A pseudo-sound constitutive relationship for the dilatational covariances in compressible turbulence: an analytical theory

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Abstract

The mathematical consequences of a few simple scaling assumptions about the effects of compressibility are explored using a simple singular perturbation idea and the methods of statistical fluid mechanics. Representations for the pressure-dilatation and dilatational dissipation covariances appearing in single-point moment closures for compressible turbulent are obtained. The results obtained, in as much as they come from the same underlying diagnostic relationship, represent a unified development for both the compressible covariances. While the results are expressed in the context of a second-order statistical closure they provide some interesting and very clear physical metaphors for the effects of compressibility that have not been seen using more traditional linear stability methods. In the limit of homogeneous turbulence with quasi-normal large scales the expressions derived are - in the low *turbulent* Mach number limit - asymptotically exact. The expressions obtained are functions of the rate of change of the turbulence energy, its correlation length scale, and the relative time scale of the cascade rate. With the appearance of the length scale the dilatational covariances are found to scale with the Mach numbers based on the mean strain and rotation rates. The expressions for the dilatational covariances contain constants which have a precise and definite physical significance; they are related to various integrals of the longitudinal velocity correlation. The pressure-dilatation covariance is found to be a non-equilibrium phenomena related to the time rate of change of the internal energy and the kinetic energy of the turbulence. Also of interest is the fact that the representation for the dilatational dissipation in a turbulence, with or without shear, features a dependence on the Reynolds number. This article is a documentation of an analytical investigation of the implications of a pseudo-sound theory for the effects of compressibility. The novelty of the analysis is in the very few phenomenological assumptions required to produce the results. Subsequent work will assess the consequences of this analysis in the context of compressible turbulence models for engineering calculations.

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1 Introduction

In the moment equations for compressible turbulence several new quantities, not seen in the incompressible form of the equations, appear. There are new terms in the equations that reflect the fact that the fluctuating dilatation, $d = u_{k,k}$, in a compressible turbulence is not zero. These effects have been divided into two categories, Lele(1994). There are contributions to the fluctuating dilatation by the the fluctuating pressure; there are also contributions to the fluctuating dilatation by fluctuations in composition or temperature which occur in situation in which of heat and mass transfer are relevant. These two sources of the fluctuating dilatation have been distinguished using the phrases compressibility effects and variable inertia effects, Lele (1994). There are also additional effects, in inhomogeneous or nonequilibrium flows, associated with the portion of density fluctuations due to gradients in the mean density; these effects might also be called variable inertia effects giving rise to the differences between Favre and Reynolds averaged variables. There are also effects associated with the transport coefficient variations due to the fluctuations in temperature. In this article effects due the occurrence of a nonzero fluctuating dilatation, the so-called compressibility effects, are treated.

In the context of single-point moment closure methods, compressibility effects due to the fluctuating divergence appear in two new terms in the kinetic energy equation of a turbulent field: the pressure-dilatation, $< pd >$, and the variance of the dilatation, $< dd >$, which is related to what has come to be called the compressible dissipation, $\epsilon_c = \frac{4}{3}\nu < dd >$. The turbulent energy equation is written as

$$< \rho > \frac{D}{Dt} k = P_k - < \rho > \epsilon_s + < pd > - < \rho > \epsilon_c + T_k. \tag{1}$$

P_k represents the production and T_k represents the transport terms and any other terms (that are not directly germane to the present analysis). There are additional terms representing the contraction of the mass flux vector on the mean flow acceleration. T_k will also be used to represent all such terms. The $< pd >$ and $< dd >$ appearing in the k equation were recognized by Zeman (1990, 1991) and Sarkar *et al.* (1991) in earlier studies of compressible turbulence closures. They are the subject of this work.

The dilatational covariances also appear in the internal energy equation, here written in terms of the mean temperature with a constant c_v:

$$< \rho > c_v \frac{D}{Dt} T = P_T - < pd > + < \rho > \epsilon_s + < \rho > \epsilon_c + T_T$$

Where T_T is the transport of the mean temperature including such effects as heat flux and the turbulent or pressure transport. The production, for a specific class of flows, is given by

$P_T = -PD + 2 < \mu > (S^2 + W^2)$ where the strain and rotation tensors are defined to be symbolically equivalent to the incompressible case, *ie.* traceless: $S_{ij} = \frac{1}{2}[U_{i,j} + U_{j,i} - \frac{2}{3}D\delta_{ij}]$, $W_{ij} = \frac{1}{2}[U_{i,j} - U_{j,i}]$. Note that $S_{jj} = 0$ since $D = U_{j,j}$. There are additional terms depending on heat or species transfer and fluctuations in fluid properties: they are not germane to the present analysis. Note that the dilatational covariances appear with opposite signs in the kinetic and mean internal energy equations. The dilatational covariances represent an irreversible, ϵ_c, and reversible transfers, $< pd >$, of energy between the mean internal energy field and the fluctuating kinetic energy field.

The pressure-dilatation covariance, $< pd >$, and the dilatational variances, $< dd >$, - which is related to the dilatational dissipation - have been the subject of several studies stressing both the fundamental issues in understanding the physics as well as obtaining models suitable for use in engineering calculations. In addition to the aforementioned works of Zeman and Sarkar, additional insight into these terms can be found in the studies of Durbin and Zeman(1992), Zeman and Coleman (1991), Zeman (1993), Erlebacher *et al.* (1990), Sarkar (1992), Sarkar *et al* (1991a, 1991b), Blaisdell *et al.* (1991), Blaisdell and Sarkar (1993), Lee (1992).

The present approach differs from the approaches of both Zeman and Sarkar. A low turbulent Mach number expansion of the equation of state, the Navier Stokes, the continuity and wave equations is conducted. The problem is recognized as a singular perturbation in as much as there are two relevant length scales: an inner scale, ℓ, associated with the turbulence field, and an outer scale $\lambda \sim \ell/M_t$ associated with a propagating "acoustic" radiation field surrounding the vortical motion producing the radiation field. The perturbation development produces an algebraic constitutive equation for the fluctuating dilatation: the continuity equation, rather than being a prognostic equation for the density, becomes a diagnostic equation for the fluctuating dilatation. Taking the relevant moments of the expression produces constitutive relations for $< pd >$ and $< dd >$. Assuming homogeneity and quasi-normality expressions *without* any undefined constants are obtained for $< pd >$ and $< dd >$.

Retaining only the lowest order isotropic contribution produces simple expressions in terms of an incompressible turbulence for the unknown covariances are found. The expressions are then given in terms of quantities carried in a single-point closure.

The analysis can be contrasted to more traditional approaches using linear stability theory. Lele (1994) provides a resume of several of these works. Despite the inherent limitations in the linear stability analysis there has been some useful light shed on the dynamical aspects of the effects of compressibility on the flow. Here a statistical approach to the problem is taken. Such an approach

accounts for the nonlinearity of the phenomena. A statistical approach approach, however, does average over the many interesting dynamical features of the fluctuation flow but delivers some very interesting insights indicating the nature of the cumulative effects of the fluctuations. Of particular note is the interpretation of the effects of compressibility as an added mechanism for the transfer of energy between the turbulence field and the mean internal energy field. The theory predicts the mechanism and the rate of this intermodal energy transfer.

It is found, in analogy with the pressure-strain covariances in the Reynolds stress equations for an incompressible flow, Ristorcelli *et al.* (1994), that the representations for the dilatational covariances have a rapid and a slow component. However, unlike the rapid-pressure in the incompressible problem, the rapid component depends on the spatial area of the rapid-pressure correlation. This behavior results in a dependence on a "gradient" Mach number: a Mach number formed using the mean velocity gradient and a length scale of the turbulence. This quantity appears to be an important parameter in distinguishing the effects of compressibility in a mixing layer from those in a boundary layer, Sarkar (1994).

Several other interesting results have been found. These are now highlighted:

1) The pressure dilatation $< pd >$ can be either positive or negative depending on the rate of change of the kinetic energy, the mean temperature, the length scale of the turbulence and the mean velocity gradients. For a near equilibrium flow, as long as the production exceeds the dissipation by an amount that scales with the square of the turbulent Mach number, $M_t^2 = \frac{2/3k}{c^2}$, and the rate of increase of the internal energy field, the pressure-dilatation will be negative transferring energy from kinetic to internal modes.

2) The representation for $< pd >$ can be shown to behave as an added mass term in the the k equation: inertia is added to the turbulence by the capacitance of the fluctuating pressure field. The fluctuating pressure field, or equivalently the mean internal energy, acts as a capacitor storing energy fed into the turbulence by the production and then transferred to the internal energy by $< pd >$.

3) The pressure dilatation, $< pd >$, scales with M_t^2 with order M_t^4 and higher corrections. The dilatational dissipation scales as M_t^4 with a Re_t^{-1} dependence; $R_t = \frac{4}{9}\frac{k^2}{\nu\epsilon}$ is the turbulent Reynolds number. For high Reynolds number flows, much higher than those seen in DNS, the dilatational dissipation is found to be small; the primary effects of compressibility are due to the pressure-dilatation. The effects of compressibility, occurring through the agency of the pressure-dilatation covariance, are found to be important in nonstationary flows.

4) The rapid portions of both $< pd >$ and $< dd >$ scale with the relative time scales of the cascade, Sk/ϵ and Wk/ϵ, (the mean strain and rotation) to the second and fourth powers. With the concomitant appearance of the turbulent Mach number M_t, the expressions are found to scale with the mean deformation and mean rotation Mach numbers $S\ell/c$ and $W\ell/c$. It is these Mach numbers that distinguish the mixing layer from the equilibrium boundary layer and thus the representation can discriminate between these two types of flows. The appearance of the two different gradient Mach numbers indicates a dependence of the dilatational covariances on the relative amounts of strain versus rotation; a fact not yet looked for in experimental or numerical results.

5) Unlike the incompressible dissipation, in which the viscosity sets the small scales but otherwise has very little effect on the cascade rate, the dilatation dissipation is, in this linear theory, dependent on the viscosity. For a fixed M_t number, as $R_t \rightarrow \infty$, the dilatational dissipation vanishes. The compressible dilatation doesn't appear to be able to be understood as a spectral cascade rate set by the large scales of the flow.

6) It is seen that the density fluctuations are related to the incompressible pressure fluctuations, $p = \gamma\rho$ which, upon rescaling, can be understood as, $\rho = M_t^2 p$. This indicates that compressible numerical simulations starting from "incompressible" initial conditions are more consistently initialized with nonzero initial density and temperature fluctuations. One can speculate that initial conditions inconsistent with the variances associated with the incompressible pressure may create a wave field that may delay the decay of the transients or that may not even decay during the course of a DNS. These transients are analogous to the transients associated with the free motion of a second order system, $y'' + y' + y = 0$ relaxing from some initial condition. This situation appears to correspond to the analysis followed by Erlebacher *et al.* (1990) and Sarkar *et al.* (1991b). The dependence on the initial conditions in compressible isotropic simulations has been thoughtfully noted in Blaisdell *et al.* (1993). The crucial point, as is made clear by the analysis, is that the initialization of any calculation with zero fluctuating temperature, density or dilatation is inconsistent with a finite non-zero turbulent Mach number.

The present treatment for the effects of compressibility can be thought of as analogous to the forced system, $y'' + y' + y = f(\omega t)$; the forcing coming from the vortical motions of a turbulence with non-zero Mach number in which the effects of transients from initial conditions has faded. The analogy can be made exact, however, doing so is not relevant to the present subject.

The present article is organized in the following fashion: governing equations, analysis, discussion of physic of the results of the analysis, discussion of limitations and assumptions. The first two sections are fundamental in laying the ground work for the representations for the covariances with the

fluctuating dilatation: first a simple heuristic picture of the physics is presented after which a system of equations consistent with the physics presumed is derived. In the subsequent section and its four subsections, the assumptions of homogeneity and isotropy are exploited in obtaining analytical expressions for the desired covariances. The methods of statistical fluid mechanics, following the inceptional works of von Karman and Howarth (1938), Batchelor (1951), and Proudman (1952), are relied on extensively. As a byproduct of the section on the rapid pressure-dilatation correlation an expression for the pressure variance in an arbitrary three-dimensional mean flow is derived.

Later sections discuss the physical implications of the representations derived. Qualitative comparison is made to the physics that is known for several simple compressible turbulent flows. It is shown that the models show no effects of compressibility in the equilibrium adiabatic wall layer as is known to be the case. The document finishes with a summary of the limitations and assumptions built into the theory which suggest future work as well as the class of flows for which the representations are expected to be useful.

This article is meant to be primarily analytical. A simple perturbation analysis and the methods of statistical fluid mechanics are used to investigate the implications of a few simple and reasonable assumptions. The results are a mathematical consequence of the initial assumptions. The article is intended to be a documentation of this procedure and its implications. The objective is to providing metaphors and nondimensional numbers with which to understand and further explore diverse issues in compressible turbulence. Testing, verifying, exploring and evolving the present analytical results into a working turbulence model suitable for engineering calculations is the subject of a sequel work now in progress. These more quantitative issues are addressed in several works planned and in progress, Ristorcelli (1995), Ristorcelli *et al.* (1995).

2.1 A physical background for the mathematics

Before presenting the mathematical development leading up to the analytical expressions for pressure dilatation and the dilatational covariances a physical picture underlying and suggesting the mathematical development is described. A more formal and mathematical presentation is given in due course.

It is useful to keep in mind the one essential and central bit of physics that forms the lynchpin of the theory and makes the present method and results possible: in the near field of an acoustic source, whose size is small with respect to the wavelength of its emission, the fluid behaves as if it were incompressible. This observation appears to have been first made by Landau and Lipschitz (1958) and is a cornerstone in the method of matched asymptotic expansions in the field of acoustics.

