}(x) = h_i(x) \quad \text{and} \quad S'_{2i+1}(x) = S_{2i}(x), \tag{39}$$

for all nonnegative integers i. (Here the primes denote differentiation with respect to x.) Then, using the first relation in (39), with $i = q-1$, we can write

$$h_q(x) = h_q(0) + \int_0^x h'_q(t)dt = h_q(0) + \int_0^x h_{q-1}(t)dt$$

$$= h_q^{(M,N)}(x) + E_q^{(M,N)}(x), \tag{40}$$

where

$$h_q^{(M,N)}(x) \equiv h_q(0) + \int_0^x h_{q-1}^{(M,N)}(t)dt,$$

$$E_q^{(M,N)}(x) \equiv \int_0^x \left\{h_{q-1}(t) - h_{q-1}^{(M,N)}(t)\right\} dt. \tag{41}$$

Using our induction hypothesis and equation (34) with $i = q-1$, we find from equations (40) and (41) that

$$\left|h_q(x) - h_q^{(M,\lambda M)}(x)\right| = \left|E_q^{(M,\lambda M)}(x)\right| \leq \int_0^{|x|} \left|h_{q-1}(t) - h_{q-1}^{(M,\lambda M)}(t)\right| dt$$

$$\leq \int_0^{|x|} \frac{K_0}{M}\frac{t^{q-1}}{(q-1)!}\left(\frac{\theta_0}{e\lambda}\right)^M dt \leq \frac{K_0}{M}\frac{|x|^q}{q!}\left(\frac{\theta_0}{e\lambda}\right)^M,$$

and hence

$$\max_{-\pi\leq x\leq\pi}\left|h_q(x) - h_q^{(M,\lambda M)}(x)\right| \leq \frac{K_0}{M}\frac{\pi^q}{q!}\left(\frac{\theta_0}{e\lambda}\right)^M. \tag{42}$$

Thus, it just remains to be shown that $h_q^{(M,N)}(x)$, defined in equations (41), has the form indicated in equations (36) and (37). To see that this is the case, if q is *odd*, we

use equation (41), equation (36) with $2p = q - 1$, and the last relation in equation (39) to write

$$h_q^{(M,N)}(x) = h_q(0) + \int_0^x \left\{ \sum_{\substack{k=q-1 \\ k \text{ even}}}^{M} A_k^{(q-1)} S_k(t) + \sum_{j=1}^{N} b_j^{(q-1,M)} \sin(jt) \right\} dt$$

$$= h_q(0) + \sum_{j=1}^{N} (1/j) b_j^{(q-1,M)} + \sum_{\substack{k=q-1 \\ k \text{ even}}}^{M} A_k^{(q-1)} S_{k+1}(x) + \sum_{j=1}^{N} (-1/j) b_j^{(q-1,M)} \cos(jx),$$

which has the form of equation (37) with $2p + 1 = q$, where

$$a_0^{(q,M)} = 2 \left\{ h_q(0) + \sum_{j=1}^{N} (1/j) b_j^{(q-1,M)} \right\},$$

$$A_k^{(q)} = A_{k-1}^{(q-1)}, \quad a_j^{(q,M)} = (-1/j) b_j^{(q-1,M)}. \tag{43}$$

If q is *even*, we use equation (41), equation (37) with $2p + 1 = q - 1$, and the fact that $h_q(0) = 0$, to write

$$h_q^{(M,N)}(x) = \int_0^x \left\{ \sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} S_k(t) + \frac{a_0^{(q-1,M)}}{2} + \sum_{j=1}^{N} a_j^{(q-1,M)} \cos(jt) \right\} dt$$

$$= \tilde{h}_q(x) + \sum_{j=1}^{N} (1/j) a_j^{(q-1,M)} \sin(jx), \tag{44}$$

$$\tilde{h}_q(x) \equiv \sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} \int_0^x S_k(t) dt + \frac{a_0^{(q-1,M)} x}{2}. \tag{45}$$

We now make some observations concerning the function $\tilde{h}_q$. First, we note that $\tilde{h}_q$ is an *odd* function of x, with $\tilde{h}_q(\pi) = \tilde{h}_q(-\pi) = 0$. The fact that $\tilde{h}_q$ is odd follows directly from equation (45) and the fact that each S_k is an even function of x whenever the index k is odd. To see that $\tilde{h}_q$ vanishes at $x = \pi$, we note first that, by letting $N \to \infty$ in equation (37) with $2p + 1 = q - 1$, we can write

$$h_{q-1}(x) \equiv \sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} S_k(x) + \frac{a_0^{(q-1,M)}}{2} + \sum_{j=1}^{\infty} a_j^{(q-1,M)} \cos(jx).$$

Integrating this expression from $x = 0$ to $x = \pi$ and using the definition of h_{q-1} (equation (33)), we find

$$\frac{\pi a_0^{(q-1,M)}}{2} = -\sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} \int_0^{\pi} S_k(x)dx. \tag{46}$$

Setting $x = \pi$ in equation (45) and then using equation (46), we find that $\tilde{h}_{q-1}(\pi) = 0$. Using this first observation, we note next that $\tilde{h}_{q-1}$ has $q-1$ continuous derivatives for $-\pi \leq x \leq \pi$, and that the only discontinuities in higher (even) order derivatives occur at $x = 0$. Thus, using the ideas of section 4, we can write

$$\tilde{h}_{q-1}(x) = \sum_{\substack{k=q \\ k \text{ even}}}^{M} \tilde{A}_k S_k(x) + \sum_{j=1}^{\infty} b_j^{(M)} \sin(jx), \tag{47}$$

where the constants $\tilde{A}_k$ are determined recursively by

$$\tilde{A}_j = \sum_{\substack{k=q-1 \\ k \text{ odd}}}^{j-1} A_k^{(q-1)} \left[S_k^{(j-1)}(0)\right] - \sum_{\substack{k=q \\ k \text{ even}}}^{j-2} \tilde{A}_k \left[S_k^{(j)}(0)\right], \quad q \leq j \leq M, \ j \text{ even},$$

and the coefficients $b_j^{(M)}$ are determined by the right side of the last equation in (22), with $x_s = 0$, $A_{k,s} = \tilde{A}_k$, and

$$b_j = \frac{(-1)^{j+1}}{j} \left(\sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} a_{k,0} + a_0^{(q-1,M)} \right) + \sum_{\substack{k=q-1 \\ k \text{ odd}}}^{M} A_k^{(q-1)} \frac{a_{k,j}}{j}.$$

Thus, when the right side of equation (47), with the upper limit of the second summation replaced by N, is substituted for $\tilde{h}_{q-1}(x)$ in equation (44), we see that $h_q^{(M,N)}$ has exactly the form of the right side of equation (36) with $2p = q$.

We now summarize these results in the following theorem.

THEOREM 2. *Let f be a 2π-periodic, "piece-wise polynomial" function, as defined in* LEMMA 1. *Then the sequence of approximations $f^{(M,\lambda M)}$ defined in section 4 converge exponentially to f, with*

$$\max_{-\pi \leq x \leq \pi} \left| f(x) - f^{(M,\lambda M)}(x) \right| \leq \frac{\tilde{K}}{M} \left(\frac{\theta_0}{e\lambda} \right)^M,$$

where

$$f^{(M,N)}(x) \equiv \frac{a_0}{2} + \sum_{s=1}^{n} \sum_{k=0}^{m} \left[f^{(k)}(x_s) \right] h_k^{(M,N)}(x - x_s)$$

and $\tilde{K}$ is a constant (which depends on the particular function f, but is independent of M) such that

$$K_0 \sum_{s=1}^{n} \sum_{k=0}^{m} \left| \left[f^{(k)}(x_s) \right] \right| \frac{\pi^k}{k!} \leq \tilde{K},$$

where K_0 is the constant which appears in the bounds (35).

Finally, we consider the case when $f(x)$ is a $2\pi-$periodic, piece-wise *analytic* function. We let the n points $x_0 = -\pi < x_1 < ... \leq x_n = \pi$ be defined such that $f(x) = f_i(x)$ for $x_{i-1} < x < x_i$, $i = 1, 2, ..., n$, where each $f_i(x)$ is analytic on the *closed* interval $x_{i-1} \leq x \leq x_i$, and such that there exists a point α_i, with $x_{i-1} \leq \alpha_i \leq x_i$, such that $(x_i - x_{i-1}) < \rho_i$, where ρ_i is the distance between α_i and the nearest singularity of f_i in the real or complex plane. Then, by Taylor's theorem, we can write

$$f_i(x) = P_{i,m}(x) + R_{i,m}(x),$$

$$P_{i,m}(x) \equiv \sum_{j=0}^{m} \frac{f_i^{(j)}(\alpha_i)}{j!}(x - \alpha_i)^j, \quad R_{i,m}(x) \equiv \sum_{j=m+1}^{\infty} \frac{f_i^{(j)}(\alpha_i)}{j!}(x - \alpha_i)^j. \tag{48}$$

We now assume (see, e.g., [9]) that

$$\left|f_i^{(j)}(\alpha_i)\right| \leq C_i \frac{j!}{\rho_i^j}, \quad 1 \leq i \leq n, \tag{49}$$

where C_i is a constant, independent of j and ρ_i. Using equations (48) and (49), we can write

$$\max_{x_{i-1} \leq x \leq x_i} |f_i(x) - P_{i,m}(x)| = \max_{x_{i-1} \leq x \leq x_i} |R_{i,m}(x)| \leq \tilde{C}_i \left(\frac{x_i - x_{i-1}}{\rho_i}\right)^{m+1}, \tag{50}$$

where $\tilde{C}_i = C_i \rho_i / (\rho_i - x_i + \alpha_i)$ and $(x_i - x_{i-1})/\rho_i < 1$, from our definition of the points $\{x_i\}$. We now define

$$\tilde{C} = \max_{1 \leq i \leq n} \tilde{C}_i, \quad \eta = \max_{1 \leq i \leq n} \left(\frac{x_i - x_{i-1}}{\rho_i}\right) < 1, \tag{51}$$

so that from equations (50) and (51) we can write

$$\max_{x_{i-1} \leq x \leq x_i} |f_i(x) - P_{i,m}(x)| \leq \tilde{C}\eta^{m+1}. \tag{52}$$

We now define $g(x)$ as the 2π-periodic, piece-wise polynomial function with $g(x) = P_{i,m}(x)$ for $x_{i-1} < x \leq x_i$, $i = 1, 2, ..., n$, and also define the approximation

$$g^{(M,N)}(x) \equiv \frac{a_0}{2} + \sum_{s=1}^{n} \sum_{k=0}^{m} \left[g^{(k)}(x_s)\right] h_k^{(M,N)}(x - x_s)$$

where $h_k^{(M,N)}(x)$ is defined in equations (36)-(37). Using Theorem 2 and equations (33) and (35), we see that

$$\left|g(x) - g^{(M,\lambda M)}(x)\right| \leq \sum_{s=1}^{n} \sum_{k=0}^{m} \left|\left[g^{(k)}(x_s)\right] \left(h_k(x - x_s) - h_k^{(M,\lambda M)}(x - x_s)\right)\right|$$

$$\leq \sum_{s=1}^{n}\sum_{k=0}^{m}\left|\left[g^{(k)}(x_s)\right]\right|\frac{K_0}{M}\frac{\pi^k}{k!}\left(\frac{\theta_0}{e\lambda}\right)^M. \tag{53}$$

We now assume that the *jumps* in $f^{(k)}$ satisfy a bound of the form

$$\left|\left[f^{(k)}(x_s)\right]\right| \leq A\xi^k k!, \quad s = 1, 2, ..., n, \tag{54}$$

where A and ξ are positive constants, independent of k and s. (Here we may assume, without loss of generality, that $\xi\pi > 1$.) Since the jumps in $g^{(k)}$ are arbitrarily close to the corresponding jumps in $f^{(k)}$, we assume that equation (54) also holds for the jumps in $g^{(k)}$, and then use this bound in equation (53) to write

$$\max_{\pi\leq x\leq\pi}\left|g(x) - g^{(M,\lambda M)}(x)\right| \leq nA\frac{K_0}{M}\left(\frac{(\pi\xi)^{m+1}-1}{\pi\xi-1}\right)\left(\frac{\theta_0}{e\lambda}\right)^M. \tag{55}$$

We now combine the bounds (52) and (55) to write

$$\max_{-\pi\leq x\leq\pi}\left|f(x) - g^{(M,\lambda M)}(x)\right| \leq \max_{-\pi\leq x\leq\pi}|f(x) - g(x)| + \max_{-\pi\leq x\leq\pi}\left|g(x) - g^{(M,\lambda M)}(x)\right|$$

$$\leq \tilde{C}\eta^{m+1} + nA\frac{K_0}{M}\left(\frac{(\pi\xi)^{m+1}-1}{\pi\xi-1}\right)\left(\frac{\theta_0}{e\lambda}\right)^M. \tag{56}$$

We now set $m = M$ in equation (56) and note that the second term in equation (56) is dominated, for large values of M, by a constant times ζ^M, where $\zeta = \pi\xi\theta_0/e\lambda$. Since we want the magnitude of this quantity to be less than 1, we must require that $\lambda > \sqrt{e^2 + 4\pi^2\xi^2}/2e$. Then, with this stipulation, we find from equation (56) that

$$\max_{-\pi\leq x\leq\pi}\left|f(x) - g^{(M,\lambda M)}(x)\right| \leq K_1\eta^M + \frac{K_2}{M}\zeta^M, \qquad \text{as } M \to \infty, \tag{57}$$

where K_1 and K_2 are constants, independent of M, while $0 < \eta < 1$ and $0 < \zeta = \pi\xi\theta_0/e\lambda < 1$. Thus, we have proved the following theorem.