Three basic ideas form the foundation of the pseudo-sound theory for the dilatational covariances.

The first is recognizing the problem as a singular perturbation problem - it has two different length scales. As it is the turbulence that creates the pressure and density fluctuations in the medium the frequencies of the compressible disturbances are the same as the frequencies of the turbulence, $c/\lambda \sim \tilde{u}/\ell$. The two length scales are: a correlation length scale associated with the fluctuations of the turbulence, ℓ, and a length scale $\lambda \sim \ell/M_t$ associated with the propagation of pressure and density fluctuations associated with the turbulent fluctuations. Here $M_t = (2k/3)^{1/2}/c$ is the turbulent Mach number where $k = 1/2 < u_j u_j >$ and $c_\infty^2 = \gamma p_\infty/\rho_\infty$ is the sound speed. The turbulent Mach number is to be used as the small parameter in expansions of the compressible equations to obtain representations for the effects of compressibility as manifested in the covariances with the fluctuating dilatation. Underlying the low M_t assumption which leads to the two disparate scales is what is called, in the sound generation problem of aeroacoustics, the *compact-source* assumption. This is equivalent to the idea that the flow structure covers a distance small with respect to the length scale of the compressible radiation it emits. Closely related to these two length scales are two time scales: one associated with the convective modes of the flow, say $\tilde{u}/\ell$, and the sound crossing time - the time it takes for a information to cross a typical scale of the turbulence, c/ℓ. Note that the conventional definition of the Mach number is used: it is the ratio of a characteristic fluctuating velocity to the (mean) sound speed. This is in concordance with the conventions of the acoustics literature from which some of our ideas are drawn.

The second idea concerns the pressure. In the sound generation problem, two pressures, an "acoustic" pressure which propagates and a pseudo-pressure associated with the convective motions of the fluid, are sometimes distinguished. The term pseudo-pressure was first coined by Blokhintsev (1956) as quoted in Ribner (1962). The term propagating pressure will be used for the term "acoustic" pressure so as not to imply that the problem is linear as the propagation of sound is assumed to be in the small disturbance limit. Without attempting to be precise, the pressure fluctuations in a fluid satisfy, to the degree suitable to the present heuristic discussion, the following wave equation, Lighthill (1952)

$$c_\infty^{-2} p_{,tt} - p_{,jj} = (\rho u_i u_j)_{,ij} \tag{2}$$

where p represents the deviations of the fluid pressure from its value at the static reference state. Care must be taken in assessing which solutions to this equation are relevant to compressible turbulence modeling. Solutions to this equation are comprised of the homogeneous solution, which obeys the sourceless wave equation and the wave equation with source due to the turbulent fluctuations. The sourceless wave equation, essentially the equation of linear acoustics, describes an

acoustic field resulting from certain specification of the boundary conditions or initial conditions. It has little to do with the vortical motions associated with the fluid turbulence that are the source of the propagating wave field. Following Ribner (1962) the fluid pressure is decomposed into its convective and propagating parts $p = p_c + p_p$ where p_c satisfies

$$- p_{c,jj} \;=\; (\rho u_i u_j)_{,ij} \tag{3}$$

and therefore p_p satisfies

$$c_\infty^{-2} p_{p,tt} \;-\; p_{p,jj} \;=\; -c_\infty^{-2} p_{c,tt}\,. \tag{4}$$

In the adiabatic limit the right hand side can be written as $(-\rho_{,tt})$ and is therefore related to changes in volume of the fluid element - the dilatation - that generates the propagating pressure. In the region of the fluid turbulence, for low turbulent Mach number, the pseudo-pressure is larger than the propagating pressure whose source is from fluid motions. Far from the turbulent portion of the fluid, the propagating pressure is the major portion of the pressure field as the pseudo-pressure, being associated primarily with the convective motions, decays rapidly. Thus, there is an inner region of scale ℓ in which the major portion of the pressure is associated with the vortical motions and an outer region, or an acoustic mantle of scale λ, in which the the propagating pressure is the major component of the pressure field. In the inner region of scale $\ell << \lambda$, the sound speed is effectively infinite: on a time scale of the flow, signals are felt throughout the region of scale ℓ effectively simultaneously. Which is to say that, in the near field, p_p satisfies the following Poisson equation

$$- p_{p,jj} \;=\; -c_\infty^{-2} p_{c,tt}\,. \tag{5}$$

These ideas, well-known in studies of sound generation, were first understood as a singular perturbation by Landau and Lipschitz (1958). A more formal presentation of these ideas is given in the following section. To obtain representations for covariances with the dilatation, only the *inner* solution of the singular perturbation problem, were the pseudo-pressure dominates, is used. This is consistent with the observations of Sarkar (1992), Blaisdell and Sarkar (1993), in which it has been found, numerically, that the incompressible portion of the pressure makes the largest contribution to the pressure-dilatation covariance.

The third idea is that equations should uniformly approach their incompressible form as the Mach number goes to zero with bounded derivatives. These facts are used to produce the gauge functions in a perturbation expansion in which the small parameter is related to the Mach number of the velocity fluctuations, the turbulent Mach number, M_t. This does not limit the theory to low mean

flow Mach numbers. In general, the aerodynamic problem requires an assessment of the effects of compressibility on a flow of arbitrary mean Mach number as felt through the compressible nature of the low Mach number turbulent fluctuations. In this way the turbulent Mach number dependence of the covariances is obtained by a systematic and consistent balance of terms in the compressible Navier Stokes equations. These are ideas have received additional amplification in the very useful and thought provoking work of Zank and Matthaeus (1991).

The use of the inner solution is a useful approximation in mediums that are finite or infinite in extent *for covariances involving at least one fluctuating quantity which does not propagate - whose source is local.* Contributions to the covariances from regions outside of the correlation length, the outer solution, are negligible. This is because there is no correlation between the local flow field in the region ℓ with quantities outside the region ℓ. Such is the case for $< pd >$ in which the major contribution to the fluctuating p is the local p_c.

This is not the case for covariances of the propagating field such as, for example, the variance $< dd >$, whose far field component may be larger than its near field if the size of the domain, D, of the flow $D/\lambda >> 1$ as is implicit in compressible homogeneous simulations. This raises some interesting and subtle ideas related to the physics of homogeneous numerical simulations: ideas relevant and important to the interpretation and use of homogeneous compressible DNS to calibrate models for flows of engineering interest occurring in finite domains.

The present article makes use of, what is called, the *compact flow* assumption: the size of the turbulent field, D, is small or on the order of the acoustic scale, $D/\lambda \leq 1$. Sound traveling through a flow on scales comparable to the wavelength of the emitted sound will begin to be scattered by the vorticity. In addition its accumulated effects on the flow will begin to modify the flow through which it is traveling. It is for this reason that acoustic analogies sometimes fail when they are used to predict the far field of an acoustic source after the signal has traversed the fluctuating medium for more than a few wavelengths. The present interest is in predicting covariances and this compact flow assumption is much less of a restriction than in acoustics as the length scale of the correlation naturally filters out signals coming from portions of the domain that are uncorrelated with the local vorticity. The compact flow assumption is still necessary in order to neglect covariances between propagating fields, such as $< dd >$.

While most flows of engineering interest can be categorized as compact flow problems, homogeneous compressible simulations do not fall into that category. Homogeneous DNS correspond, locally, to a turbulence immersed in a general random background wave field which will make contributions to the variances of a propagating field such as $< dd >$, even though the coherence between the

local turbulent field and the background field that propagates through it is small. For that matter, the local flow could be exactly incompressible and it would still experience a net drain of energy from the background dilatational wave field in which it is immersed. The work of Sarkar *et al.* (199b) has made progress on problems of this type. The model problem they appear to have solved in their approach is that of a turbulence of scale ℓ irradiated by an infinite external acoustic field generated by a turbulence whose statistics are the same as those of the local turbulent region. Their findings are related to flows in which the size of the turbulent field is, at least, λ and the field is also homogeneous on the λ scale (at least) because the assumption of homogeneity made in compressible DNS. While such simulations shed much basic insight on the effects of compressibility (this paper could not have been written without those insights) application of such results to models for compact flows of general engineering interest should be done with these possibilities in mind.

The assumption of homogeneity is made throughout the mathematical development: this is an assumption of homogeneity on the scale ℓ which is to say $\ell/L < 1$ where L is the scale of the region of homogeneity. This assumption coupled with the compact source and compact flow assumptions means that the theory is applicable to flows in which $\ell/\lambda < 1, L/\lambda < 1, D/\lambda \leq 1$. The article of Sarkar *et al.* (1991a) and compressible homogeneous simulations in general appears to be relevant to the problem in which $\ell/\lambda < 1$, and $L/\lambda > 1, D/\lambda > 1$.

This completes an intuitive background of the physics of the problem and is a useful perspective from which to view subsequent developments and more subtle side issues. A mathematically more formal statement of these ideas is now carried out.

2.2 The governing equations: a mathematical foundation

The following equations are used to describe the portion of the flow of interest:

$$\rho_{,t} + u_p \rho_{,p} = -\rho u_{p,p} \tag{6}$$

$$\rho u_{i,t} + \rho u_p u_{i,p} + p_{,i} = 0 \tag{7}$$

$$p/p_\infty = (\rho/\rho_\infty)^\gamma \tag{8}$$

For clarity of exposition the viscous terms are not carried: they can be shown to be of higher order for the compressible portions of the field, see for example Zank and Matthaeus (1991). This reflects the fact that the inner solution of the sound generation problem, on the small length scale, ℓ, is being sought and at these scales viscous effects which attenuate wave propagation are unimportant. Moreover, a spectral Mach number exhibits an approximate $\kappa^{-1/3}$ dependence and the scales of the motion responsible for the fluctuating dilatation will not be the scales of the flow influenced by viscosity.

The momentum and continuity equations can be combined to give the following equation

$$\rho_{,tt} - p_{,jj} = (\rho u_i u_j)_{,ij} \tag{9}$$

which becomes a wave equation for ρ or p if the gas law is used to eliminate one in favor of the other. The equation is left in this form for subsequent purposes. There are, of course, some limitations regarding the application of this set of equations to a general compressible flow. The most substantial is the assumed form of the gas law valid for isentropic flows: heated wall-bounded flows have strong temperature effects and as such the application of the theory in the near-wall region at high Mach number is, strictly speaking, not valid. The development presented here suggests a procedure for handling the problem in more complex flows with heat transfer.

Perturbing about a quiescent state, (p_∞, ρ_∞), the nondimensional forms of the pressure and density are taken as $p = p_\infty(1 + p')$, $\rho = \rho_\infty(1 + \rho')$. After rescaling the independent variables with $\ell/\tilde{u}$ and ℓ, and dropping the primes, the equations become

$$\rho_{,t} + u_p \rho_{,p} = -(1+\rho) u_{p,p} \tag{10}$$

$$(1+\rho) u_{i,t} + (1+\rho) u_p u_{i,p} + \epsilon^{-2} p_{,i} = 0 \tag{11}$$

$$p - \gamma\rho = 1/2\ \gamma(\gamma - 1)\rho^2 \tag{12}$$

$$\rho_{,tt} - \epsilon^{-2} p_{,jj} = [(1+\rho) u_i u_j]_{,ij} \tag{13}$$

where $\epsilon^2 = \gamma M_t^2$ and $M_t = \tilde{u}/c_\infty$ where $\tilde{u} = 2k/3 =< u_j u_j > /3$ and $c_\infty^2 = \gamma p_\infty/\rho_\infty$. Note that the choice of time scales is determined by the energy containing scales of the motion: it is a coarse grained time scale. The fine grained time scale of the problem includes some very interesting physics, but relevant only in a cumulative, to the construction of a statistical model. A meaningful balance, giving bounded first derivatives of the velocity, is established if $p \sim \epsilon^2$. It then follows that $\rho \sim \epsilon^2$ also. The conventional definition, in concordance with the acoustics literature from which some of our ideas are drawn, of the Mach number is used. It is the small parameter that emerges naturally in the relevant nondimensionalization of the compressible equations. The use of this symbol as the small parameter is only for this section; it will be used subsequently to denote the dissipation. Note that the conventional definition of the turbulent Mach number $M_t = (2k/3)^{1/2}/c$ means that it is a factor 0.816 or 0.577 smaller than the Mach numbers defined using k or $q^2 =< u_j u_j >$. Expansions of the form

$$p = \epsilon^2 \left[p_1 + \epsilon^2 p_2 + \ldots\right. \tag{14}$$

$$\rho = \epsilon^2 \left[\rho_1 + \epsilon^2 \rho_2 + \ldots\right. \tag{15}$$

$$u_i = v_i + \epsilon^2 \left[w_i + \epsilon^2 w_{2i} + \ldots\right. \tag{16}$$

are chosen. The gauge functions for the velocity are determined by the boundedness condition on the second derivative of the velocity, following the methodology of Zank and Matthaeus (1991), with different results. The only meaningful balance using a perturbation series in unit powers produces the same results. Note that this is not the linear acoustic scaling in which pressure and density disturbance scale as $p' \sim \rho_\infty \tilde{u}^2 \sim \rho_\infty w' c_\infty$, $\rho' \sim \rho_\infty M^2$ and thus the fluid velocity of a fluid particle associated with the passage of a wave is $w' \sim M\tilde{u}$.