THEOREM 3. *Let f be a 2π-periodic, "piece-wise analytic" function, as defined above. Then, for $\lambda > \sqrt{e^2 + 4\pi^2\xi^2}/2e$, the sequence of approximations $f^{(M,\lambda M)}$ defined in section 4 converge exponentially to f, i.e.,*

$$\max_{-\pi\leq x\leq\pi}\left|f(x) - f^{(M,\lambda M)}(x)\right| \leq K_1\eta^M + \frac{K_2}{M}\zeta^M, \quad \textit{where} \quad \zeta = \pi\xi\theta_0/e\lambda < 1.$$

Here K_1 and K_2 are constants, independent of M, η is related to the lengths of the intervals $[x_{i-1}, x_i]$ and the singularities of f_i (see equation (51)), ξ is related to the jumps in the derivatives of f (see equation (54)), and $\theta_0 = 1/\left(2\sqrt{1 - 1/(4\lambda^2)}\right)$.

Before we illustrate the results of this theorem, we make two observations. First, if the jumps $\left[f^{(k)}(x_s)\right]$ satisfy a bound of the form of (54) *and* $\xi\pi < 1$, then it follows that THEOREM 3 holds with $\zeta = \theta_0/e\lambda$. (This conclusion follows from the estimates derived

above and the fact that $(\pi\xi)^{M+1} \to 0$, as $M \to \infty$.) Secondly, we observe that if the jumps in $f^{(k)}$ satisfy a bound of the form $\left|\left[f^{(k)}(x_s)\right]\right| \leq A\xi^k$, i.e., of the form of equation (54), but without the factor $k!$ present, then, again, THEOREM 3 holds with $\zeta = \theta_0/e\lambda$. (This follows from the fact that, for this case, the "$k-$summation" in equation (53) is bounded by a constant times $\exp(\pi\xi)$, as $m = M \to \infty$.)

We now summarize these observations in the following corollary.

COROLLARY 4. *If the jumps* $\left[f^{(k)}(x_s)\right]$ *satisfy a bound of the form of (54) and* $\xi\pi < 1$, *or if they satisfy a bound of the form* $\left|\left[f^{(k)}(x_s)\right]\right| \leq A\xi^k$, $s = 1, 2, ...n$, *where* A *and* ξ *are positive constants, independent of* k *and* s, *then the results of* THEOREM 3 *hold with* $\zeta = \theta_0/e\lambda$. *In either of these cases,* ζ *will be strictly less than one if* $\lambda > (1/2e)\sqrt{1+e^2} \doteq 0.533$.

6. More examples. In this section we consider three examples which serve to illustrate some of the convergence properties discussed in the previous section.

Example 3: Let ω be a *non-integer* constant and define $f(x) = \cos(\omega(x-\pi))$, for $-\pi < x \leq \pi$, and $f(x+2\pi) = f(x)$ for all x. We note that, since the cosine function is analytic in the entire finite part of the complex plane, the only singularities in f arise from the fact that $\cos(\omega(x-\pi))$ is *not* $2\pi-$*periodic*. In this case, we set $n = 1$, with $x_0 = -\pi$ and $x_1 = \pi$. Then the quantity η in equation (52) can be made arbitrarily small and, hence, the convergence of our approximations is dominated by the second term on the right side of equation (57). Also, for $k = 0, 1, 2, ...$, $f^{(k)}$ has only one singularity in the interval $(-\pi, \pi]$ and this singularity lies at $x = \pi$, with the corresponding jumps given by

$$\left[f^{(k)}(\pi)\right] = \begin{cases} (-1)^{1+k/2}\omega^k\left(1-\cos(2\omega\pi)\right), & \text{if } k \text{ is } \textit{even}, \\ (-1)^{(k-1)/2}\omega^k \sin(2\omega\pi), & \text{if } k \text{ is } \textit{odd}. \end{cases} \tag{58}$$

Thus, the jumps in $f^{(k)}$ satisfy a bound of the form specified in equation (54) with $\xi = \omega$, but without the factor of $k!$. Thus, if follows from COROLLARY 4 that our approximations converge exponentially to f at a rate which is bounded by $\zeta = \theta_0/e\lambda$.

The Fourier coefficients of f are given by

$$a_j = \frac{(-1)^{j+1}\omega\sin(2\omega\pi)}{\pi(j^2-\omega^2)}, \quad b_j = \frac{(-1)^{j+1}j(1-\cos(2\omega\pi))}{\pi(j^2-\omega^2)} \tag{59}$$

while the coefficients $a_j^{(M)}$ and $b_j^{(M)}$ are defined by

$$a_j^{(0)} = a_j, \quad a_j^{(2k+1)} = a_j^{(2k+2)} = \frac{(-1)^{j+1}\omega\sin(2\omega\pi)}{\pi(j^2-\omega^2)}\prod_{i=0}^{k}\left(\frac{4\omega^2-(2i+1)^2}{4j^2-(2i+1)^2}\right),$$

$$b_j^{(2k)} = b_j^{(2k+1)} = \frac{(-1)^{j+1}j(1-\cos(2\omega\pi))}{\pi(j^2-\omega^2)}\prod_{i=0}^{k}\left(\frac{4\omega^2-(2i+1)^2}{4j^2-(2i+1)^2}\right), \tag{60}$$

for $k = 0, 1, 2, \ldots$. Following the same arguments given in section 5, we find for this example,

$$\left|E^{(M,\lambda M)}(x)\right| \leq \frac{\tilde{K}}{M}\left(\frac{\theta_0}{\lambda e}\right)^M = \frac{\tilde{K}}{M}\left(\frac{1}{2e\sqrt{\lambda^2 - 1/4}}\right)^M .$$

In Figure 9 we have plotted the normalized errors $M\left(2e\sqrt{\lambda^2 - 1/4}\right)^M E^{(M,\lambda M)}(x)$ for $\lambda = 1$ and several values of M, for the representative case when $\omega = 7/5$. In Figure 10 we have plotted the normalized L_∞−errors $M\left(2e\sqrt{\lambda^2 - 1/4}\right)^M \max\left|E^{(M,\lambda M)}(x)\right|$ as a function of M for several values of λ, also for $\omega = 7/5$. We shall discuss these results in section 7.

Example 4: We let ϵ be a positive constant and define $f(x) = 1/(x^2 + \epsilon^2)$, for $-\pi \leq x \leq \pi$, and $f(x + 2\pi) = f(x)$, for all x. For this example, f has singularities in the complex plane at $x = \pm i\epsilon$, which lie "close to" the interval $[-\pi, \pi]$, if ϵ is small. Following the remarks before Theorem 3, for this example we set $n = 2$ and define $x_0 = -\pi$, $x_1 = 0$, and $x_2 = \pi$. Then we can select $\alpha_1 = -\pi$ and $\alpha_2 = \pi$ and find

$$\eta = \frac{\pi}{\sqrt{\pi^2 + \epsilon^2}}. \tag{61}$$

In addition, each $f^{(k)}(x)$ has a singularity in the interval $(-\pi, \pi]$, which lies at $x = x_2 = \pi$, with the corresponding jumps given by

$$\begin{aligned} \left[f^{(k)}(\pi)\right] &= \frac{2k!}{(\pi^2 + \epsilon^2)^{k+1}} \sum_{j=0}^{(k-1)/2} (-1)^j \binom{k+1}{2j+1} \pi^{k-2j}\epsilon^{2j} \\ &= \frac{2k!}{\epsilon(\sqrt{\pi^2 + \epsilon^2})^{k+1}} \sin\left((k+1)\tan^{-1}(\epsilon/\pi)\right), \quad k \text{ odd}, \end{aligned} \tag{62}$$

and $\left[f^{(k)}(\pi)\right] = 0$, if k is even. Using the fact that $|\sin(z)| \leq 1$ for all real values of z, it follows from equation (66) that

$$\left|\left[f^{(k)}(\pi)\right]\right| \leq \frac{2}{\epsilon\sqrt{\pi^2 + \epsilon^2}}\left(\frac{1}{\sqrt{\pi^2 + \epsilon^2}}\right)^k k!, \tag{63}$$

for all values of k. Thus, the jumps in $f^{(k)}$ satisfy a bound of the form specified in equation (54), with $A = 2/(\epsilon\sqrt{\pi^2 + \epsilon^2})$ and $\xi = (\pi^2 + \epsilon^2)^{-1/2}$. Since $\pi\xi < 1$ for any real value of ϵ, it follows from COROLLARY 4 that our approximations again converge exponentially to f at a rate bounded by the right side of equation (57), with η given by equation (61) and $\zeta = \theta_0/e\lambda$. Thus, for "small" values of ϵ, the rate of convergence is dominated by the value of η, while for "large" values of ϵ it is dominated by the value of ζ.

To illustrate the rate of convergence of our approximations for this example, in Figure 11 we have plotted the quantities

$$\zeta_M \equiv \left[\left(\max\left|E^{(M,\lambda M)}(x)\right|\right) / \left(\max\left|E^{(M-2,\lambda(M-2))}(x)\right|\right)\right]^{1/2} \tag{64}$$

as a function of $1/M$ for $\lambda = 1$ and various values of ϵ between 0.1 and 2. As $M \to \infty$, ζ_M approaches the asymptotic rate of convergence of our approximations. For the "small" value of $\epsilon = 0.1$, the asymptotic rate of convergence $\eta = \pi/\sqrt{\pi^2 + \epsilon^2} \doteq 0.999$ is indicated by the short dotted line near $\zeta_M = 1$, while for the "large" value of $\epsilon = 2$, the asymptotic rate $\zeta = \theta_0/e\lambda \doteq 0.212$ is also indicated by a short dotted line. We note that, in both of these "extreme" cases, the figure seems to indicate that the estimate of the rate of convergence predicted by the theory is overly conservative, with the actual rate of convergence being smaller (better) in each case than the theory predicts. We shall discuss this example further in the next section.

Example 5: Finally, let ϵ be a real, positive number and define the function $f(x) = ((\pi + \epsilon)^2 - x^2)^{-1}$, for $-\pi \le x \le \pi$, and $f(x + 2\pi) = f(x)$, for all x. For this example, f has singularities at $x = \pm(\pi + \epsilon)$, which lie "close to" the interval $[-\pi, \pi]$, if ϵ is small. For this example, we set $n = 1$ and define $x_0 = -\pi$ and $x_1 = \pi$, and also set $\alpha_1 = 0$. Then

$$\eta = \frac{\pi}{\pi + \epsilon}. \tag{65}$$

In addition, each $f^{(k)}(x)$ has a singularity in the interval $(-\pi, \pi]$, which lies at $x = x_1 = \pi$, with the corresponding jumps given by

$$\begin{aligned} \left[f^{(k)}(\pi)\right] &= -\frac{2k!}{(\epsilon(2\pi + \epsilon))^{k+1}} \sum_{j=0}^{(k-1)/2} \binom{k+1}{2j+1} \pi^{k-2j} (\pi + \epsilon)^{2j} \\ &= -\frac{k!}{\pi + \epsilon} \left\{ \frac{1}{\epsilon^{k+1}} - \frac{1}{(2\pi + \epsilon)^{k+1}} \right\}, \quad k \text{ odd}, \end{aligned} \tag{66}$$

and $\left[f^{(k)}(\pi)\right] = 0$, if k is even. Using equation (66), we find

$$\left|\left[f^{(k)}(\pi)\right]\right| \le \frac{k!}{\epsilon\pi} \left(\frac{1}{\epsilon}\right)^k, \tag{67}$$

and, hence, the bound (54) holds with $\xi = \epsilon^{-1}$. Thus, for this example, since $\xi\pi > 1$ when $\epsilon < \pi$, THEOREM 3 guarantees exponential convergence only for $\lambda > \sqrt{e^2 + 4\pi^2\xi^2}/2e \to \pi/e\epsilon \doteq 1.1557/\epsilon$, as $\epsilon \to 0$. Alternatively, for a fixed value of λ, the requirement that $\zeta < 1$ will be satisfied only if $\epsilon > \epsilon_0 \equiv \pi/(e\sqrt{\lambda^2 - 1/4})$, with $\epsilon_0 \doteq 1.335$ when $\lambda = 1$.

To illustrate the rate of convergence of our approximations for this example, in Figure 12 we have plotted the quantities ζ_M defined by equation (64) as a function of $1/M$ for $\lambda = 1$ and $\epsilon = 0.25$, 0.375, 0.5, and 1. As $M \to \infty$, ζ_M approaches the asymptotic rate of convergence of our approximations. It is obvious from the plot that the lower bound on ϵ obtained above ($\epsilon_0 \cong 1.335$) is too conservative, with the actual value of ϵ_0 being closer to 0.4 (for $\lambda = 1$). We shall discuss these results further in the next section.

7. Discussion and Conclusions. We now make some observations concerning the convergence properties of the approximations we have constructed, particularly in light of the examples we have considered.

For the function of Example 1, each Fourier coefficient is just a *monomial* in $(1/j)$, i.e., $b_j = -2/j$. In fact, this function is just a constant multiple of the function h_0, which was the basic "building block" for the convergence proofs of section 5. In this case, explicit expressions for the Fourier coefficients $b_j^{(M)}$ (equation (30)) were obtained, where their $O(1/j^{M+2})$ behavior is evident. Since $f^{(k)}(x) \equiv 0$, for $k \geq 2$, the "need" to include the basis functions S_k with $k \geq 2$ in the expressions for R_k is due solely to the jump in the kth derivative of S_l, for $l < k$ (see, e.g., equation (23)). This observation suggests that it might be worthwhile to try to construct a new set of basis functions, say, $\left\{\tilde{S}_k\right\}$, which have the property that $\tilde{S}_k$ has a jump in *only* its kth derivative at $x = 0$. Such functions can be constructed in a straightforward manner, by expressing $\tilde{S}_k$ as an infinite linear combination of the S_l, with $l \geq k$. However, this series representation of $\tilde{S}_k$ fails to converge for all x in the range $-\pi \leq x \leq \pi$, and hence is not useful. Even if these expressions for $\tilde{S}_k$ did converge over the entire interval, it is not clear that they would have any *practical* advantage over the functions $\{S_k\}$ which we have introduced.

The function of Example 2 is a piece-wise polynomial function, as defined in LEMMA 1, with $m = 2$, and has $n = 4$ points of singularity in the interval $-\pi < x \leq \pi$. Consequently, each of its Fourier coefficients is a *polynomial* in $(1/j)$. Even though this function has *multiple* singularities, the approximations we constructed are able to handle them in a straightforward manner.

Each Fourier coefficient of the function of Example 3 can be expressed as an *infinite series* in powers of $(1/j)$. However, the construction of our sequence of exponentially convergent approximations follows exactly the procedure as in the polynomial case. In fact, the only technical difference is the inclusion of the jump in $f^{(k)}$ in the definition of the constants $A_{k,j}$ (see equation (21)). For this example, the function f is an entire function, with no singularities in the finite part of the complex $x-$plane. In particular, it is analytic on $[-\pi, \pi]$, but is *not* $2\pi-$periodic. Consequently, the only "singularities" in the $2\pi-$periodic extension of f occur at $x = \pi$. Because of these properties, the rate of convergence of our approximations for this example is the same as the rate of convergence for the function h_0, i.e., $\zeta = \theta_0/e\lambda$.

Each Fourier coefficient of the function f of Example 4 is again an infinite series in $(1/j)$, although these coefficients must now be computed numerically. As in Example 3, f is analytic at each value of x in the interval $[-\pi, \pi]$, but it is not $2\pi-$periodic. However, for this case, f has (purely imaginary) singularities in the finite part of the complex plane at $x = \pm i\epsilon$. As a consequence, the rate of convergence of the approximations depends upon the value of ϵ, with the rate of convergence approaching 1 as $\epsilon \to 0$. However, as $\epsilon \to \infty$, the rate of convergence is again the same as that for the function h_0. Also, for *any* real, positive value of ϵ, the approximations converge exponentially for any value of $\lambda > \lambda_0 = (1/2e)\sqrt{1+e^2}$.

The function of Example 5 is also analytic on $[-\pi, \pi]$, but, again, is not $2\pi-$periodic. In this case, however, f has *real* singularities at $x = \pm(\pi+\epsilon)$, and the rate of convergence of our approximations again depends upon ϵ. In this case, however, for sufficiently small values of ϵ the value of λ must be greater than a specific lower bound, which depends upon ϵ and which increases as $\epsilon \to 0$, in order to ensure exponential convergence.

In comparing the results of Examples 4 and 5, we note that, in both cases, there are singularities of f which lie a distance ϵ from the interval $[-\pi, \pi]$. In Example 5, when ϵ is "small", these singularities lie *close to* the points where the 2π-periodic extension of f has *real* singularities, namely, at $x = \pm\pi$. In this case, the presence of these "near-by" singularities seems to adversely affect the convergence of our approximations, in the sense that we are forced to require that λ increase as ϵ decreases. By contrast, in Example 4 the singularities lie "far" from the points where the 2π-periodic extension of f has *real* singularities, even when ϵ is small. In this case, although the rate of convergence is affected by the value of ϵ through the parameter η, it is *not* necessary to require that λ increase as ϵ becomes smaller. Thus, the "nearness" of the singularities of f to the points $\{x_j\}$ seems to be of more significance than simply the nearness of the singularities to the interval $[-\pi, \pi]$.

In the formulas above, we have consistently used (because of its simplicity) the simple (loose) bound θ_0 in our estimates of the rates of convergence of our approximations. These estimates could be improved somewhat by using instead the sharper estimate derived in the Appendix. For example, the plots in Figure 8 of the L_∞-errors for Example 1, which were normalized using θ_0, clearly show, by the downward trends as $M \to \infty$, that the actual rate of convergence is smaller than the rate $\theta_0/\lambda e$ which was used in the normalization. In Figure 13, we have plotted the maximum (L_∞) errors $\max\left|E^{(M,\lambda M)}(x)\right|$ for Example 1, normalized by $M(\lambda e/D_0\theta_0)^M$, which involves the sharper error bound derived in the Appendix, where D_0 is defined by equation (73). Each of these plots are clearly bounded as M becomes large, and appear to be approximately linear as $M \to \infty$, with a slope that approaches zero as λ becomes large. These observations are consistent with an asymptotic rate of convergence given by $D_0\theta_0/\lambda e < \theta_0/\lambda e$.

As mentioned in the introduction, Gottlieb and several of his co-investigators [3],[4],[5],[6],[7] have proposed and analyzed a technique using the Gegenbauer polynomials which also leads to a series which converges exponentially to f in the maximum norm (and, hence, also eliminates the Gibbs phenomena). Their technique requires a knowledge of only the first $2N+1$ Fourier coefficients of f, whereas our method requires a knowledge of the location and magnitude of the jumps of f and its first N derivatives within the interval $[-\pi, \pi]$. Consequently, since our technique requires more "information" about the function than Gottlieb's method, it is not surprising that our approximations converge much more rapidly than his do. In particular, the error in his approximations contains a term proportional to $N^2(q_T)^N$, where $q_T = (\beta+2\alpha)^\beta(2\pi e\alpha)^{-\alpha}\beta^{-\beta} < 1$, and α and β can be chosen so that the convergence rate is optimized. The corresponding term in our approximations (see THEOREM 2) is $N^{-1}(\theta_0/e)^N$, where we have set $\lambda = 1$ and, hence, $M = N$. For the function f of our Example 1 (which is also the first example in [3]), these terms are (approximately) $N^2(0.4763)^N$ and $N^{-1}(0.2124)^N$, respectively. In Figure 14 we have plotted $\log_{10}\left[\max\left|E^{(N,N)}(x)\right|\right]$ as a function of N for Example 1 and Example 3 using our approximations and Gottlieb's approximations. The difference in the rates of convergence is apparent from the figure

We should also point out that in [3] the assumption was made that $\rho > 1$, where

ρ is the distance of the nearest singularity of f from the basic interval $[-1, 1]$. (The corresponding assumption for our analysis would be that $\rho > \pi$.) This condition is relaxed somewhat in [5]. However, we impose no such restriction on our functions f.

Several other methods have been proposed to accelerate the convergence of Fourier series and/or to eliminate (or at least mitigate) the Gibbs' phenomena. These methods include the construction of Fourier-Padé approximations [2] (mentioned in the introduction), Fejér's method [1], based on Cesàro sums, and Lanczos' "sigma" method [11]. In Figure 15, we have compared these three approximations for the function $f(x) = x$ of Example 1, along with the original Fourier series partial sum $F^{(4)}$ (equation (4)), Gottlieb's approximation [3], and our approximation $f^{(1,4)}$ (equation (14)). It is interesting to note that, although the approximations of Fejér and Lanczos have essentially eliminated the Gibbs phenomena, these approximations are of poor quality in an L_2 sense. The simple reason for this is that each of these methods retains the same *form* of the approximation and just replaces the original Fourier coefficients with certain other, modified coefficients. It is well known (see [1], for example) that, for any linear combination of a finite number of sine and cosine functions, the original Fourier coefficients are optimal in an L_2 sense. Thus, the approximations of Fejér and Lanczos can be no better in the L_2 sense than the original Fourier series partial sum.

Currently, we are investigating several topics related to the approximations we have proposed, and they will be discussed in future papers. These topics include the problem of determining estimates of the locations and the magnitude of the jumps in f and its derivatives in the interval $[-\pi, \pi]$ directly from the Fourier coefficients of f. Such estimates will allow the basis functions defined here to be used in a manner similar to that described above, when only a finite number of the Fourier coefficients of f are known. Some preliminary investigations have indicated that the new approximations will still retain the property of exponential convergence, although the rate of convergence of the new approximations will undoubtedly not be as rapid as the approximations described here. Applications of these ideas directly to the solution of differential equations (particularly, boundary value problems) are also being investigated, as well as an extension of the ideas presented here to other orthogonal basis (eigenfunction) expansions.

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Appendix: Derivation of θ_0

In this appendix, we first present a brief derivation of the bound

$$\tilde{\theta}^M \equiv \frac{1}{2^M}\prod_{i=0}^{m}\left(1-\left(\frac{2i+1}{2j}\right)^2\right)^{-1} \le \hat{K}\theta_0^M, \quad \theta_0(\lambda) \equiv \frac{1}{2\sqrt{1-1/(4\lambda^2)}}, \tag{68}$$

where $M = 2m+1$ and $j \ge \lambda M + 1$, which was used in connection with the discussion of the function of Example 1, as well as the function h_0 which was used in the proof of Theorems 2-3. To show that equation (68) holds, we first observe that

$$\frac{2i+1}{2j} \le \frac{2m+1}{2(\lambda M+1)} < \frac{1}{2\lambda},$$

for $0 \le i \le m$ and $j \ge \lambda M + 1$. Using this result, we can write

$$\prod_{i=0}^{m}\left(1-\left(\frac{2i+1}{2j}\right)^2\right) \ge \prod_{i=0}^{m}\left(1-\frac{1}{4\lambda^2}\right) = \left(1-\frac{1}{4\lambda^2}\right)^{m+1} = \left(\sqrt{1-1/(4\lambda^2)}\right)^{M+1}.$$

Using this bound in equation (68) we find, for all $\lambda \ge \hat{\lambda} > 1/2$,

$$\tilde{\theta}^M \le \hat{K}\theta_0^M, \quad \hat{K} \equiv \left(\sqrt{1-1/(4\hat{\lambda}^2)}\right)^{-1}. \tag{69}$$

In the special case when λM is an integer, we can obtain a sharper bound for $\tilde{\theta}^M$. (Recall that M is an *odd* integer.) To do this, we first define $J = 2(\lambda M + 1)$ and then note that

$$\begin{aligned}\prod_{i=0}^{m}\left(1-\left(\frac{2i+1}{2j}\right)^2\right)^{-1} &\le \prod_{i=0}^{m}\left(1-\left(\frac{2i+1}{J}\right)^2\right)^{-1} \\ &= \prod_{i=0}^{m}\left(\frac{J^2-(2i+1)^2}{J^2}\right)^{-1} \\ &= J^{M+1}\frac{(J-M-2)!!}{(J+M)!!}.\end{aligned} \tag{70}$$

(Here we have used that usual notation $k!! = 1 \cdot 3 \cdot 5 \cdots k$, for any *odd* integer k.) Then, using Stirling's approximation to the factorial, we find that $k!! \to \sqrt{2k}(k/e)^{k/2}$, as $k \to \infty$. Using this approximation in equation (70), we find that

$$\tilde{\theta}^M \leq \hat{K} \cdot (D_0\theta_0)^M, \quad \text{for } \lambda \geq \hat{\lambda} > 1/2, \tag{71}$$