Inserting the expansions into the equations produces, to the lowest two orders, the incompressible equations

$$v_{i,t} + v_p v_{i,p} + p_{1,i} = 0 \tag{17}$$

$$p_{1,jj} = -(v_i v_j)_{,ij} \tag{18}$$

$$p_1 = \gamma \rho_1 \tag{19}$$

where $v_{i,i} = 0$. Note that if the pressure fluctuations are scaled with velocity fluctuations then the last equations can be written as $\rho_1 = M_t^2\, p_1$. The correction for the compressibility of the flow, which does not involve a wave equation on the inner scales, is

$$\rho_{1,t} + v_p \rho_{1,p} = -w_{k,k} \tag{20}$$

$$w_{i,t} + v_p w_{i,p} + w_p v_{i,p} + p_{2,i} = \rho_1 (v_{i,t} + v_p v_{i,p}) \tag{21}$$

$$-p_{2,jj} = (w_i v_j + w_j v_i + \rho_1 v_i v_j)_{,ij} - \rho_{1,tt} \tag{22}$$

$$p_2 - \gamma \rho_2 = 1/2\, \gamma(\gamma - 1)\rho_1^2. \tag{23}$$

This is a statement of the fact that, over a region of size ℓ, the pressure signal is felt, effectively, instantaneously. Reflect on the fact that ϵ is a small parameter and that analysis will not adequately represent the effects of compressibility when shocklets are important. The analysis represents the effects of compressibility as a linear correction to the nonlinear zeroeth-order of incompressible turbulence problem. This completes the derivation of the evolution equations for the inner expansion. The full problem is the sound generation of acoustics and it requires matching the inner solution to an outer solution. For the single-point turbulence closure problem for the dilatational covariances the outer solution is not required. The above equations will be used to develop representations for the unknown terms. Some of the above relations may also be used to specify initial conditions for DNS that seek to investigate compressibility effects whose source is the turbulent velocity field and not initial conditions that reflect some other generation mechanism (for example, passage through a shock).

The terms that are sought in this study are various moments of the fluctuating dilatation $d = w_{j,j}$. The zeroeth-order equations show that the density fluctuations are given by the pressure

fluctuations, $\gamma\rho_1 = p_1$. The evolution equation for the density fluctuations now becomes a *diagnostic* relation for fluctuating dilatation,

$$-\gamma d \quad = \quad p_{,t} + v_p p_{,p} \,. \tag{24}$$

The subscript has been dropped. It is seen that one does not need to obtain a solution to the evolution equation for the compressible velocity field, w_i, in order to obtain its dilatation. A very nice result indeed †; it forms the kernel of the present pseudo-sound theory. The dilatation is diagnostically related to the *local* fluctuations of the pressure and velocity; it is the rate of change of the incompressible pressure field $p_{1,jj} = (v_i v_j)_{,ij}$, following a fluid particle. The pressure fluctuations which originate as a constraint to keep the eddies incompressible drives the near field dilatation. Note that scaling the fluctuating pressure with the mean energy of the velocity fluctuations indicates $d = M_t^2 \, [\, p_{,t} + v_p p_{,p} \,]$.

Constitutive relations for the pressure-dilatation and the dilatational squared covariances can be written by taking the appropriate moment of the fluctuating dilatation equation to produce, dropping the subscript,

$$-2\gamma < pd > \quad = \quad < pp >_{,t} \tag{25}$$

$$\gamma^2 < dd > \quad = \quad < \dot{p}\dot{p} > + 2 < \dot{p} v_q p_{,q} > + < v_p p_{,p} \, v_q p_{,q} > \tag{26}$$

for a homogeneous turbulence. The overdot is used to represent the time derivative when it appears within the brackets. It should be noted that the near field compressibility effects, as manifested in $< pd >$ and $< dd >$, have been directly linked to the incompressible velocity fields. This fact will be exploited to obtain expressions for the dilatational covariances in a turbulent flow.

3.1 Analysis for the pressure dilatation covariance in isotropic turbulence

The simplest form of the problem is now solved: expressions for the dilatational covariances for an isotropic turbulence without any mean deformation are obtained. For those familiar with the pressure strain covariance modeling in incompressible turbulence, this is analogous to the slow pressure component of the representation. The constitutive relationship for $< pd >$ is the starting point. Using the assumptions of isotropy, homogeneity and quasi-normality, an expression *with no undefined constants* can be obtained. The methods of statistical fluid mechanics similar to those of Batchelor (1951, 1953), Proudman (1952), and especially works of von Karman and Howarth (1938) are used.

†A similar expression in different contexts with different assumptions has been obtained independently by both S. Girimaji (1995) and S.Crow (1970).

Batchelor (1951) has obtained a representation for the pressure variance, $< pp >$, in an isotropic incompressible turbulence. Here, a simpler Greens function method, following Kraichnan (1956), is used. The pressure of interest satisfies the Poisson equation: $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$ from which it follows that the two-point pressure variance obeys

$$< p(x,t)p(x',t) >_{,jjp'p'} = \quad < v_i v_j v'_p v'_q >_{,ijp'q'} \tag{27}$$

which can be written as a differential equation in $r_i = x'_i - x_i$. Following the usual methods for translationally invariant random processes,

$$< p(x,t)p(x',t) >_{,jjpp} = \quad < v_i v_j v'_p v'_q >_{,ijpq} \tag{28}$$

Using the assumption of translational invariance (homogeneity) where now $< p(x,t)p(x',t) >=< pp' > (r)$. The Greens function for the equation is $-\frac{1}{8\pi} \mid r' - r \mid$ and the solution is expressed as

$$< pp' >= \quad -\tfrac{1}{8\pi} \int < v_i v_j v'_p v'_q >_{,ijpq} \mid r' - r \mid d^3 r' \tag{29}$$

The quasi-normal assumption is used to relate the fourth-order moment to the second-order moments. The adequacy of the quasi-normal assumption have been investigated over several years. Batchelor (1951) has presented evidence of its adequacy when invoked with respect to the large scales of the flow. A spectral version of this assumption is used in the EDQNM theory which since its inception, as presented in Orzag (1970), has produced very useful results. The adequacy of the assumption for the large scales of the flow has been documented in the several experimental works. McComb (1990) gives a summary of these results. In compressible flows the adequacy of the quasi-normal assumption for the large scales has been investigated in Sarkar *et al.* (1991a) and Blaisdell *et al.* (1993). The assumption produces

$$< v_i v_j v'_p v'_q >=< v_i v_j >< v'_p v'_q > + < v_i v'_p >< v_j v'_q > + < v_i v'_q >< v'_p v_j > . \tag{30}$$

The definition for the correlation, $< v_i v'_j >= \frac{2k}{3} R_{ij}(r)$, where $k = \frac{1}{2} < v_j v_j >$, is used to obtain

$$< v_i v_j v'_p v'_q >_{,ijpq} = 2 \ < v_i v'_p >_{,jq} < v_j v'_q >_{,ip} = \ 2 \, (\frac{2k}{3})^2 R_{ip,jq} R_{jq,ip}. \tag{31}$$

The pressure variance becomes

$$< pp' >= \quad -2 \, (\tfrac{2k}{3})^2 \tfrac{1}{8\pi} \int R_{ip,jq} R_{jq,ip} \mid r' - r \mid d^3 r'. \tag{32}$$

Continuity, the fact that $R_{ij,j}(r) = 0$, has been used. For isotropic turbulence the integral can be written in terms of the longitudinal correlation function. ¿From the inceptional paper of von Karman and Howarth (1938), the longitudinal correlation, $< v_1(0) v_1(r) >=< v_1 v_1 > f(r) =$

$\frac{2k}{3}f(r) = \frac{2k}{3}R_{11}$, allows the general two-point correlation to be written as $R_{ij} = -\frac{r_i r_j}{2r}f' + (f + \frac{1}{2}rf')\delta_{ij}$. The integrand can be written in terms of the scalar function $f(\xi)$. Following Batchelor's (1951) development the fourth-order two-point correlation can be expressed as

$$\begin{aligned} R_{ip,jq}R_{jq,ip} = & \quad 2\,[2f''^2 + 2f'f''' + \frac{10}{r}f'f'' + \frac{3}{r^2}f'^2] \\ = & \quad 2\,\frac{1}{\xi^2}\frac{d}{d\xi}[\frac{1}{\xi}\frac{d}{d\xi}(\xi^3 f'^2)]. \end{aligned} \tag{33}$$

Inserting into the integrand and applying integration by parts successively produces, returning to dimensional variables, the Batchelor (1951) result:

$$< pp > = \; 2\,(\frac{2k}{3})^2\rho_\infty^2\int_0^\infty \xi f'^2(\xi)d\xi \; = \; \frac{8}{9}\rho_\infty^2 k^2 I_1^s \tag{34}$$

where

$$I_1^s = \int_0^\infty \xi f'^2(\xi)d\xi \; . \tag{35}$$

Here, and henceforth, $\xi = r/\ell$ is the nondimensional spatial coordinate such that $\int f(\xi)d\xi = 1$. Inserting the result into the constitutive relation for the pressure dilatation,

$$-2\gamma < pd > = \; \frac{D}{Dt} < pp > \tag{36}$$

produces, in dimensional quantities, the following expression for the slow pressure dilatation,

$$< pd >^s = \; -\frac{2}{3}I_1^s\,\frac{D}{Dt}\,[< \rho > M_t^2 k] \tag{37}$$

after accounting for the normalization employed. Here, the undisturbed density and pressures have been replaced by by the local mean density and pressures. The turbulent Mach number is defined as $M_t^2 = \frac{2k}{3}/c^2$ where $c^2 = \gamma < \rho > /P$ is the local mean speed of sound.

3.2 Analysis for the variance of the dilatation in isotropic turbulence

The quasi-normal form of the constitutive relation for the variance of the dilatation is

$$\gamma^2 < dd > = \; < \dot{p}\dot{p} > + \; < v_p p_{,p}\, v_q p_{,q} > \; . \tag{38}$$

Starting, once again, from the nondimensional Poisson equation for zeroeth-order pressure field, $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$, an equation similar to the two-point variance of the pressure derived above can be obtained for the variance of the time derivative of the pressure by differentiation:

$$< \dot{p}(x,t)\dot{p}(x',t) >_{,jjpp} \; = \; < (v_i v_j)_{,t}\,(v'_p v'_q)_{,t} >_{,ijpq} \; . \tag{39}$$

The assumption of homogeneity has been used and the equation is written in terms of the usual spatial difference coordinate, r_i. Expanding the products of the time derivatives produces,

$$< (v_i v_j)_{,t}\,(v'_p v'_q)_{,t} >_{,ijpq} \; = \; 4 < \dot{v}_i v_j \dot{v}'_q v'_p >_{,ijpq} \tag{40}$$

and the differential equation for the variance becomes

$$< \dot{p}(x,t)\dot{p}(x',t) >_{,jjpp} \quad = \quad 4 \ < \dot{v}_i \dot{v}_p{}' >_{,jq} < v_j v_q' >_{,ip} . \tag{41}$$

The fact that $< \dot{v}_j v_q' >= 0$ for homogeneous isotropic turbulence, as can be seen by using the Navier Stokes equations to rewrite $\dot{v}_j$, has been used. The tensor $< \dot{v}_i \dot{v}_j' >$ can be written in terms of the correlation function, $< \dot{v}_i \dot{v}_j' >=< \dot{v}\dot{v} > \dot{R}_{ij}$ which can be rewritten in terms of the longitudinal correlation, f_1, where as usual, $\dot{R}_{ij} = -\frac{r_i r_j}{2r} f_1' + \ (f_1 + \frac{1}{2} r f_1')\delta_{ij}$ to produce $< \dot{v}_i \dot{v}_i' >=< \dot{v}\dot{v} > [3f_1 + \ r f_1'] =< \dot{v}\dot{v} > r^{-2}(r^3 \ f_1)'$. The bi-harmonic equation for the variance of the time derivative of the pressure becomes

$$< \dot{p}\dot{p}' >_{,jjpp} = \quad 8 \ \frac{2k}{3} < \dot{v}\dot{v} > \frac{1}{r^4}[\frac{1}{r^3}(r f' f_1')']'. \tag{42}$$

Using the Greens function method and integrating by parts produces, in dimensional form,

$$< \dot{p}\dot{p} >= \quad 4 \ \rho_\infty^2 \frac{2k}{3} < \dot{v}\dot{v} > \int_0^\infty \xi f' f_1' \ d\xi \tag{43}$$

An expression for the two-point variance of the acceleration, $< \dot{v}\dot{v} > f_1'$ is required. Little is known of the longitudinal correlation for the acceleration. The Navier Stokes equations can be used to obtain an equation relating the acceleration correlation, f_1, to f, the well-known longitudinal correlation of the two-point velocity correlation [‡]. The dynamical equations of the inviscid portions of the motion, in the absence of a mean velocity field, can be used to produce the following equation for the two-point covariance of the acceleration:

$$< \dot{v}_i \dot{v}_i' >= -\rho_\infty^{-2} < pp' >_{,jj} - < v_i v_j v_i' v_k' >_{,jk} \tag{44}$$

where the usual nomenclature of the homogeneous turbulence is in effect and the independent variable is the two-point separation r_i.

The quasi-normal expression for the last term on the right in the evolution equation for the two-point acceleration produces

$$\begin{aligned} < v_i v_j v_i' v_k' >_{,jk} = & \quad [< v_i v_j >< v_i' v_k' > + \ < v_i v_i' >< v_j v_k' > + \ < v_i v_k' >< v_i' v_j >]_{,jk} \\ = & \quad (\frac{2k}{3})^2 \ \frac{1}{r^2} \ [r^3 \ (f f'' - \frac{1}{2} f'^2 + \frac{4}{r} f f')]' \end{aligned} \tag{45}$$

after substituting in terms of the longitudinal correlation. For operations on functions of r, the Laplacian can be written $< pp' >_{,jj} = r^{-2} \frac{d}{dr}(r^2 \frac{d}{dr} < pp' >)$ and the dynamical equation for the two-point acceleration becomes, after one integration,

$$< \dot{v}\dot{v} > f_1 = -\frac{1}{\rho_\infty^2} \frac{1}{r} \frac{d}{dr} < pp' > - (\frac{2k}{3})^2 (f f'' - \frac{1}{2} f'^2 + \frac{4}{r} f f'). \tag{46}$$

[‡]This development was indicated to me by Y. Zhou.

¿From the expression for the two-point covariances for the pressure, Batchelor (1951), the following expression can be derived

$$\frac{1}{r}\frac{d}{dr} < pp' >= -4\rho_\infty^2 (\frac{2k}{3})^2 \int_r^\infty \frac{1}{r'} f'^2 dr' \tag{47}$$

Inserting Batchelors expression in the equation for f_1 and taking the derivative produces the quantity required,

$$< \dot{v}\dot{v} > f_1' = -(\frac{2k}{3})^2 (ff''' + \frac{4}{r}ff'' + \frac{8}{r}f'f' - \frac{4}{r^2}ff'). \tag{48}$$

Inserting the expression for $< \dot{v}\dot{v} > f_1'$ into the variance, $< \dot{p}\dot{p} >= \ 4\ \rho_\infty^2 \frac{2k}{3} < \dot{v}\dot{v} > \int_0^\infty \xi f'(\xi) f_1'(\xi) d\xi$ produces, after some manipulations,

$$< \dot{p}\dot{p} >= 4\ \rho_\infty^2 (\frac{2k}{3})^3 \ \frac{1}{\ell^2}\ I_2^s = \ \ \frac{9}{\alpha^2}\ \rho_\infty^2 (\frac{2k}{3})^2 \frac{\epsilon^2}{k^2} I_2^s = \ \ \frac{4}{\alpha^2}\ \rho_\infty^2 \epsilon^2 I_2^s, \tag{49}$$

where

$$I_2^s = \ \ \int_0^\infty \xi f'[ff''' + \tfrac{4}{\xi}ff'' + \tfrac{8}{\xi}f'f' - \tfrac{4}{\xi^2}ff']d\xi\ . \tag{50}$$

In the above expression the usual and empirically verified scaling $\epsilon \sim \tilde{u}^3/\ell$ has been used. The characteristic velocity fluctuation will be taken to be $\tilde{u}^2 = \frac{2}{3}k$ in which case $\epsilon = \alpha(\frac{2k}{3})^{3/2}/\ell$. Note that the coefficient of proportionality is twice as large as that when $\tilde{u}^2 =< u_1 u_1 >$ is used. The integral length scale is identified with the longitudinal correlation, $\ell = L_{11}$.

Work by Sreenivasan (1984) has indicated the utility of this expression for turbulent flows. A more recent and very timely article, Sreenivasan (1995), assesses the accuracy of this expression in several canonical (incompressible) simple shear flows. For homogeneous shear the data indicates $\alpha \sim 1-2$. For the log layer or wake flows $\alpha \sim 4$. For flows with smaller microscale Reynolds numbers Sreenivasan (1984, 1994) shows that $\alpha \sim R_\lambda^{-1}$. There is also a weak dependence on nondimensional shear rate.

The fourth-order moments in the constitutive expression $\gamma^2 < dd >=< \dot{p}\dot{p} > + \ < v_p p_{,p}\, v_q p_{,q} >$ are now treated. Beginning with the two-point statistic and writing it as a function of the separation distance, r_i,

$$\begin{aligned} < v_p p_{,p}\, v_q' p'_{,q'} >= \quad & -[< v_k v_q' >< pp' > + < v_p p >< v_q' p' > + < v_p p' >< v_q' p >]_{,pq} \\ = \quad & - < v_k v_q' >< pp' >_{,pq} \end{aligned} \tag{51}$$

where continuity, $< v_p v_q' >_{,p} = 0$, and the fact that any isotropic vector is zero have been used. Further manipulations and setting $r = 0$ produces

$$- < v_k v_q >< pp >_{,pq} = -\frac{2k}{3} < pp >_{,pp} - 2k b_{pq} < pp >_{,pq} \tag{52}$$

where b_{pq} is the anisotropy tensor, $b_{ij} =< v_i v_j > /2k - \frac{1}{3}\delta_{ij}$. It is zero in an isotropic turbulence. A theory including the contribution of the anisotropy of the turbulence would be possible assuming otherwise. Expressing the two-point covariance in terms of its longitudinal correlation function and performing the appropriate differentiations of $< pp' >=< pp > P(r)$ produces for the fourth-order moment:

$$-\frac{2k}{3} < pp >_{,pp} = \quad -\frac{2k}{3} < pp > 3P_0'' \tag{53}$$

The second derivative of Batchelors solution for the two-point pressure variance can be used to show that $P_0'' = -\frac{4}{\ell^2} I_3^s$ where

$$I_3^s = \int_0^\infty \frac{1}{\xi} f'^2 d\xi \tag{54}$$

The fourth-order moment can be written as

$$< v_p p_{,p}\, v_q p_{,q} >= \quad \frac{2k}{3} < pp > \frac{12}{\ell^2}\, I_3^s = 16\rho_\infty^2 (\frac{2k}{3})^2\, \frac{k}{\ell^2} I_1^s I_3^s = \frac{54}{\alpha^2}\rho_\infty^2 (\frac{2k}{3})^2\, \frac{\epsilon^2}{k^2} I_1^s I_3^s. \tag{55}$$

Batchelors result for the pressure variance and the empirically validated scaling $\epsilon = \alpha(\frac{2k}{3})^{3/2}/\ell$ have been used. The particular form of the expression is chosen in anticipation of later manipulations.

The above results are substituted into the constitutive equation for the variance of the dilatation - $\gamma^2 < dd >= < \dot{p}\dot{p} > + \ < v_p p_{,p}\, v_q p_{,q} >$ - to obtain the following simple expression for the slow portion of the representation for the variance of the dilatation

$$< dd >^s = \quad \frac{9}{\alpha^2} M_t^4 (\frac{\epsilon}{k})^2\, [I_2^s + 6 I_1^s I_3^s]. \tag{56}$$

The variance of the dilatation scales with the time scale of the large eddies of the spectral cascade. The integrals are, typically, order one quantities. Zhou (1995) has determined their value from high Reynolds number wind tunnel data. Values are given in the appendix. Following present conventions, in which the compressible dissipation is defined as $< \rho > \epsilon_c = \frac{4}{3} < \mu >< dd >$, the model can be put in a form more pertinent to the evolution of the kinetic energy of the turbulence. Using the definition of the Reynolds numbers, $R_t = \frac{4k^2}{9\epsilon < \nu >}$ to eliminate the viscosity the model can be written in a form appropriate for the kinetic energy equation

$$\epsilon_c^s = \frac{16}{3\alpha^2} \frac{M_t^4}{R_t}\, \epsilon_s\, [I_2^s + 6 I_1^s I_3^s]. \tag{57}$$

A complete summary of the models is given in a subsequent section and in the appendix.

3.3 Analysis for the pressure dilatation subject to mean velocity gradients

In this section an expression for the pressure dilatational covariances for a general homogeneous mean velocity gradient with no mean dilatation is derived. A similar procedure, in which frequent recourse is made to the Poisson equation for pressure, is followed. The Poisson equation now involves the mean velocity gradient. The nomenclature used in the pressure-velocity covariance modeling in incompressible second-order closures will be followed: the covariance will be called the rapid component of the pressure-dilatation covariances.

The constitutive relation for the pressure-dilatation is

$$-2\gamma < pd >= \frac{D}{Dt} < pp > \tag{58}$$

The velocity field is partitioned according to the Reynolds decomposition $V_i + v_i$; the upper case denoting a steady mean velocity field with constant gradients the lower case will continue to indicate the fluctuating field. The mean strain and rotation tensors are $S_{ij} = \frac{1}{2}\,[V_{i,j} + V_{j,i}]$, $W_{ij} = \frac{1}{2}\,[V_{i,j} - V_{j,i}]$; W^2 and S^2 denote the traces of the squares of these matrices. The nondimensional form of the Poisson equation for pressure is $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$. The fluctuating pressure is know given by the following Poisson equation:

$$p(x,t)_{,jj} = -\,(v_i V_j + V_i v_j + v_i v_j)_{,ij}\,.$$

Multiplying this equation by a similar Poisson equation for $p(x',t)$ and averaging produces

$$< p(x,t)p(x',t) >_{,j'j'qq} = \; 4V_{i,j}\,V_{p,q'} < v_{j,i}\,v'_{q,p'} > + < v_i v_j v'_p v'_q >_{,ijp'q'}\,.$$

Expressing the differential equation in terms of the spatial separation, r_i, produces a biharmonic equation for the two-point pressure variance

$$< pp' >_{,jjqq} = \; -\,4V_{i,j}\,V_{p,q} < v_j v'_q >_{,ip}\,. \tag{59}$$

The fourth order moment which represents the slow pressure contribution obtained in a previous section have been dropped. The Greens function method produces the following solution

$$< pp' > (r) = \; 4V_{i,j}\,V_{p,q}\,\frac{1}{8\pi}\int < v_j v'_q >_{,ip} |r - r'| d^3r' = \; 4V_{i,j}\,V_{p,q}\,I_{jqip}(r). \tag{60}$$

The dependence of the solution on the mean flow gradients has been expressed. The pressure variance is known once a representation for the integral I_{jqip} is found. For a class of turbulent flows a tensor polynomial in the anisotropy tensor is a suitable approximation for I_{jqip}. Ristorcelli *et al.* (1994, 1995a) includes a discussion of issues related to this assumption. Here, only the zeroeth-order term in such a polynomial will be retained for the purpose understanding the physics and obtaining

scalings for the compressibility effects. Higher order terms which scale with the anisotropy of the turbulence, $b_{ij} =< v_i v_j > /2k - \frac{1}{3}\delta_{ij}$ are neglected in these zeroeth-order expressions. Note that $b_{12} \sim 0.16$ is not atypical. At this point, and considering the purpose of the article, the additional algebra necessary to obtain the anisotropic contributions to the integral is not warranted. A fourth-order isotropic tensor possessing the proper symmetry and satisfying the continuity relation $I_{jiip} = 0$ at $r = 0$ is

$$I_{jqip} = A_1^r \, [\delta_{jq}\delta_{ip} - \frac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})]$$

where

$$A_1^r = \frac{2}{15} I_{jjii} = \frac{2}{15}\frac{1}{8\pi} \int < v_j v_q' >_{,ip} r' d^3 r' = \frac{1}{15}\frac{2k}{3}\ell^2 I_1^r. \tag{61}$$

Expressing the integrand in terms of the longitudinal correlation in the normalized coordinate, $< v_j v_j' >_{,ii} = \frac{2k}{3}\, \xi^{-2}[\xi^3 f''' + 7\xi^2 f'' + 8\xi f']$. The facts $< v_j v_j' >= \frac{2k}{3}[r f' + \ 3f] = r^{-2}\frac{d}{dr}(r^3 f)$ and, that in spherical coordinates, the Laplacian is $\nabla^2 = r^{-2}\frac{d}{dr}r^2\frac{d}{dr}$ have been used. It is also possible to integrate by parts allowing the integrand to be expressed in lower order derivatives for more accurate computation from experimental data. Thus

$$\begin{aligned} A_1^r = & \ \tfrac{1}{15}\tfrac{2k}{3}\ell^2 \textstyle\int_0^\infty \xi^2[\xi^2 f''' + 7\xi f'' + 8f']d\xi \\ = & \ \tfrac{1}{15}\tfrac{2k}{3}\ell^2 \textstyle\int_0^\infty \xi \, \tfrac{d}{d\xi}(\xi^2 \tfrac{d}{d\xi}(\xi^{-2}\tfrac{d}{d\xi}(\xi^3 f))) \, d\xi \\ = & \ \tfrac{2}{15}\tfrac{2k}{3}\ell^2 \textstyle\int_0^\infty \xi f d\xi = \tfrac{1}{15}\tfrac{2k}{3}\ell^2 I_1^r \end{aligned} \tag{62}$$

and the solution for the rapid pressure variance in an arbitrary three-dimensional mean velocity gradient can be expressed as

$$< pp >^r = \ \tfrac{1}{15}\rho_\infty^2 \tfrac{2k}{3}\ell^2 \, [3S^2 + 5W^2] \, I_1^r. \tag{63}$$

Note that the integral has dimensions of a characteristic correlation area: the rapid pressure contribution to the pressure variance will vary according to the spatial scale of the turbulence unlike the slow pressure contribution given. This dependence on the spatial scale was first noted by Kraichnan (1956) who solved the problem of the pressure fluctuations in an isotropic turbulence in unidirectional shear. The results here extend Kraichnans (1956) results for a planar unidirectional shear to an arbitrary mean deformation. A very interesting, and more modern, paper highlighting the physical and wavespace aspects of the results is George *et al.* (1984). The results are now substituted into the constitutive equations given above and the rapid portion of the pressure-dilatation covariance in dimensional variables is then given by

$$< pd >^r = -\frac{1}{30} \, I_1^r \frac{D}{Dt} \, [\rho_\infty \frac{2k}{3}\frac{\ell^2}{c_\infty^2} \, [3S^2 + 5W^2] \,]$$

Note the appearance of the quantity $S\ell/c_\infty$; the dependence of the compressibility effects on a deformation rate Mach number indicated by Lele (1994) and Sarkar (1994) and substantiated phenomenologically in Sarkar (1994). Note that the theory also predicts a dependence on a Mach number based on the mean rotation, $W\ell/c_\infty$. The expression is recast in terms of the turbulent Mach number and the representation for the rapid component of the pressure-dilatation covariance becomes

$$< pd >^r = -\frac{1}{30}\, I_1^r\, \frac{D}{Dt}\, [< \rho > M_t^2 \ell^2\, [3S^2 + 5W^2]\,] \tag{64}$$

It is seen that two effects contribute to the pressure-dilatation covariance: one due to the exchange between potential and kinetic modes of energy (since $M_t^2 \sim k$) and the other due to changes in the scale area of the correlation. Increases in the kinetic energy results in a transfer of the mechanical energy to the fluctuating pressure field. Similarly, increases in length scale, implying a decreased rate of cascade to the smaller scales, also transfers energy to the fluctuating pressure. Unlike the slow pressure-dilatation, however, the rapid-pressure-dilatation does not always have the opposite sign of the growth of kinetic energy but now depends on the rate of increase of the area of the correlation, ℓ^2.