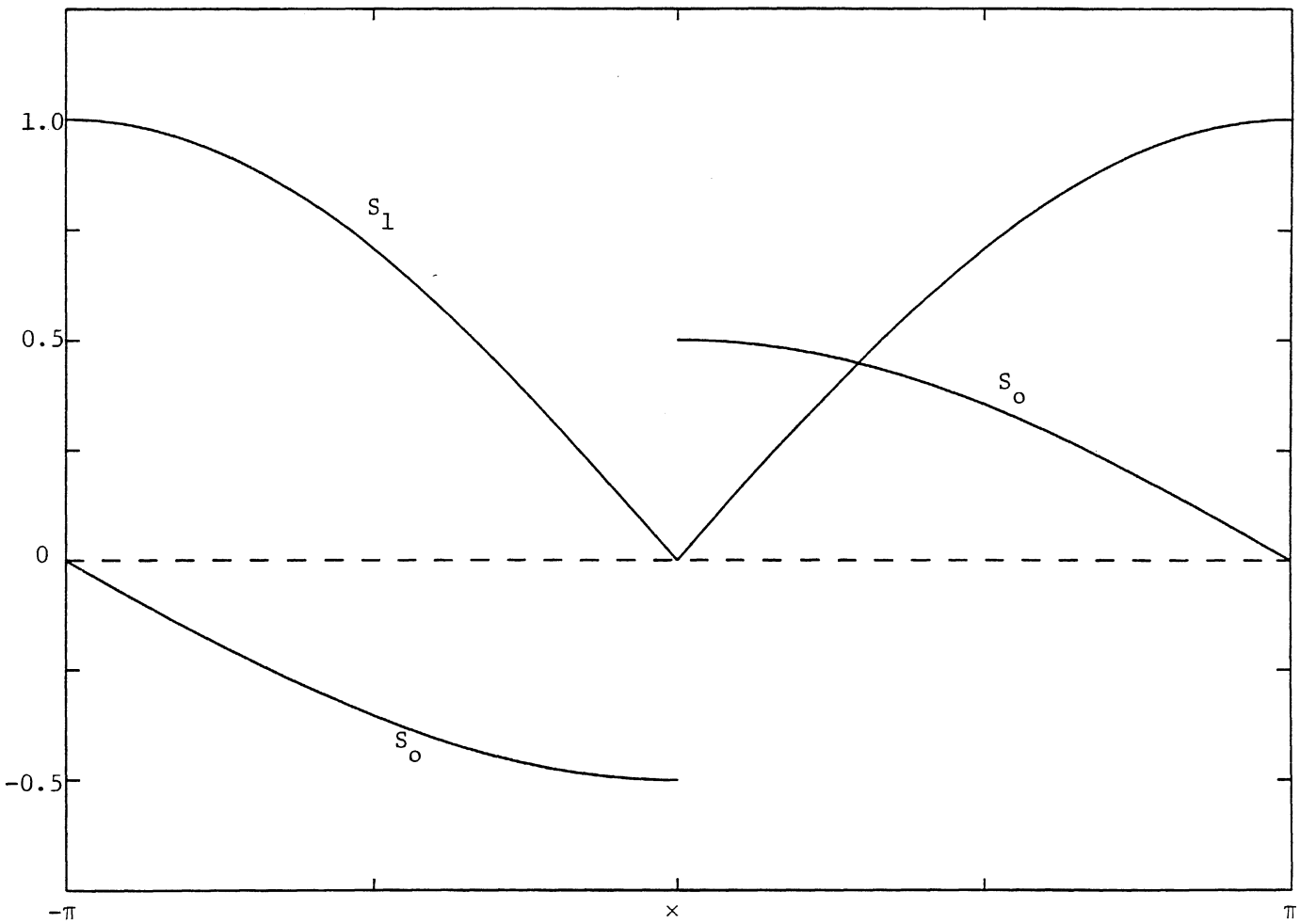
where

$$\hat{K} = \hat{K}(\lambda) \equiv \left(\frac{2\lambda+1}{2\lambda-1}\right)^{\lambda} e^{1/\lambda} \cdot \frac{1}{\sqrt{1-1/(4\lambda^2)}} < \frac{3}{\sqrt{1-1/(4\hat{\lambda}^2)}}, \tag{72}$$

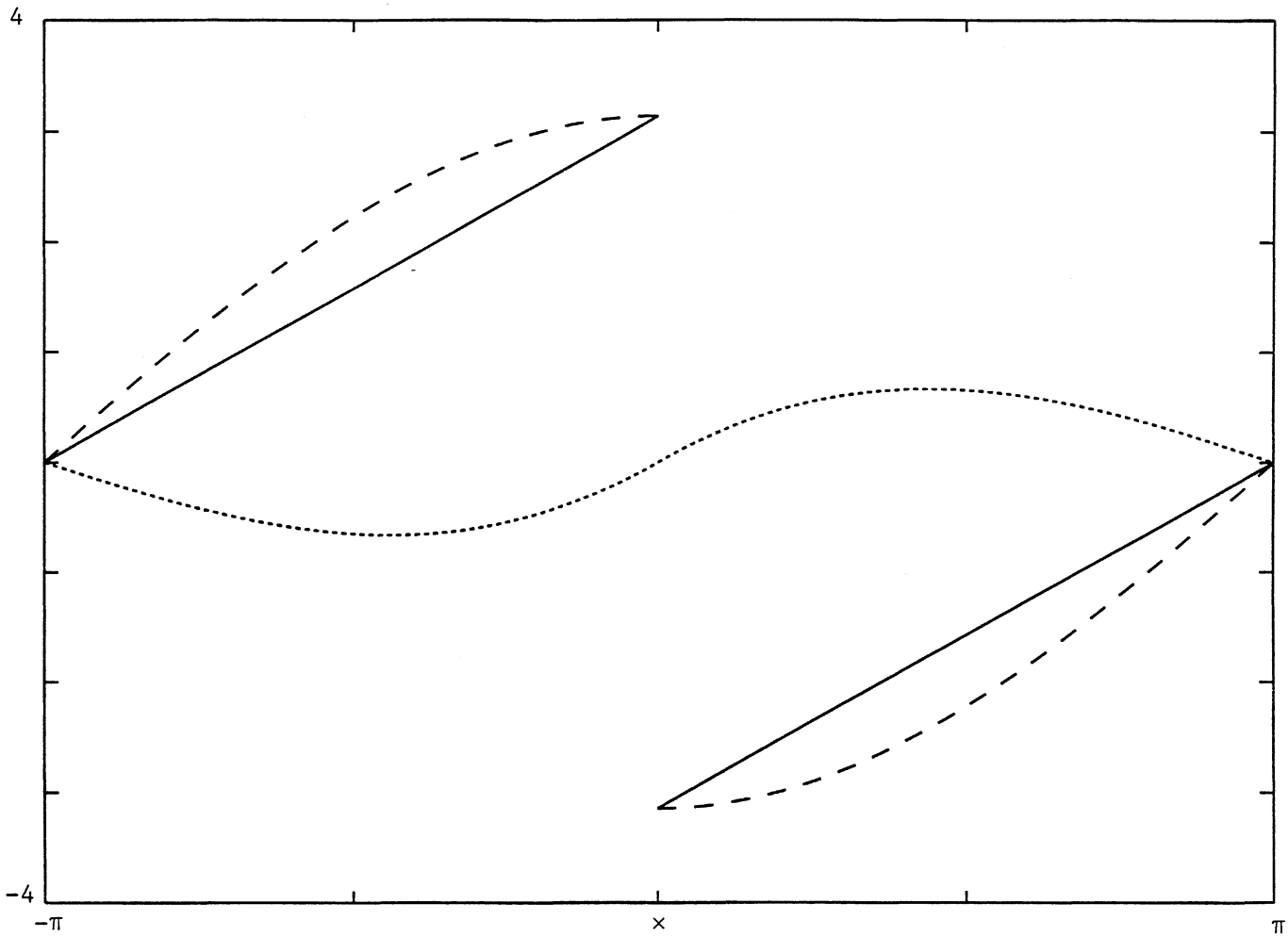
and

$$D_0 = D_0(\lambda) \equiv e\left(\frac{2\lambda-1}{2\lambda+1}\right)^{\lambda} \to 1, \quad \text{as } \lambda \to \infty. \tag{73}$$

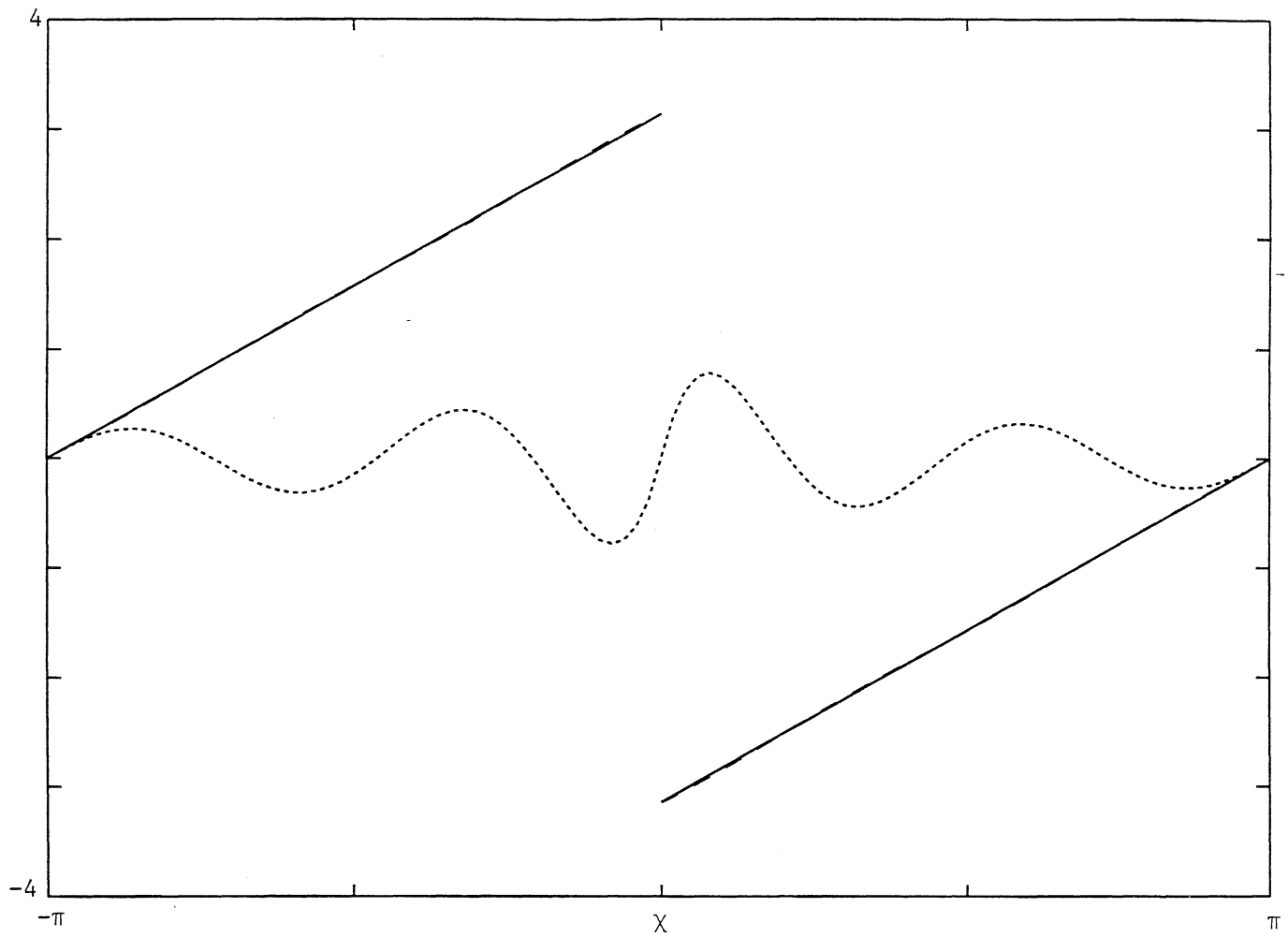
We remark that, although our derivation is valid only for the case when λM is an integer, the final result (equations (71)-(73)) holds for *any* positive value of λ. The reason for this is that the expression for $k!!$ can be written as $(k!)/(2^{(k-1)/2}((k-1)/2)!)$, and this expression can be defined for noninteger values of k in terms of the Gamma function. Since the estimate we used for $k!!$ (following equation (70)) also holds for noninteger values of k, it follows that the expressions (71)-(73) hold for noninteger values of λM as well.