In order to close the representation it is necessary to have an expression for ℓ. This introduces an element of empiricism; up to this point no phenomenological assumptions, other than the very reasonable quasi-normal assumption for the large scales, had been made. The usual heuristic approximation $\epsilon = \alpha(2k/3)^{3/2}/\ell$ where $\alpha \simeq 1 - 4$ produces

$$< pd >^r = -\frac{1}{30}\,(\frac{2}{3})^3 I_1^r\, \alpha^2 \frac{D}{Dt}\, [< \rho > k M_t^2 [3\hat{S}^2 + 5\hat{W}^2]\,] \tag{65}$$

Here the quantities with a carat are nondimensional deformation and rotation rates *eg.*, $\hat{S}^2 = (Sk/\epsilon)^2$.

3.4 Analysis for the variance of the dilatation subject to mean velocity gradients

In the constitutive relationship for the variance of the dilatation the time derivative is replaced by the mean advective derivative, $\frac{D}{Dt} = (\ )_{,t} + V_k(\ )_{,k}$ which comes from the mean portion of the advective terms in the expressions for the dilatational covariances. Carrying the substantial derivative as part of the time derivative term involves no approximation and follows quite naturally from the Reynolds decomposition. However it is necessary to carry out the development in a way that preserves Galilean invariance. The quasi-normal form of the constitutive relation for the pressure dilatation is

$$\gamma^2 < dd > = \quad <\mathring{p}\mathring{p}> \; + < v_p p_{,p}\, v_q p_{,q} >$$

The small circle will be used to indicate the mean convective derivative; for example in the equation above $<\overset{\circ}{p}\overset{\circ}{p}>=<\frac{Dp}{Dt}\frac{Dp}{Dt}>$.

An expression for $<\overset{\circ}{p}\overset{\circ}{p}>$ appearing in the the constitutive relations for the variance of the dilatation is obtained first. For the convenience of the presentation the two contributions to the variance of the dilatation will be denoted $\gamma^2 < dd >_1 = <\overset{\circ}{p}\overset{\circ}{p}>$ and $\gamma^2 < dd >_2 = < v_p p_{,p}\, v_q p_{,q} >$. Applying the Reynolds decomposition to the nondimensional form of the Poisson equation for pressure, $p(x,t)_{,jj} = -(v_i v_j)_{,ij}$, and taking the appropriate derivatives and dropping the terms quadratic in the fluctuating velocities (which were treated in an earlier section) produces

$$\overset{\circ}{p}(x,t)_{,jj} = -(\overset{\circ}{v}_i V_j + V_i \overset{\circ}{v}_j)_{,ij} = -2V_{i,j}\overset{\circ}{v}_{j,i}\,. \tag{66}$$

Multiplying this by a similar Poisson equation for $\overset{\circ}{p}(x',t)$ and averaging produces the biharmonic equation for the two-point pressure variance

$$<\overset{\circ}{p}(x,t)\overset{\circ}{p}(x',t)>_{,j'j'qq} = 4V_{i,j}V_{p,q'} <\overset{\circ}{v}_{j,i}\overset{\circ}{v}{}'_{q,p'}> = 4V_{i,j}V_{p,q} <\overset{\circ}{v}_j\overset{\circ}{v}{}'_q>_{,ip}\,. \tag{67}$$

The last equality has been written in terms of the separation variable, r_i. The Greens function solution procedure produces the following representation for the two-point variance

$$<\overset{\circ}{p}\overset{\circ}{p}{}'> = 4V_{i,j}V_{p,q}\frac{1}{8\pi}\int <\overset{\circ}{v}_j\overset{\circ}{v}{}'_q>_{,ip}|r-r'|d^3r' = 4V_{i,j}V_{p,q}I_{jqip}(r). \tag{68}$$

It is the variance that is required. Following the method discussed in the the previous section, the fourth order tensor, neglecting higher order corrections for anisotropy, is represented at $r=0$, as an isotropic tensor

$$\begin{aligned} I_{jqip} &= A_2^r[\delta_{jq}\delta_{ip} - \frac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})] \\ A_2^r &= \frac{\ell^2}{15}\int_0^\infty <\overset{\circ}{v}_j\overset{\circ}{v}{}'_j>_{,pp}\xi^3 d\xi = \frac{\ell^2}{15}\frac{2k}{3}I_2^r \end{aligned} \tag{69}$$

As has been noted little appears to be know about the two-point statistics of the acceleration. If the acceleration correlation were known it would be a simple matter to show that the integrand is given by $< v_j v'_j >_{,ii} = <\overset{\circ}{v}\overset{\circ}{v}> [\xi^3 f_1''' + 7\xi^2 f_1'' + 8\xi f_1']\xi^{-2}$ in the normalized coordinate. Unfortunately this is not the case and an expression for f_1 in terms of f is sought. The Navier Stokes equations without the viscous terms, which describe the energy containing range of the flow, will be used to obtain an expression for the integral, $\int_0^\infty <\overset{\circ}{v}_j\overset{\circ}{v}{}'_j>_{,pp}\xi^3 d\xi$. Taking the equation for $-\overset{\circ}{v}_i = v_k V_{i,k} + v_k v_{i,k} + p_i$, and multiplying it by a similar equation for $\overset{\circ}{v}{}'_j$, averaging and taking the trace produces, in the r_i coordinate,

$$\begin{aligned} <\overset{\circ}{v}_j\overset{\circ}{v}{}'_j> = & -V_{i,k}V_{i,q} < v_k v'_q > - < pp' >_{,jj} - [V_{i,k} < v_k v'_q v'_i >_{,q} + V_{i,q} < v_k v_i v'_q >_{,k}] \\ & - [< p v'_k v'_i >_{,ik} + < p' v_i v_k >_{,ik}] - [V_{i,k} < v_k v'_q v'_i >_{,q} + V_{i,q} < v_k v_i v'_q >_{,k}] \\ & - < v_j v_k v'_q v'_j >_{,kq} \end{aligned}$$

The two-point triple covariances are zero for homogeneous isotropic turbulence and the fourth-order correlation was treated in a previous section. The equation yields, after taking the Laplacian, the quantity sought:

$$<\mathring{v}_j \mathring{v}_j'>_{,pp} = \; - V_{i,k} V_{i,q} < v_k v_q' >_{,pp} - < pp' >_{,jjpp} . \tag{70}$$

In the previous section it had been shown that the two-point pressure variance satisfied the biharmonic equation: $< pp' >_{,jjqq} = \; 4V_{i,j} V_{p,q} < v_j v_q' >_{,ip}$. Thus

$$<\mathring{v}_j \mathring{v}_j'>_{,pp} = \; - \; V_{i,k} V_{i,q} < v_k v_q' >_{,pp} - \; 4V_{i,j} V_{p,q} < v_j v_q' >_{,ip} , \tag{71}$$

which upon multiplication by ξ^3 and integration produces the desired result for $\int_0^\infty <\mathring{v}_j \mathring{v}_j'>_{,pp} \xi^3 d\xi$ in the definition of I_2^r:

$$\frac{2k}{3} I_2^r = \; - \; V_{i,k} V_{i,q} I_{kq} - \; 4V_{i,j} V_{p,q} I_{jqip} \tag{72}$$

and I_2^r is seen to be related to the two integrals, $I_{kq} = \int_0^\infty < v_k v_q' >_{,pp} \xi^3 d\xi$, and $I_{jqip} = \int_0^\infty < v_j v_q' >_{,ip} \xi^3 d\xi$. The isotropic portion of these tensors are related to an earlier integral, I_1^r, defined in the previous section. The tensors have the following representations

$$\begin{aligned} I_{jqip} = & \; \frac{2}{15} I_1^r \; [\delta_{jq}\delta_{ip} - \frac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})] \\ I_{jq} = & \; \frac{1}{3} I_1^r \; \delta_{jq} . \end{aligned}$$

Inserting these expressions into the equation for I_2^r produces an expression for the two-point acceleration correlation integral in terms of the two-point velocity correlation integrals:

$$I_2^r = \frac{1}{30} \; I_1^r \; [13S^2 + 15W^2 \;] , \tag{73}$$

and the rapid pressure variance becomes

$$<\mathring{p}\mathring{p}>^r = \frac{1}{15}\frac{1}{30} \; [3S^2 + 5W^2] \; [13S^2 + 15W^2 \;] \; I_1^r \ell^2 \; \frac{2k}{3}. \tag{74}$$

Substituting $\ell = \alpha(2k/3)^{3/2}/\epsilon$, and inserting into $\gamma^2 < dd >_1 = <\mathring{p}\mathring{p}>$ which is related to the dilatational dissipation by $\epsilon_{c1}^r = \frac{4}{3}\nu < dd >_1 = <\mathring{p}\mathring{p}>$ produces

$$\epsilon_{c1}^r \; = (\frac{1}{15})^2 (\frac{2}{3})^5 \; \frac{M_t^4}{R_t} \; \epsilon \; [3\hat{S}^2 + 5\hat{W}^2] \; [13\hat{S}^2 + 15\hat{W}^2 \;] \; \alpha^2 I_1^r \tag{75}$$

after accounting for the nondimensionalizations employed.

An expression for the fourth-order moment, $< v_p p_{,p} \, v_q p_{,q} >$, appearing in $< dd >_2$ is now sought. In a previous section it was seen that under the quasi-normal and isotropic approximations that $< v_p p_{,p} \, v_q p_{,q} > = -\frac{2k}{3} < pp >_{,qq}$. The Greens function method produces

$$< pp' >_{,jj} = \; 4V_{i,j} V_{p,q} \; \frac{-1}{4\pi} \int < v_j v_q' >_{,ip} \frac{d^3 r'}{|r - r'|} = \; 4V_{i,j} V_{p,q} \, I_{jqip}(r) \tag{76}$$

where the biharmonic equation for the pressure variance from the previous section has been used. Following the usual procedures, at $r = 0$, with $I_{jqip} = A_3^r\,[\delta_{jq}\delta_{ip} - \frac{1}{4}(\delta_{ji}\delta_{qp} + \delta_{jp}\delta_{iq})]$ produces

$$< pp >_{,jj} = \quad \frac{2}{15}\frac{2k}{3}\,[3S^2 + 5W^2]I_3^r. \tag{77}$$

where

$$A_3^r = \frac{2}{15}I_{jjii} = \frac{2}{15}\frac{2k}{3}I_3^r \tag{78}$$

Using the facts $< v_j v_j' >= \frac{2k}{3}[rf' + \ 3f] = r^{-2}\frac{d}{dr}(r^3 f)$ and, that in spherical coordinates the Laplacian is $\nabla^2 = r^{-2}\frac{d}{dr}r^2\frac{d}{dr}$, produces $I_3^r = -\int_0^\infty \xi^2 f''' + 7\xi f'' + 8f'\,d\xi$. Reflection on the integrand will show that it is suitable for application of Gauss's theorem: the exact result $I_{jjii} =< v_j v_j >= \frac{2k}{3}3f(0)$ is possible and $I_3^r = 3$. The integral is nonetheless carried symbolically in the light of computations extending this theory to anisotropic turbulence. More pragmatically, the integral expression and its exact value, have been used to evaluate the accuracy of the numerical integration technique of experimental data.

The fourth-order moment becomes

$$< v_p p_{,p}\, v_q p_{,q} >= \ -\frac{2k}{3} < pp >_{,qq} = \frac{2}{15}(\frac{2k}{3})^2\,[3S^2 + 5W^2]I_3^r \tag{79}$$

and thus

$$< dd >_2 = \frac{2}{15}M_t^4\,[3S^2 + 5W^2]I_3^r. \tag{80}$$

which allows the second portion of the rapid dilatational dissipation to be expressed as

$$\epsilon_{c2}^r \ = \frac{3}{5}(\frac{2}{3})^5\,\frac{M_t^4}{R_t}\,\epsilon\,[3\hat{S}^2 + 5\hat{W}^2]\,I_3^r. \tag{81}$$

The rapid portion of the dilatation dissipation can be written as the sum $\epsilon_c^r = \epsilon_{c1}^r + \epsilon_{c2}^r$ and thus

$$\epsilon_c^r \ = (\frac{2}{3})^5\,\frac{M_t^4}{R_t}\,\epsilon_s\,[3\hat{S}^2 + 5\hat{W}^2][\frac{3}{5}I_3^r \ + \ (\frac{1}{15})^2[13\hat{S}^2 + 15\hat{W}^2]\,\alpha^2\,I_1^r]. \tag{82}$$

This concludes the analytical development of the representations for the pressure dilatation and the dilatational dissipation covariances for compact flows. There will be additional mathematical manipulations of the expressions obtained section in order to understand the implications of the analysis.