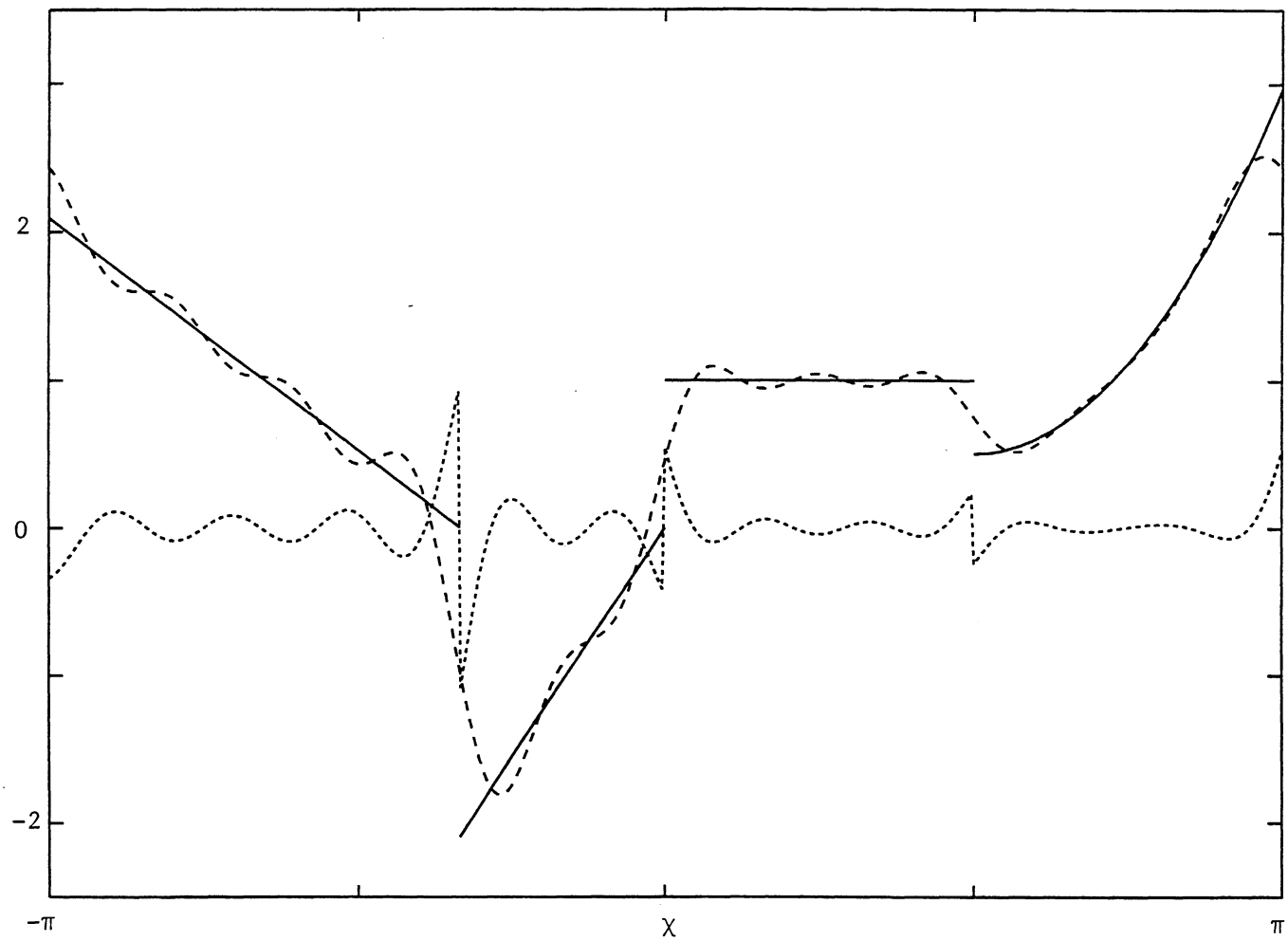
1) The functions $S_0(x)$ and $S_1(x)$, as defined in equation (1). Note that S_0 has a jump of magnitude 1 at $x = 0$, while S_1' has a jump of magnitude 1 at $x = 0$.



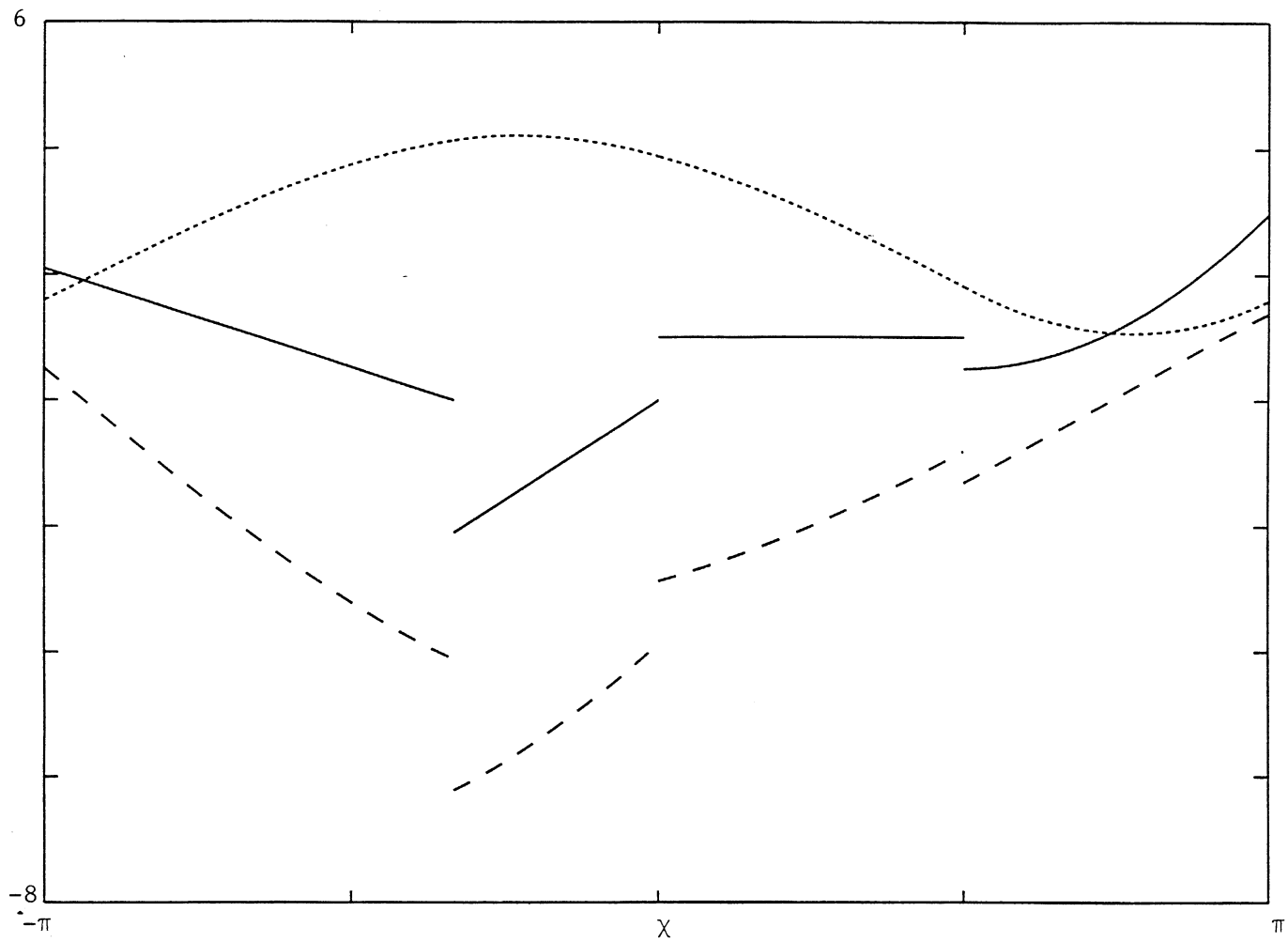
2) The function f (solid line), the function R_1 (dashed line), and the difference $f_1 = f - R_1$ (dotted line), for Example 1.



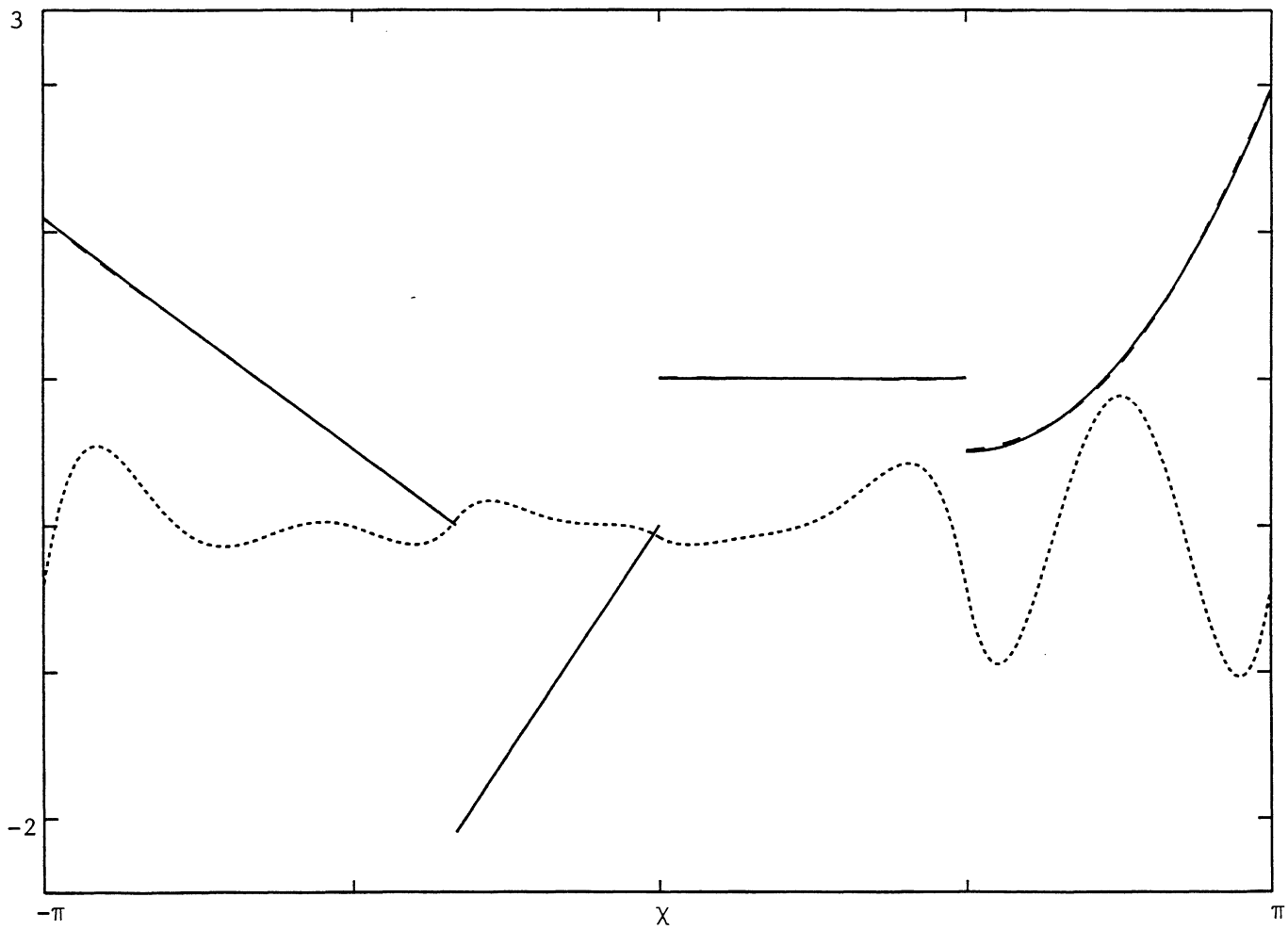
3) The function f (solid line), the function $f^{(1,3)}$ (the barely visible dashed line), and the magnified error $50(f - f^{(1,3)})$ (dotted line), for Example 1.