3.5 Summary of the dilatational covariance representations

The section is ended with a summary of the results of the analysis. The dilatational dissipation is comprised of a slow and a rapid part: $\epsilon_c = \epsilon_c^r + \epsilon_c^s$ where

$$\epsilon_c^s = \frac{16}{3\alpha^2}\frac{M_t^4}{R_t}\,\epsilon_s\,[I_2^s + 6I_1^s I_3^s]. \tag{83}$$

$$\epsilon_c^r = (\frac{2}{3})^5\,\frac{M_t^4}{R_t}\,\epsilon_s\,[3\hat{S}^2 + 5\hat{W}^2][\frac{3}{5}I_3^r\ +\ (\frac{1}{15})^2[13\hat{S}^2 + 15\hat{W}^2]\,\alpha^2\ I_1^r] \tag{84}$$

It is useful to reflect on the results of Blaisdell *et al.* (1993), Figure 12, in which it appeared that the dilatational dissipation could not be parameterizable solely in terms of the turbulent Mach number. The present analysis suggests its dependence, in simple shear flows, on two additional parameters, R_t and Sk/ϵ_s. The pressure-dilatation covariance is a sum of similar terms, $< pd >=< pd >^s + < pd >^s$,

$$< pd >^s = -\frac{2}{3}I_1^s\,\frac{D}{Dt}\,[< \rho > M_t^2 k] \tag{85}$$

$$< pd >^r = -\frac{1}{30}\,(\frac{2}{3})^3 I_1^r\,\alpha^2\frac{D}{Dt}\,[< \rho > kM_t^2\,[3\hat{S}^2 + 5\hat{W}^2]\,] \tag{86}$$

The constants, denoted by the I_i, in the these expressions are given by integrals of the longitudinal correlation:

$$\begin{aligned}
I_1^s &= \int_0^\infty \xi f'^2 d\xi \\
I_2^s &= -\int_0^\infty \xi f'[ff''' + \frac{4}{\xi}ff'' + \frac{8}{\xi}f'f' - \frac{4}{\xi^2}ff']d\xi \\
I_3^s &= \int_0^\infty \frac{1}{\xi}f'^2 d\xi \\
I_1^r &= 2\int_0^\infty \xi f d\xi. \\
I_3^r &= -\int_0^\infty \xi^2 f''' + 7\xi f'' + 8f'\,d\xi
\end{aligned}$$

Except for two very reasonable phenomenological assumptions the results presented above are a mathematical consequence of the assumptions that led to the diagnostic relationship: $-\gamma d = p_{,t} + v_p p_{,p}$. The assumptions used in developments subsequent to the diagnostic relationship are the quasi-normal behavior of the large scales and relationship relating length scale to dissipation. The analysis, apart from the quasi-normal behavior, verified in Blaisdell *et al.* (1993), has produced an exact but unclosed, in the context of single-point moment methods, result. The quasi-normal approximation relates the fourth-order moments of the velocity distribution to the second-order moments as if the large scales of the turbulence were Gaussian. To achieve closure an expression for the length scale is required; the very well established phenomenological relationship between

turbulence length scale and dissipation: $\ell = \alpha(2k/3)^{3/2}/\epsilon_s$ is used. With these qualifications in mind the expressions derived are, in the limit of a homogeneous isotropic turbulence, mathematically precise. The expressions here may be viewed as the leading order term in a more general expression in which successive terms scale with the anisotropy and inhomogeneity of the flow. It is expected that such an analysis for the dilatational covariances will, at the very least, predict the fundamental nondimensional parameters and scalings in the characterization of the effects of compressibility.

4.1 The physics embodied in the dilatational dissipation representations

Various aspects of the compressible dissipation representations are now discussed. The dilatational dissipation is comprised of a slow and a rapid part: $\epsilon_c = \epsilon_c^r + \epsilon_c^s$ where

$$\epsilon_c^s = \frac{16}{3\alpha^2}\frac{M_t^4}{R_t}\,\epsilon_s\,[I_2^s + 6I_1^s I_3^s]. \tag{87}$$

$$\epsilon_c^r = (\frac{2}{3})^5\,\frac{M_t^4}{R_t}\,\epsilon_s\,[3\hat{S}^2 + 5\hat{W}^2][\frac{3}{5}I_3^r \;+\; (\frac{1}{15})^2[13\hat{S}^2 + 15\hat{W}^2]\,\alpha^2\,I_1^r] \tag{88}$$

Immediately apparent, in contradistinction to other models for these terms, is the fact that the analysis predicts a dependence of the compressible dissipation on mean flow gradients and Reynolds number. The dependence on the Reynolds number suggests that assessing the importance of the dilatational dissipation on the basis of low Reynolds number numerical simulations may be misleading when applied to higher Reynolds number flows. Computation done with these models, for example Ristorcelli *et al* (1995) indicates that the major reduction in spread rate in the mixing layer, for example, is due to the pressure dilatation. The Mach number dependence is also stronger than the M_t^2 dependence in Sarkars model for the dilatational dissipation, and less steep than the exponential dependence of Zemans model.

The importance of the dilatational terms is difficult to assess *a priori* ; their scaling with M_t and R_t suggests that they are negligible. This is probably the case for the slow term. This is however not the case for the rapid term. Terms like $\hat{S}$ appearing in ϵ_c^r are typically in the range $0 - 10$; depending on the mean velocity gradients the dilatational dissipation may or may not contribute to a flow. The equilibrium (incompressible) homogeneous shear, for example, has $Sk/\epsilon \simeq 6$; the equilibrium log-layer $Sk/\epsilon \simeq 3.3$. For these planar flows $\hat{S}$ and $\hat{W}$ are the same. Using $\hat{S} \sim 10$ as an upper bound the quantity $[3\hat{S}^2 + 5\hat{W}^2] \sim 10^3$. When squaring it again, as occurs in the I_1^r term, the fact of its M_t^4 dependence is easily compensated.

The present mathematical development also shows the importance, in mean fields solely characterized by the mean strain and rotation, of a Mach number based on the mean velocity gradient.

Sarkar (1994) has defined a gradient Mach number as $M_g = S\ell/c$ where, for simple planar flows, $U_{i,j} = U_{1,2} = S$ and the length scale he used is transverse two-point correlation of the longitudinal velocity. In this article ℓ will be taken as the traditionally defined integral scale. Using the scaling $\ell = \alpha(2k/3)^{3/2}/\epsilon_s$ the dependence on the mean deformation can be related to the turbulent Mach number and the ratio of strain to correlation times: $S\ell/c = \alpha\frac{2}{3}\frac{Sk}{\epsilon_s}M_t = \alpha\frac{2}{3}\hat{S}M_t = M_S$. For a general three-dimensional flow the theory makes a distinction between a Mach number based on the mean strain and the mean rotation. A second gradient Mach number is defined $W\ell/c = \alpha\frac{2}{3}\frac{Wk}{\epsilon_s}M_t = \alpha\frac{2}{3}\hat{W}M_t = M_W$. For simple planar flows the mean rotation and strain are the same, $M_S = M_W$ and the mean gradient parameterization, $S\ell/c$ is complete. For flows characterized by mean pressure gradients and bulk dilatation additional work is required.

The rapid portion of the dilatational dissipation can be rewritten in terms of these two mean gradient Mach numbers as

$$\epsilon_c^r \sim \frac{M_t^2}{R_t}\,\epsilon_s\,[3M_S^2 + 5M_W^2][\frac{3}{5}I_3^r + (\frac{1}{15})^2[13\hat{S}^2 + 15\hat{W}^2]\,\alpha^2\,I_1^r]. \tag{89}$$

The structure of the model is seen to be similar to the Sarkar model, $\epsilon_c \propto M_t^2\epsilon_s$, but now with a coefficient that is not a constant but depends on the Reynolds number and the mean strain and rotation Mach numbers.

Sarkars (1994) heuristic reasoning can be used with success to indicate the behavior of the dilatational terms in different mean flows. Though Sarkars (1994) subject is the changes in the anisotropy of the turbulence due to compressibility, as indicated by the work of Abid (1993), his arguments are equally applicable to both the dilatational dissipation and the pressure-dilatation covariances. Sarkars (1994) arguments indicate that the effects of compressibility are much larger in the mixing layer than in the equilibrium boundary layer: the mixing layer is stabilized with respect to the boundary a layer by compressibility. The difference between the compressible mixing layer and the boundary layer flow can be parameterized in terms of a gradient Mach number, M_g. In Sarkars (1994) examples M_g (proportional to M_S) for the mixing layer can be an order of magnitude larger than that for the boundary layer. The same reasoning using the a mean gradient Mach number applied to the dilatational dissipation indicates that compressibility dissipation effects are substantially more important for the mixing layer than for the wall boundary layer. Using Sarkars (1994) values and definition of the gradient Mach numbers, $M_g \sim 6$ in a mixing layer while in the boundary layer $M_g \sim 1$ and the effects of the compressible dilatation are an order 6^2 more important in the mixing layer than in the boundary layer. A similar variation is seen using M_S to characterize the effects of compressibility.

Gradient Mach number type quantities have been identified in the works of Durbin and Zeman (1992), Cambon *et al.* (1993), and Lele (1994). The quantity has been given various physical interpretations. Durbin and Zeman (1993) have, in the context of a RDT theory for compressed turbulence, interpreted it as the change in Mach number over an eddy of scale ℓ. It can also be understood as the ratio of the acoustic propagation time across an eddy to the mean deformation time scale, Lele (1994). Reviewing the analysis indicates a more thermodynamic, rather than kinematic, origin and interpretation of the gradient Mach number. The $S\ell$ in the gradient Mach number appears because of the scaling of the rapid pressure integral with the area, as was first noticed by Kraichnan (1956). The c in $S\ell/c$ appears because of the linearization of the gas law relating the fluctuating p and ρ about the local mean pressure and density state. It sets the magnitude of the proportionality constant in the dimensional form of the diagnostic equation relating d and p. With these ideas in mind one is led to interpret the gradient Mach number as an indication of the relative magnitude of the pressure fluctuations (due to shear) to the dilatational fluctuations as set by the local mean density and pressure (which need not be adiabatically related).

Mention should also be made of the Wilcox (1992) analysis of the sensitivity of the flat plate friction coefficient, c_f, to models for the dilatational dissipation. His arguments show that models, such as the Sarkar *et al.* model (1991b) or Zeman (1990), predict effects of compressibility when in fact there are none. Sarkars (1991b) model, for example, undesirably reduces the skin friction in the compressible flat plate flow because of the modification of the effective von Karman constant. (It should be made clear that Sarkars model was intended for use in free shear flows such as the mixing layer: it was after all calibrated using the homogeneous DNS of a compressible shear). Wilcox's analysis has been repeated for the current model. The modifications to the von Karman constant of the present model are smaller than that of the incompressible form of the modeled equations. This is because of the near wall M_t^4 dependence.

The thoughtful reader will have noticed that the analysis has produced a representation for the dissipation that depends on the Reynolds number. The magnitude of the dilatational dissipation depends on the viscosity: for a fixed M_t, as $R_t \to \infty$, the dilatational dissipation vanishes. This is a rigorous consequence of the diagnostic relationship, $-\gamma d = p_{,t} + v_p p_{,p}$, derived from the perturbation method and subsequently employed to obtain the results. The initial assumptions lead to an expression for the dilatation that is related to the pressure fluctuations, an essentially inviscid phenomena. The dependence on the viscosity arises when one computes the compressible dilatation from its definition using the variance of the dilatation: $\epsilon_c = \frac{4}{3}\nu < dd >$.

It appears that, in the small turbulent Mach number limit, that the usual interpretation of dissipation type quantities as spectral fluxes is not appropriate for the dilatational dissipation. This is a finding which has caused some consternation and much consideration; it is, however, a mathematical consequence of the initial assumptions. A portion of the ideas arising from diverse discussions with colleagues will serve to make this result plausible; as will, perhaps, the fact that the results are consistent with EDQNM results. The utility of the results of the analysis, in the context of their application to computing engineering flows, is another and very different subject to be treated in a subsequent work.

That the compressible dissipation might not be interpreted as a spectral flux (as is the case of the solenoidal dissipation) is suggested by results given in the EDQNM of Bataille (1994). In Batailles (1994) simulation the energy spectrum is divided into its incompressible components and compressible components. The solenoidal spectrum, E_{ss}, is found to scale, as is usual, $\kappa^{-5/3}$; the compressible spectrum, for small Mach number, is much steeper and scales as $E_{cc} \sim \kappa^{-11/3}$. Multiplying by κ^2 the solenoidal and dilatational dissipations are found to scale as $\kappa^{1/3}$ and $\kappa^{-5/3}$. The negative power law scaling of the dilatational dissipation indicates that, unlike the solenoidal dissipation, the dilatational spectrum peaks in the lower wavenumber regions of the spectrum.

Such a point of view can also be understood by more heuristic arguments involving a spectral Mach number, $M_t^2(\kappa) \sim E(\kappa)\kappa/c^2$. Using the incompressible spectrum, as the compressible spectrum falls off faster, produces $M_t\ (\kappa) \sim k^{-1/3}$ suggesting that the dilatational dissipation is a result of a combination or competition of effects that are important at different scales of the motion: the energy in fluctuating dilatation at the large scales and the sharp gradients necessary for viscous dissipation at the small scales. This is intuitively consistent with the fact that, for fixed M_t, increasing R_t by decreasing the viscosity adds more small scales to the field that are also more divergence free. Thus, with the length at which the gradients are strong enough to undergo viscous dissipation becoming smaller and simultaneously more divergence free, there results a net reduction of the dilatational dissipation.