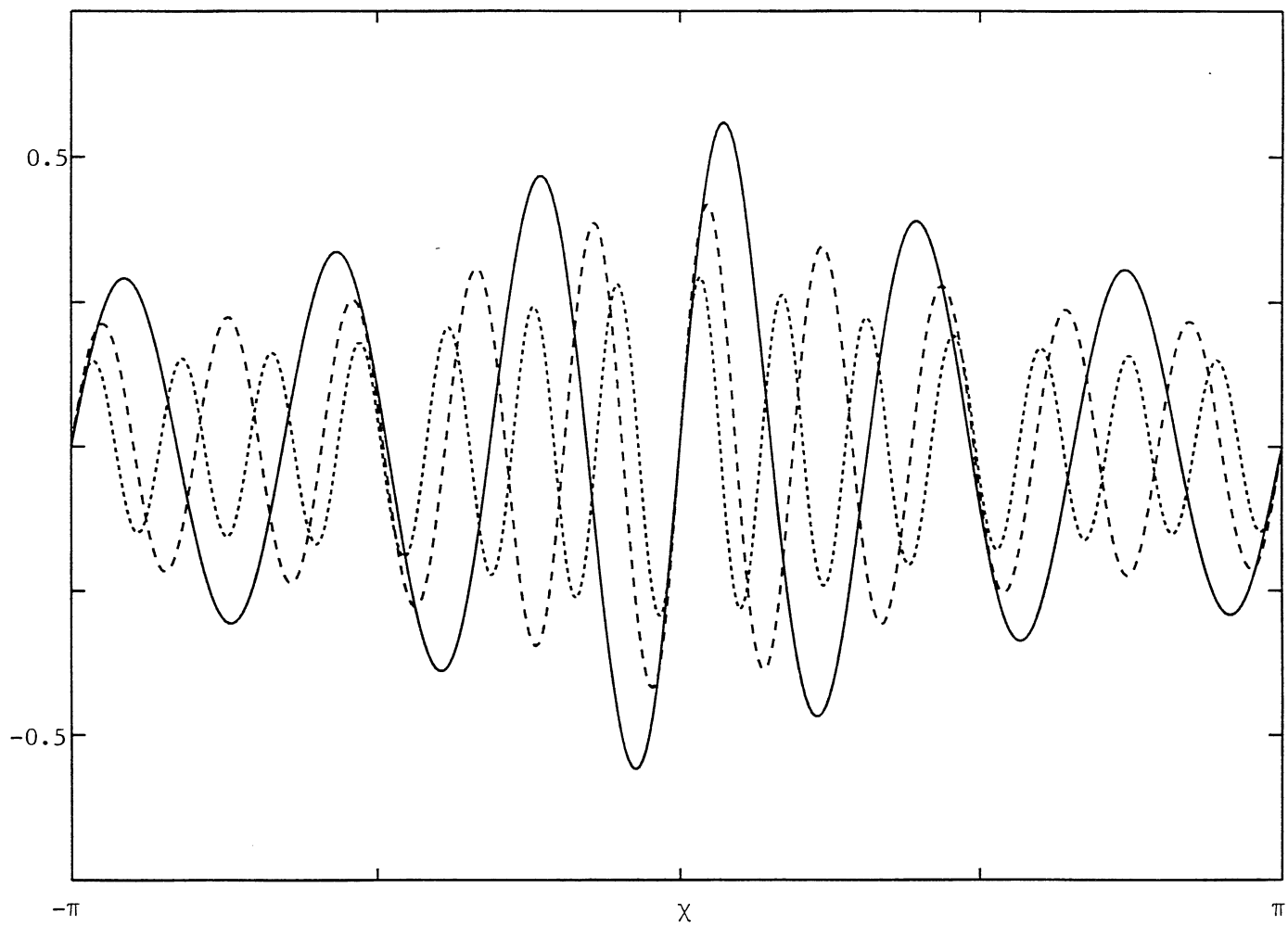
4) The function f for Example 2 (solid lines), along with the Fourier series partial sum $F^{(N)}$ of f (see equation (4)) with $N = 10$ (dashed line), and the error $f - F^{(10)}$ (dotted line).



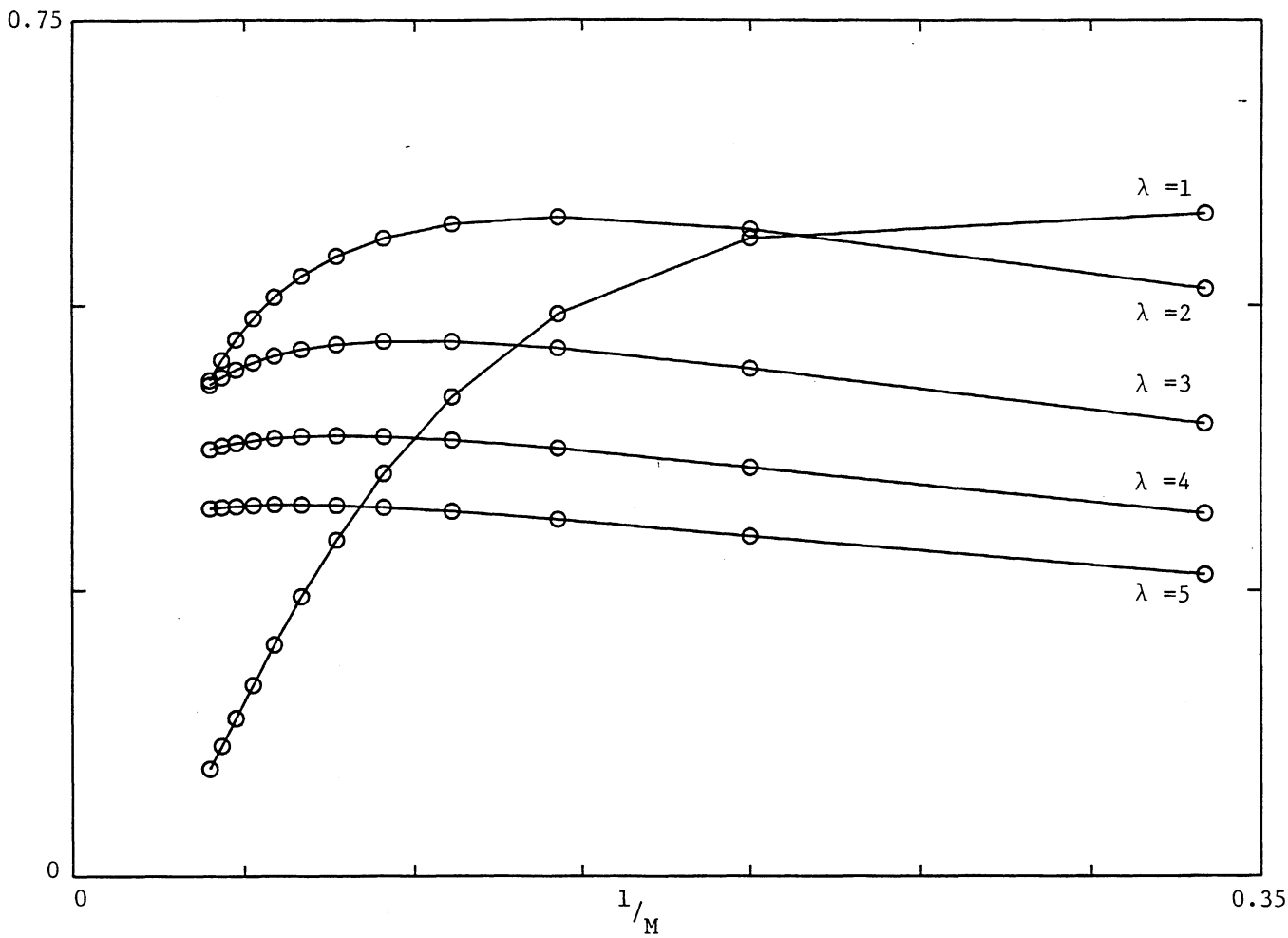
5) The function f (solid line), the function R_1 (dashed line), and the difference $f_1 = f - R_1$ (dotted line), for Example 2.



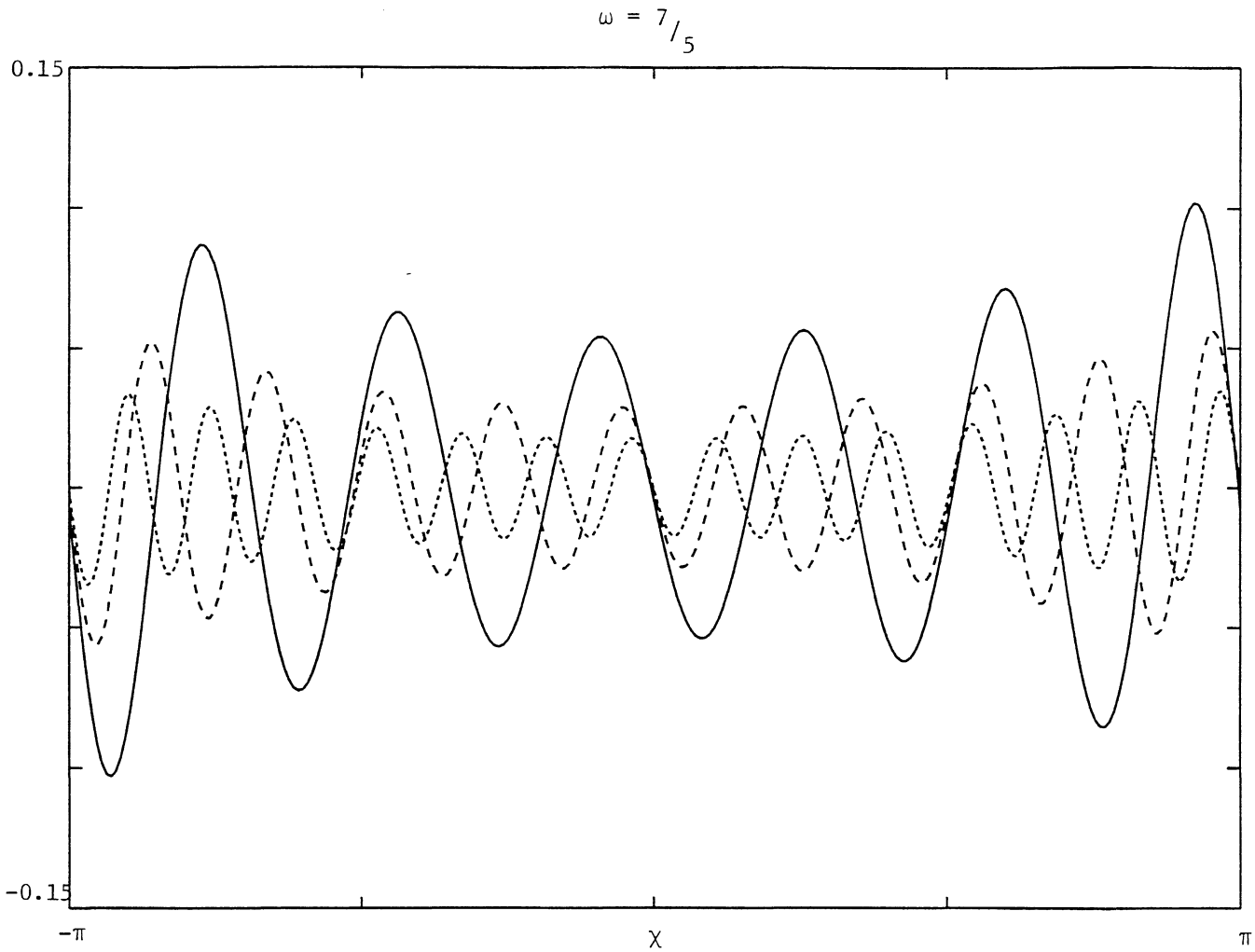
6) The function f (solid line), the function $f^{(1,3)}$ (the barely visible dashed line), and the magnified error $50(f - f^{(1,3)})$ (dotted line), for Example 2



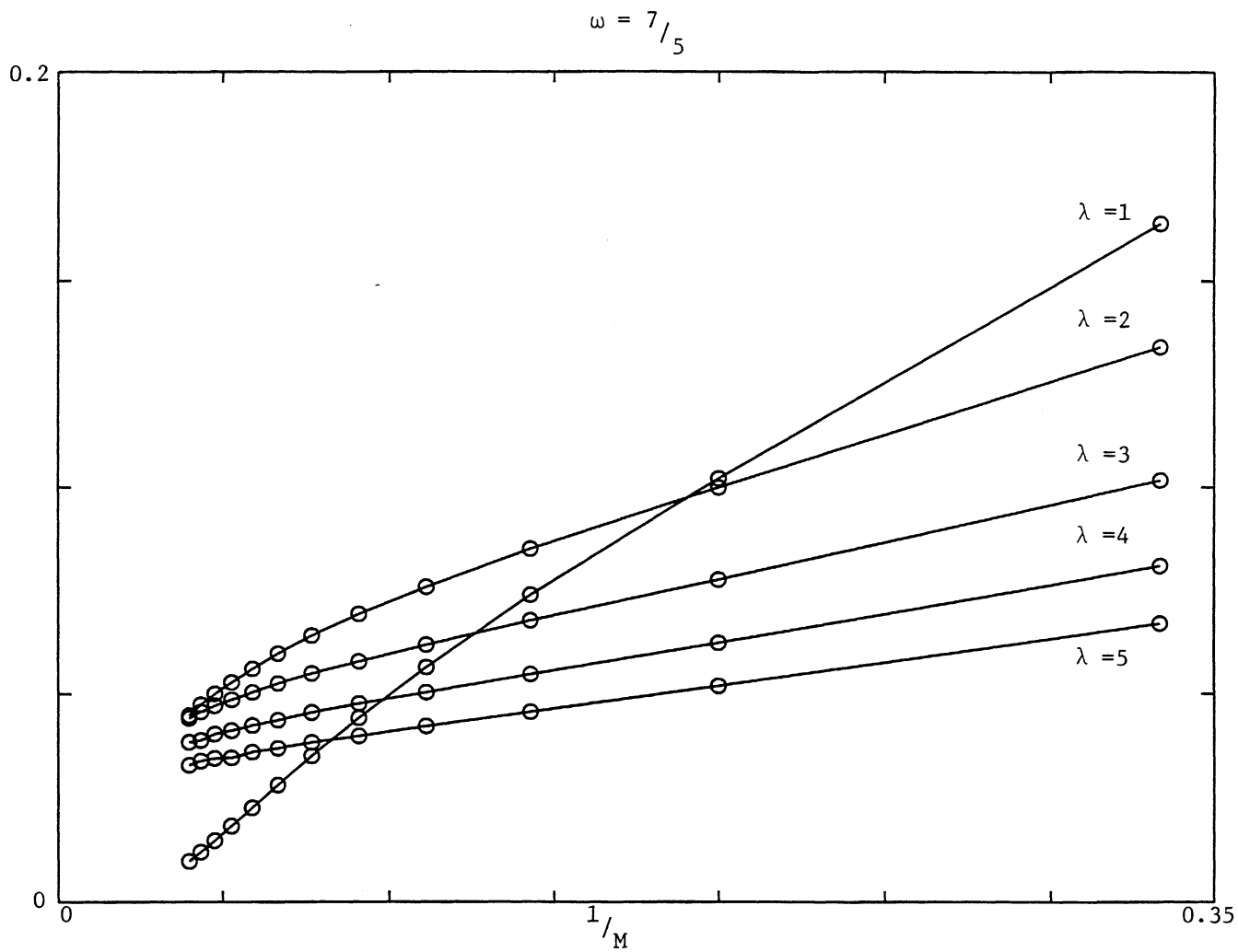
7) The normalized errors $M(\lambda e/\theta_0)^M E^{(M,\lambda M)}(x)$ for Example 1, when $\lambda = 1$ and $M = 5$ (solid line), $M = 9$ (dashed line), and $M = 13$ (dotted line). Note the more or less "uniform" distribution of the error over the interval $[-\pi, \pi]$ (especially as M increases), as opposed to the error in the original Fourier series partial sums, which is highly concentrated near $x = 0$.



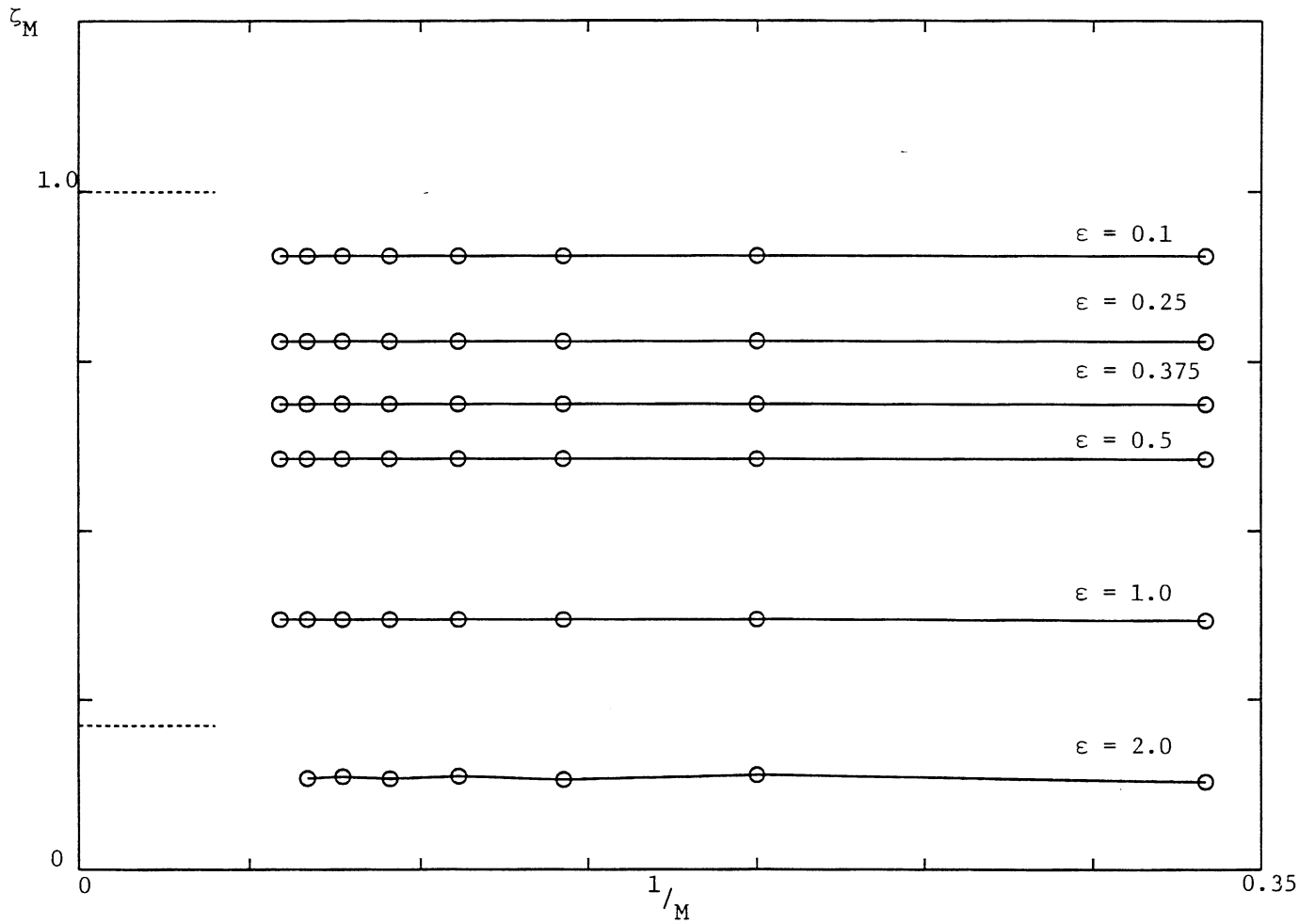
8) The normalized maximum (L_∞) errors $M(\lambda e/\theta_0)^M \max\left|E^{(M,\lambda M)}(x)\right|$ for Example 1 plotted as a function of $1/M$, for $\lambda = 1$, 2, 3, 4, and 5.



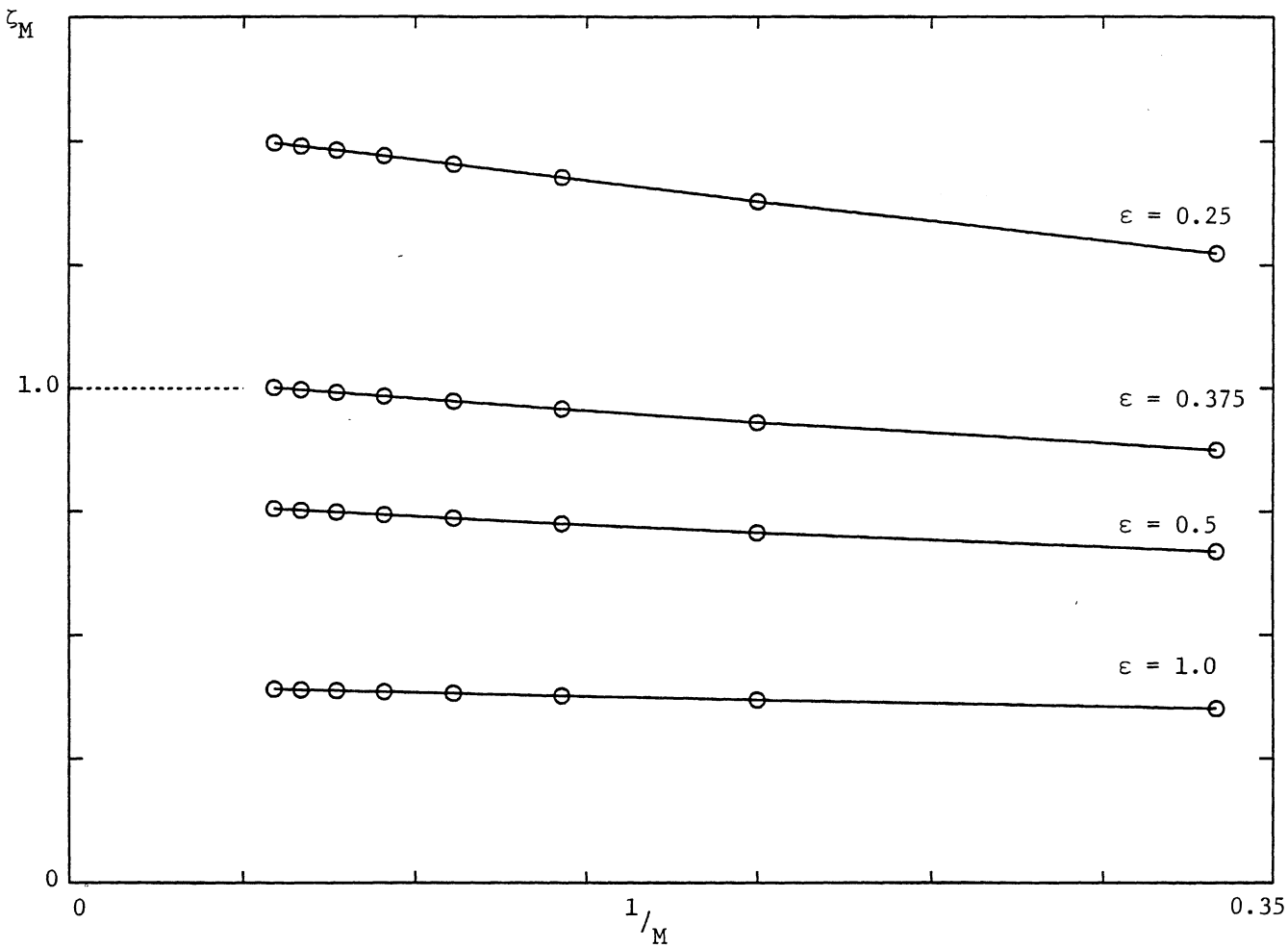
9) The normalized errors $M(\lambda e/\theta_0)^M E^{(M,\lambda M)}(x)$ for Example 3, when $\lambda = 1$ and $M = 5$ (solid line), $M = 9$ (dashed line), and $M = 13$ (dotted line). Note the more or less "uniform" distribution of the error over the interval $[-\pi, \pi]$ (especially as M increases), as opposed to the error in the original Fourier series partial sums, which is highly concentrated near $x = \pi$.