These ideas must be tempered with the fact that these results come from a low turbulent Mach number perturbation method - a linear perturbation about the nonlinear incompressible problem. This is not to say that physics related linearly to the velocity field exhibit a steeper spectral slope than the energy spectrum. The linearly related spectrum of a passive scalar temperature, which scales with $\sim \kappa^{-5/3}$, Tennekes and Lumley (1972), doesn't. The problem is more complicated than this and the analogy is inappropriate. While a low turbulent Mach number analysis is expected to be appropriate for most flows of aerodynamic interest, it must be remembered that the nonlinear self-

interaction terms of the compressible velocity field are absent in this analysis. As $M_t \rightarrow 1$ nonlinear terms and shocklets are expected to become important. This analysis has no relevance to flows in which shocklets are a major portion of the dissipation. On this point the ones attention might go to the homogeneous shear results done as part of G. Blaisdells thesis, published in Blaisdell *et al.* (1993); it was found that shocklets, for moderate turbulent Mach numbers, contributed very little to the dilatational dissipation. Note the difference in definition of the turbulent Mach numbers.

Batailles (1994) EDQNM results show that as the M_t increases the slope of E_{cc} decreases. Though outside the range of the validity of the simulation, the EDQNM simulation shows that as $M_t \rightarrow 1$, the slope of E_{cc} is approaches the slope of E_{ss}. These speculations concern phenomena outside of the range of validity of the linear theory and the EDQMN; within the range of validity the results of the perturbation theory are consistent with the EDQNM simulations. These EDQNM results have been found to be insensitive to the form of the small scale damping terms.

4.2 The physics of the pressure-dilatation covariance representations

The phenomenological implications of the pseudo-sound assumptions for the pressure-dilatation covariance are now explored. Unlike the compressible dissipation, which represents an irreversible transfer of energy, the pressure-dilatation represents a reversible transfer of energy between kinetic and internal modes. This reversible rate of transfer is proportional to the departure of the flow from equilibrium and with a simple rearrangement of terms the pressure-dilatation is seen to be equivalent to an increase in the flows inertia. Subsequent to this discussion the representations for the pressure-dilatation covariance are further manipulated to produce a final expression that is more easily understood and applied. The pressure-dilatation is seen to be proportional to the net imbalance of production, transport and dissipation of k and T. The pressure dilatation is also seen to be a function of the how rapidly the eddy turnover rate, k/ϵ_s, tracks then mean deformation and rotation, S and W.

The pressure-dilatation covariance is a sum of the slow and rapid terms $< pd >=< pd >^s + < pd >^s$ already given. The full pressure-dilatation covariance representation can be written as

$$< pd > \;=\; -I_{pd}\,\frac{D}{Dt}\,[< \rho > k M_t^2\,] \;-\; I_{pd}^r\,\frac{D}{Dt}\,[< \rho > k\;\; M_t^2\;\mathcal{T}] \tag{90}$$

where

$$\begin{aligned}
\mathcal{T} &= [3\hat{S}^2 + 5\hat{W}^2] \\
I_{pd} &= \frac{2}{3} I_1^s + I_{pd}^r\,\mathcal{T} \\
I_{pd}^r &= \frac{1}{30}(\frac{2}{3})^3\,\alpha^2 I_1^r.
\end{aligned}$$

Let $\mathcal{T}$ be called the relative cascade or eddy turnover rate. By expanding the differentials according to the product rule the expression for $< pd >$ can be suggestively rearranged in the k equation to produces terms representing an "added mass" effect as well as two additional source terms:

$$[1 + I_{pd} M_t^2] < \rho > \frac{D}{Dt} k \quad = \quad P_k - < \rho > \epsilon \; - k \; I_{pd} \frac{D}{Dt} [< \rho > M_t^2] - < \rho > k \; M_t^2 I_{pd}^r \frac{D}{Dt} \mathcal{T}.$$

The evolution of the kinetic energy experiences additional inertia. The flow appears to act with an added mass equal to M_t^2 times the weighted sum of the integrals: I_{pd} and I_{pd}^r. The additional inertia is due to the reversible transfer and storage of energy in the internal energy (mean temperature) field. This role of the pressure dilatation, as a transfer between internal and kinetic modes of energy, appears to have first been noticed by Zeman (1991) and explored further in homogeneous shear by Sarkar *et al.* (1991a). Thus the effect of the pressure-dilatation is to reduce the effects of production and dissipation unbalance by a factor $1 + I_{pd} M_t^2$.

Note also the appearance of the nonequilibrium and history effect, $\overset{\circ}{\mathcal{T}}$ and $\overset{\circ}{M}_t$. The first reflects how rapidly the eddy turnover time tracks the mean velocity gradients. While the second, $\overset{\circ}{M}_t$ reflects how rapidly the kinetic energy and mean internal energy adjust to each other. Thus $\overset{\circ}{k}$ is influenced by the rate at which the pressure dilatation can equilibrate the "potential" difference between the k and T fields. The appearance of relaxational effects incompressible flows have been noted in the calculations of Abid *et al.* (1995).

The mean deformation and mean rotation rate Mach numbers play a role in the added mass term. They appear in the product of $I_{pd} M_t^2$. Thus, the gradient Mach numbers, as manifested in the term $M_t^2 \mathcal{T}$, also effects the development of the flow by influencing the pressure-dilatation terms. The representation distinguishes the mixing layer and the boundary layer not only through the substantial derivative terms but also through the mean gradient Mach numbers. These terms are expected to make a difference primarily in flows with streamwise variations.

Additional analysis will now produce a final form for the pressure-dilatation representation. The fact that the pressure-dilatation covariance depends on the rate of change of M_t^2 shows that representation couples the kinetic energy equation to the internal energy equation. The pressure-dilatation covariances can be expressed in terms of k and T through the definition of M_t^2; this suggests that $\overset{\circ}{M}_t^2$ in the representation can be eliminated. The substantial derivative of M_t^2 is easily found from its definition and the ideal gas law to be

$$\overset{\circ}{M}_t^2 = M_t^2 \; [\frac{\overset{\circ}{k}}{k} - \frac{\overset{\circ}{T}}{T}]. \tag{91}$$

The mean temperature and k equations are, with the assumption of constant c_v,

$$< \rho > c_v \frac{D}{Dt} T = P_T - < pd > + < \rho > \epsilon_s + < \rho > \epsilon_c + T_T$$

$$< \rho > \frac{D}{Dt} k = P_k + < pd > - < \rho > \epsilon_s - < \rho > \epsilon_c + T_k$$

will be used to eliminate $\overset{\circ}{M}_t$ from the equations. Examination of the two equations for k and T shows some important effects. Both T and k receive energy from the mean flows kinetic energy through the production terms, P_T and P_k. The signs of PD and P_k can be of either sign in which case energy from the mean flow can be diminished or increased by interactions with the turbulence or with the mean internal energy of the flow. Note that if one insists on a Boussinesq eddy viscosity approximation for the Reynolds stresses that P_k allows only a one way transfer of energy from the mean to the turbulence. The dissipation terms, $< \rho > \epsilon_s + < \rho > \epsilon_c$, are always positive representing a flow of energy from the turbulence to the internal energy of the fluid. The pressure-dilatation, exchanging energy between kinetic and internal modes, on the other hand can be of either sign.

The pressure-dilatation covariance, for a flow with negligible mean dilatation, can be written as

$$< pd > = - < \rho > I_{pd} M_t^2 \, k \, [\, 2\frac{\overset{\circ}{k}}{k} - \frac{\overset{\circ}{T}}{T} \,] - < \rho > k \;\; M_t^2 \, I_{pd}^r \, \overset{\circ}{\mathcal{T}} \tag{92}$$

Using the equations for $\overset{\circ}{k}$ and $\overset{\circ}{T}$ and the definition of the Mach number in terms of temperature, $\frac{k}{c_v T} = \frac{3}{2} M_t^2 \gamma(\gamma - 1)$ produces a simpler and final and *almost* algebraic expression for the pressure-dilatation covariance:

$$\begin{aligned} < pd > = -\chi_{pd} M_t^2 \, [P_k - < \rho > \epsilon + T_k \;\; - \tfrac{3}{4} M_t^2 \gamma(\gamma - 1)(P_T + < \rho > \epsilon + T_T)] \\ - < \rho > k \;\; M_t^2 \, \chi_{pd}^r \, \overset{\circ}{\mathcal{T}}. \end{aligned} \tag{93}$$

The streamwise adjustment of the eddy turnover time scale to the mean velocity gradients remains. Here ϵ stands for the combined solenoidal and compressible contributions to the dissipation and

$$\chi_{pd} = \frac{2 I_{pd}}{1 + 2 I_{pd} M_t^2 + \frac{3}{2} I_{pd}^r M_t^4 \gamma(\gamma - 1)}$$

$$\chi_{pd}^r = \frac{I_{pd}^r}{1 + 2 I_{pd} M_t^2 + \frac{3}{2} I_{pd}^r M_t^4 \gamma(\gamma - 1)}$$

The χ coefficients are thus functions of the turbulent Mach number, M_t^2, and the relative turnover rate $\mathcal{T}$ through I_{pd}.

One of the failures of early models for compressibility effects in turbulent flows is their lack of universality. Current models capture the compressibility effects in the mixing layer, but the same

models, when applied to the near equilibrium boundary layer, add undesirable compressibility effects. Here the dependence of the results on the mean flow parameters, such as the mean strain and rotation Mach numbers and in D/Dt - as manifested in the production-dissipation balance and in $\overset{\circ}{\mathcal{T}}$ - show that the analytical representations do distinguish between these two classes of flows.

The effects of compressibility have been observed to be negligible in the unidirectional near equilibrium boundary layer flow. In near equilibrium flows there is an approximate balance between production, dissipation and transport and therefore $D/Dt \simeq 0$: the model then predicts, consistent with observations, $< pd > \simeq 0$. Particularly noteworthy are the predictions for the equilibrium log layer in which the transport terms negligible: for the log layer $P_k \simeq < \rho > \epsilon$ and $< pd > \simeq 0 + \mathcal{O}(M_t^4)$ is a further indicating that these compressibility effects are not important in equilibrium flows. In the mixing layer or jet, on the other hand, where production is an important quantity, the pressure-dilatation will shunt energy from the turbulence into the mean temperature thus reducing the level of the kinetic energy. This has been seen in laboratory experiments an also in calculations done using these analytical results. These calculations are the subject of studies addressing issues relevant to turbulence modeling and computations.

For flows in which production is not important the analysis indicates that the net effect of the pressure-dilatation is (if the contributions of the dissipation and transport to the energy budget have the same sign) is to *increase* the level of the the kinetic energy of the turbulence. Such situations arise in wake flows with and without momentum defects.

Expressions for the pressure-dilatation in two simple flows, the isotropic decay and the homogeneous shear are worth considering. Consider first an isotropic decaying turbulence. The pressure-dilatation covariance is

$$< pd > = \chi_{pd} M_t^2 \; [1 + \frac{3}{4} M_t^2 \gamma(\gamma - 1)] < \rho > \epsilon. \tag{94}$$

It is seen that the pressure-dilatation is positive indicating a net transfer of energy from the internal modes to the turbulence. After which the energy is, of course, dissipated by viscosity and returned to the internal energy of the fluid increasing its temperature. A portion of this energy, proportional to M_t^2 and the extent of the departure from equilibrium, can then once again transferred to the turbulence.

The $\overset{\circ}{T}$ and $\overset{\circ}{k}$ equations for the case of an isotropic decaying turbulence are written

$$\frac{D}{Dt} k = - (1 - \chi_{pd} M_t^2) \epsilon + \frac{3}{4} \chi_{pd} M_t^4 \gamma(\gamma - 1) \epsilon \tag{95}$$

$$c_v \frac{D}{Dt} T = + (1 - \chi_{pd} M_t^2) \epsilon - \frac{3}{4} \chi_{pd} M_t^4 \gamma(\gamma - 1) \epsilon \tag{96}$$

and the dependence on the production-dissipation balance and the turbulent Mach number are more readily seen. Note that the factor multiplying the dissipation is always positive, $1 - \chi_{pd} M_t^2 > 0$. Here $\epsilon = \epsilon_c + \epsilon_s$ and that the density has been set to unity for convenience of presentation. Note that the energy decay rate is explicitly dependent on the Reynolds number through ϵ_c. It is, of course, still implicitly dependent on the Reynolds number by the dependence of ϵ_s on the Reynolds number in a decaying turbulence. This issue has been inconclusively explored in Blaisdell *et al.* (1991) in an attempt to assess the effects of compressibility on the decay law.

The simulations of the isotropic decay by Blaisdell *et al.* (1991) and Sarkar *et al.* (1991) have indicated a strong dependence on the initial conditions. Neither of these simulations have used initial conditions consistent with the pseudo-sound analysis. This would require initial conditions in which $\gamma < \rho\rho >= \frac{\gamma}{\gamma-1} < \theta\theta >=< pp >$ where $< pp >$ is the variance of the incompressible pressure field. Which is to say that the compressible fluctuations are generated by the turbulence as opposed to imposed on the flow as an arbitrary initial condition. Even the choice of socalled "incompressible" initial conditions $< \theta\theta >=< \rho\rho >= 0$ is asymptotically inconsistent with finite initial M_t. This can be verified by expressing $\gamma < \rho\rho >=< pp >$ in primitive variables in which pressure is nondimensionalized by $\rho_\infty k$.

Blaisdell *et al.* (1993) has investigated the possibility a polytropic gas law, $n < \rho\rho >=< pp >$ where $< pp >$ is the total pressure variance. It is found that $n \simeq \gamma$ in the homogeneous shear and that, in the isotropic decay, n is initial condition dependent. One may well conjecture that a set of initial conditions as specified by this pseudo-sound theory might show $n \simeq \gamma$ subsequent to intialization. In which case Blaisdell *et al.* (1993) conclusion that algebraic models for the dilatational dissipation are inadequate may have to be qualified. We do concur with the Blaisdell *et al.* (1993) conclusion that in situations where the compressible component of the flow is arbitrarily specified by the initial conditions that such algebraic models will not work.