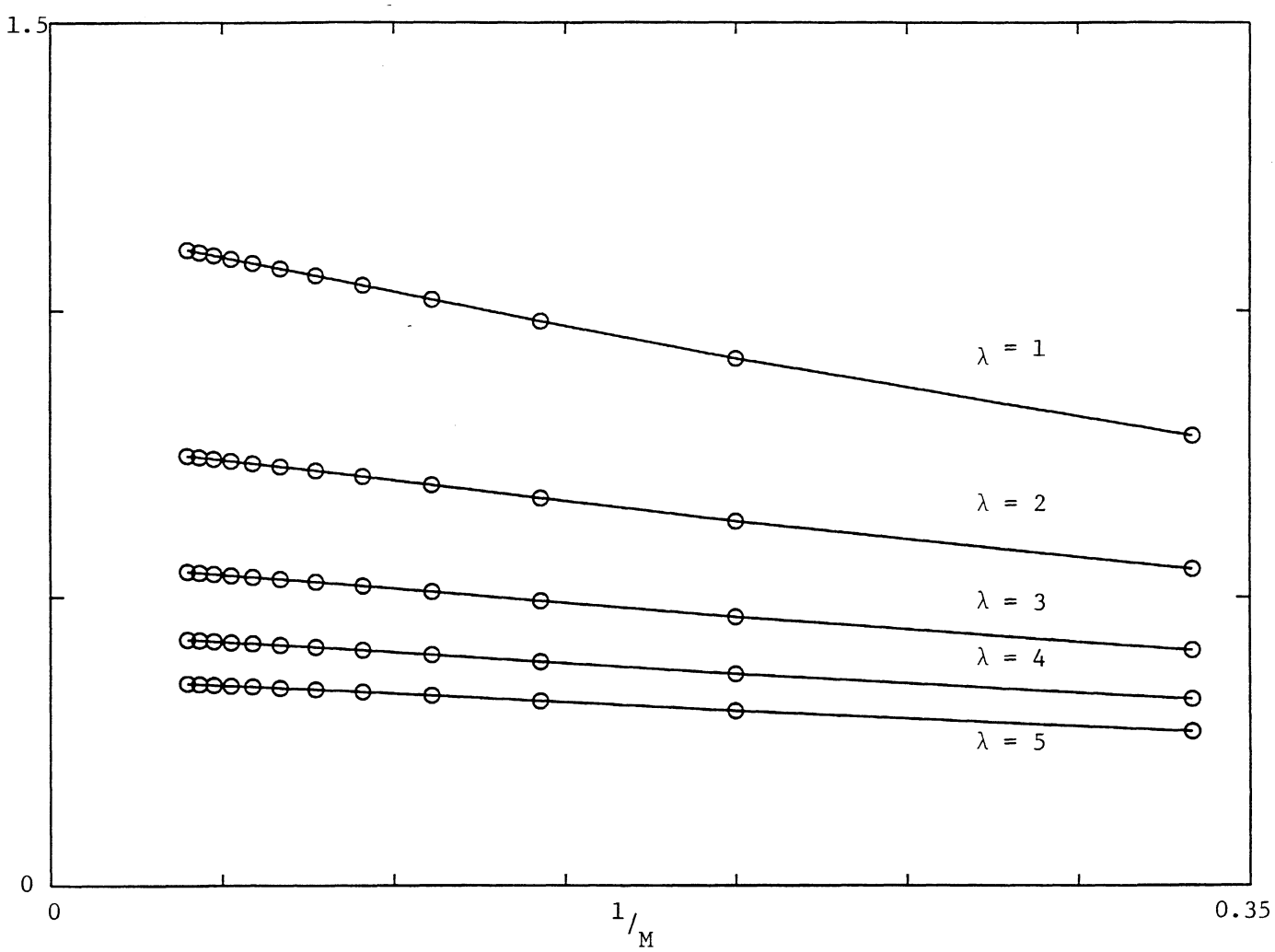
10) The normalized maximum (L_∞) errors $M(\lambda e/\theta_0)^M \max\left|E^{(M,\lambda M)}(x)\right|$ for Example 3 plotted as a function of $1/M$, for $\lambda = 1$, 2, 3, 4, and 5.



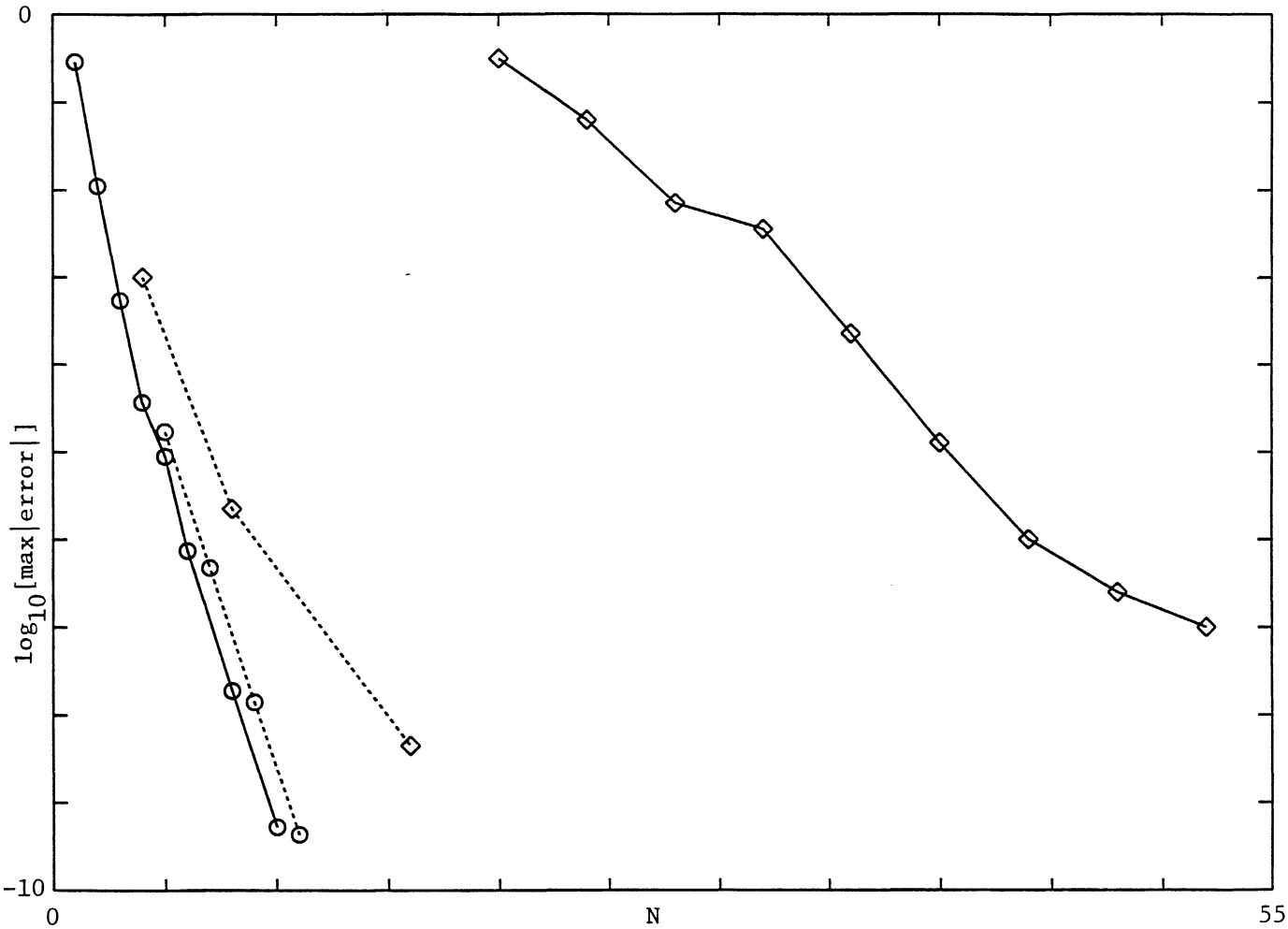
11) The quantities $\zeta_M \equiv \left[\left(\max\left|E^{(M,\lambda M)}(x)\right|\right) / \left(\max\left|E^{(M-2,\lambda(M-2))}(x)\right|\right)\right]^{1/2}$ for Example 4 plotted as a function of $1/M$ for $\lambda = 1$ and for $\epsilon = 0.1, 0.25, 0.375, 0.5, 1$, and 2. As $M \to \infty$, ζ_M approaches the asymptotic rate of convergence of our approximations. The theoretical asymptotic rates of convergence $\eta = \pi/\sqrt{\pi^2 + \epsilon^2} \doteq 0.999$ (for $\epsilon = 0.1$) and $\zeta = \theta_0/e\lambda \doteq 0.212$ are each indicated by a short dotted line.



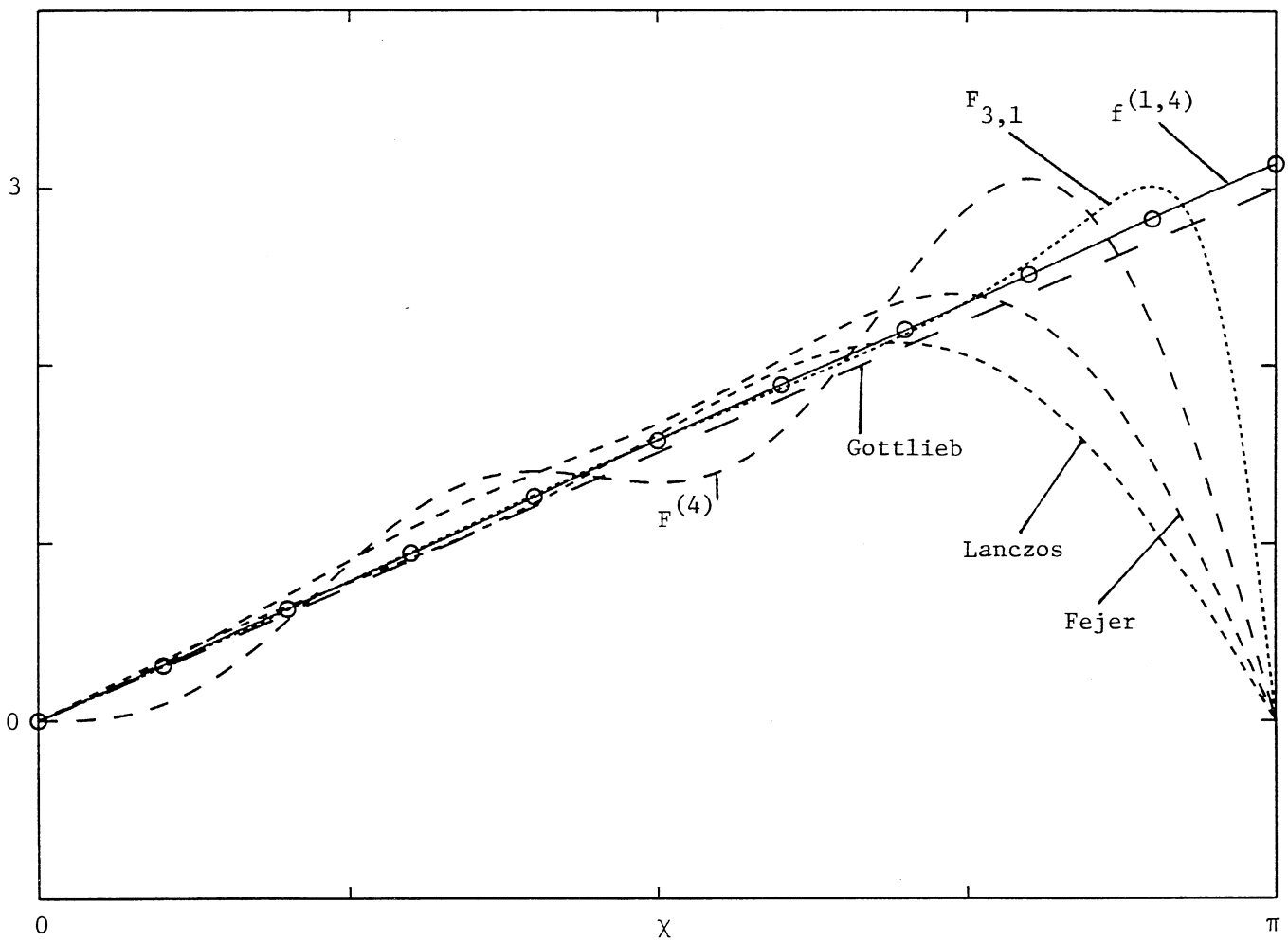
12) The quantities $\zeta_M \equiv \left[\left(\max\left|E^{(M,\lambda M)}(x)\right|\right) / \left(\max\left|E^{(M-2,\lambda(M-2))}(x)\right|\right)\right]^{1/2}$ for Example 5 plotted as a function of $1/M$ for $\lambda = 1$ and for $\epsilon = 0.25$, 0.375, 0.5, and 1. As $M \to \infty$, ζ_M approaches the asymptotic rate of convergence of our approximations. The critical value $\zeta_M = 1$ is indicated by the short dotted line.



13) The maximum (L_∞) errors $M(\lambda e/D_0\theta_0)^M \max\left|E^{(M,\lambda M)}(x)\right|$ for Example 1, normalized by the sharper error bound derived in the Appendix, plotted as a function of $1/M$, for $\lambda = 1$, 2, 3, 4, and 5.



14) A plot of the $\log_{10}\{\max|\text{Error}|\}$ for the functions of Example 1 (dotted lines) and Example 3 (solid lines) using Gottlieb's approximation [3] (diamonds) and the approximations $f^{(N,N)}$, equation (22) (circles), plotted as a function of N.



15) A comparison of the function $f(x) = x$ of Example 1 (denoted by circles) and the original Fourier series partial sum $F^{(4)}$ (equation (4)), the Fourier-Padé approximation $F_{3,1}$ [2], Fejér's approximation [1], Lanczos' "sigma" approximation [11], Gottlieb's approximation [3], and the approximation $f^{(1,4)}$ (equation (14)). All of these approximations are based on only the first 4 terms in the original Fourier partial sum.

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13. ABSTRACT *(Maximum 200 words)*
A family of simple, periodic basis functions with "built-in" discontinuities are introduced, and their properties are analyzed and discussed. Some of their potential usefulness is illustrated in conjunction with the Fourier series representation of functions with discontinuities. In particular, it is demonstrated how they can be used to construct a sequence of approximations which converges *exponentially in the maximum norm* to a piece-wise smooth function. The theory is illustrated with several examples and the results are discussed in the context of other sequences of functions which can be used to approximate discontinuous functions.

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