Sarkar *et al.* (1991) have modeled $- < pd > +\epsilon_c = \alpha_1 M_t^2 \epsilon_s$ and the turbulence energy equation can be rewritten

$$< \rho > \frac{D}{Dt} k \; =< pd > - < \rho > \epsilon_s - \; < \rho > \epsilon_c \; = -(1 + \alpha_1 M_t^2)\epsilon_s.$$

Keeping only order M_t^2 terms the present analysis gives for the turbulence energy equation, in apparent contradiction

$$\frac{D}{Dt} k \; = - \; (1 - \chi_{pd} M_t^2 \;)\epsilon_s \; .$$

The pressure-dilatation covariance is more important than the dilatational dissipation and will act to *slow* the rate of decrease of k by shunting energy from the internal modes (mean temperature)

where it is stored into the kinetic modes. The results of the present analysis and Sarkar *et al.* (1991) are both internally consistent: they treat two different problems. The present analysis treats a turbulence in which the compressible portions of the flow are generated by the turbulent motions. Sarkar *et al* (1991) treat a turbulence on which is superposed, by the initial conditions, an M_t^2 compressible velocity field. The postulated M_t^2 initial condition gives rise to an M_t^2 dilatational field. The effects of compressibility will reflect the evolution of the compressible field in the presence of an incompressible turbulence and the dilatational field will then be order M_t^2 causing an increase in the k decay. The dilatational field is an order M_t^4 effect when the dilatational fluctuations are generated, not by the initial conditions, but by the vortical fluctuations. These very interesting and potentially contradictory issues need to be investigated more closely with a DNS in which special care is taken with the implementation of the initial conditions.

Consider now a near equilibrium, $\overset{\circ}{T} \simeq 0$, homogeneous, $T_k = T_T = 0$, high Reynolds number, $P_T \simeq 0$ shear flow. In such a flow the pressure-dilatation is now

$$< pd >= -\chi_{pd} M_t^2 \left[P_k - < \rho > \epsilon - \frac{3}{4} M_t^2 \gamma(\gamma - 1) < \rho > \epsilon \right] . \tag{97}$$

There are several things worth noting. The first is the change of sign of $< pd >$ noted by Sarkar *et al.* (1991a). For flows with small turbulence production the pressure dilatation is positive. If the production is large and exceeds dissipation by a certain amount the pressure-dilatation covariance is negative and there is a net transfer of energy from the turbulence field to the mean internal energy. This is consistent with numerical results of Sarkar *et al.* (1991a) only here the analysis indicates when the change of sign of $< pd >$ occurs.

The $\overset{\circ}{T}$ and $\overset{\circ}{k}$ equations for the homogeneous shear flow can be written

$$\frac{D}{Dt} k = (1 - \chi_{pd} M_t^2)[P_k - \epsilon] + \frac{3}{4} \chi_{pd} M_t^4 \gamma(\gamma - 1)\epsilon \tag{98}$$

$$c_v \frac{D}{Dt} T = \chi_{pd} M_t^2 \left[P_k - \epsilon \right] + \epsilon - \frac{3}{4} \chi_{pd} M_t^4 \gamma(\gamma - 1)\epsilon \tag{99}$$

and the dependence on the production-dissipation balance and the the turbulent Mach number are more readily seen. Note that $\chi_{pd} M_t^2 < 1$. Thus, to lowest order, $\mathcal{O}(M_t^2)$, the effects of the compressible dilatation is to reduce the excess production over dissipation in the k equation by a factor $\chi_{pd} M_t^2$; this energy is transferred to internal modes and the rate of increase of the mean temperature is amplified by an additive factor $\chi_{pd} M_t^2 \left[P_k - \epsilon \right]$. The reduction of the turbulence related quantities seen in compressible flows appears to be attributable to the transfer of energy from kinetic to internal modes. Computations with this pressure-dilatation representation in the compressible mixing layer have shown it to be primarily responsible for the substantial reduction

in growth rates. Additional contributions come from the $\overset{\circ}{\mathcal{T}}$ relaxational term not carried in the equations above. These computations, as they represent the use of these representations for practical calculations, are the subject of articles now in progress whose objective is to evolve the present mathematical results into a computational model.

If in the representation for $< pd >$ one sets $< pd >= 0$ one can predict the critical M_{tc}^2, for a given shear rate, turbulent Reynolds number and anisotropy, at which the pressure-dilatation changes sign. Below this critical turbulent Mach number there is a net transfer of energy from the mean temperature field in which it is stored. Above this critical turbulent Mach number there is a transfer and storage of energy in the temperature of the fluid rather than increasing the kinetic energy of the turbulence. Using $P_k = \frac{1}{2} b_{12} k S$ and $P_T = 2 < \mu > S^2$ the critical Mach number as a function of the anisotropy and shear rate can be obtained. To zeroeth-order

$$M_t^2 \;=\; \frac{4}{3} \frac{\frac{1}{2}\, k\, b_{12}\, S - < \rho > \epsilon}{\gamma(\gamma-1)[2 < \mu > S^2 + < \rho > \epsilon]} \tag{100}$$

Recall that $\epsilon = \epsilon_s + \epsilon_c$ and that in fact the equation is a quadratic for M_t^2 if the dilatational dissipations contribution is included. If the dilatational dissipation is not distinguished a little more algebraic manipulation produces

$$M_t^2 \;=\; \frac{4}{3} \frac{\frac{1}{2}\, b_{12}\, \hat{S} - 1}{\gamma(\gamma-1)[1 + \frac{18}{4R_t} \hat{S}^2]} \tag{101}$$

This is essentially a statement of the fact that as long as the production exceeds the dissipation by and order M_t^2 quantity

$$P_k \;\geq\; [1 \;+ \frac{3}{4} M_t^2 \gamma(\gamma-1)] \;< \rho > \epsilon \tag{102}$$

that there is a net transfer and storage of energy in the mean temperature field. Which immediately suggests a numerical simulation of a homogeneous shear with an isotropic initial condition. At a critical Mach number $< pd >$ will change sign as a function of the anisotropy and Reynolds number of the flow.

5. Discussion and clarification of limitations and assumptions

A few assumptions have been made to obtain representations for the dilatational covariances. The assumptions are not in anyway unreasonable but do limit the applicability of the results to specific classes of flows. This section is a compendium of the assumptions; it exists in order to insure that the applications of these representations be made with an awareness of their limitations. It should also help asses how much of the physics these representations capture or neglect in any specific

flow. It is hoped that full disclosure of the assumptions will suggest future work to account for their potential shortcomings.

1) To model the effects of compressibility on the turbulence it has been assumed that the turbulent Mach number of the fluctuations, $M_t^2 = \frac{2}{3}\frac{k}{c^2}$, is small. This appears to be the case for several classes of supersonic flows of current engineering interest. The expressions derived are not expected to be useful for hypersonic flows in which eddy shocklets are important; such a flow situation will, in all likelihood, require a very different analysis.

As has been discussed, the low M_t assumption is equivalent to the *compact source* assumption of aeroacoustics. In the present context, this means that the correlation length scale of the flow structures producing the dilatational field is much smaller than the wavelength of the propagating field produced. This allows the dilatation to be algebraically related to the instantaneous material derivatives of the pressure fluctuations of the solenoidal field.

The low *turbulent* Mach number assumption should not be understood to imply a low *mean* flow Mach number.

It should also not, necessarily, imply a low gradient Mach number. The leading order contribution to the Reynolds stresses, in the low M_t limit, is from the solenoidal field and the lack of signal communication across an eddy, when the gradient Mach number is high, effects higher order corrections to the Reynolds stresses. The present analysis treats the leading order dilatational fluctuations. The form of the equations derived suggests a 'thermodynamic' rather than signal propagation interpretation of the the gradient Mach number; the gradient Mach number scales the relative magnitude of the dilatational fluctuations to the pressure fluctuations generated by the mean velocity gradients. Interpreting it as a quantity characterizing the propagation of information is expected to be important in large turbulent Mach number situations.

2) The scalings employed imply that the equations derived do not account for effects associated with phenomena that have coherences on much larger length scales. This would include variances and covariances of quantities that propagate. Such is the case with dilatation which has propagated into the local turbulence volume from regions more distant than the local integral scale. Though the correlation with the local turbulence is expected to be negligible, such signals are correlated with themselves and will make a contribution to the local dilatational dissipation. These effects are not accounted for in the present development; as such the development is limited to turbulence fields that are on order of or smaller than the acoustic scale of the flow. This is the *compact flow* assumption, $D/\lambda \leq 1$, and constitutes a limitation to the current representation. The limitation

may be nominal for many flows of current engineering interest in which the compact flow assumption is expected to be useful.

The compact flow assumption also excludes the cumulative effects, over scales large with respect to the wavelength, the propagating field on the turbulence, as sometimes occurs in the sound generation and propagation problem.

3) The covariances of the fluctuating dilatational field are assumed to be generated by and evolve in accordance with the turbulence field. They do not result from any externally imposed "acoustical" fields, or radiation from far field turbulence. Nor are they an adjustment of the flow to initial conditions with an acoustical or compressible component not generated by, or otherwise unrelated to the turbulent flow.

The scalings and analysis employed imply that the source of compressibility in the flow is due to the turbulence within an integral scale of the position in question. The effects of $q_c^2/q^2 \sim \mathcal{O}(M_t^2)$ compressible velocity fields, as seen in Sarkar *et al.* (1991), superposed on the flow by the initial conditions will produce much higher dilatational dissipation rates. This sensitive dependence of the dilatational dissipation rates on the initial conditions has been seen in the simulations of Blaisdell *et al* (1993). There are, in all likelihood, complex flow situations in which sizable compressible fields, q_c^2/q^2 are generated. This, however, is not the situation for which the present pseudo-sound theory has been developed.

Here compressibility effects due to the finite Mach number of the vortical fluctuations are studied. IN studies of such compressibility effects resulting from the turbulence appropriate initial conditions are required for DNS. Consistency requires that the density and temperature variances be related to the incompressible field according to $\gamma < \rho\rho >= \frac{\gamma}{\gamma-1} < \theta\theta >=< pp >$. Where, in dimensional terms, for an isotropic turbulence with no mean velocity gradients the pressure variance is

$$< pp >^s = \frac{8}{9}\rho_\infty^2 k^2 I_1^s \tag{103}$$

while for a turbulence with divergence free mean velocity gradients, to leading order,

$$< pp >^r = \frac{1}{15}\rho_\infty^2 \frac{2k}{3}\ell^2 \left[3S^2 + 5W^2\right] I_1^r = \frac{1}{15}\rho_\infty^2 (\frac{2}{3})^4 k^2 \left[3\hat{S}^2 + 5\hat{W}^2\right] I_1^r. \tag{104}$$

Or, in the context of a DNS starting from incompressible initial conditions, a point-wise proportionality between the fluctuating pressure and the density and temperature is required. The proper initial condition on the dilatational field is more difficult but may be much less important.

4) The mean pressure and mean density has been assumed locally constant - constant over a length scale over which the turbulence is correlated. This is also equivalent to the statement that the

sound speed is also locally constant; the stochastic nature of the fluctuations in the sound speed has been neglected. A corollary to the locally constant mean density assumption, is the fact that the mean dilatation is negligible. Thus, for flows in which the mean density and mean pressure vary appreciably over an integral scale the present representation is only a zeroeth-order theory.

5) The turbulent fluctuations that contribute to the dilatational covariances have been assumed to obey the adiabatic gas law. This uncouples the problem from noisentropic aspects of a compressible flow which may be important in wall bounded flows. A scale analysis of the fluctuating dilatation in the near wall region does appear to indicate that the nonisentropic contributions are higher order. This may not be the case for walls with large heat transfer.

6) All the expressions presented have been obtained assuming that the major contribution to the quantities come from the isotropic portions of the statistics of the fluctuating field. The expressions obtained are the lowest order expressions in a series expansion in powers of the single-point anisotropy of the turbulence. Higher order terms allowing for contributions from the anisotropy are straightforward in concept but complicated in execution. This has been found to be the case in a few cases for which such parameterizations have been worked out. Perhaps, if the results are to be used for engineering calculations, the constants derived should be viewed as requiring some modifications for the anisotropy of the flow. A DNS might be useful to see which of the several contributions to the dilatational covariances are most sensitive on the anisotropy of the turbulence field.

7) Throughout the development homogeneity has been invoked to make the statistical manipulation tractable. In practice this requires that the turbulent field be homogeneous on a scale $\ell/L < 1$ where L is the scale of the inhomogeneity. Clearly few engineering flows meet this requirement; however, any representation that is created must at least be consistent with results obtained using this state.

8) The assumption of quasi-normality has been made to achieve the statistical closure for the large scales of the flow. This involves the neglect of third-order moments with respect to second and fourth moments. This is an assumption that is extensively used and discussed throughout the literature. Corrections to the derived relations including the third-order moments are thought to be minor.

9) The spectrum of the turbulence is assumed to have a negative power law behavior implying that a spectral Mach number of the fluctuations decays with wave number. This suggests that the portion of the spectrum exhibiting compressibility effects are at the lower wave numbers. It

is for this reason that the viscous effects have not been carried in the analysis for the dilatational covariances. Such effects may well need to be incorporated in near wall flows.

10) The scale relation $\epsilon = \alpha(\frac{2k}{3})^{3/2}/\ell$ has been used several times. While having substantial empirical support in incompressible flows it is now being used in situations in which its validity must be assessed. Is $\epsilon = \alpha(\frac{2k}{3})^{3/2}/\ell$ valid for a compressible turbulence ? If not how must it be modified ? What is the best definition of a length scale with which to define a gradient Mach number ? Implicit in the present derivations has been a longitudinal length scale. Sarkar (1994) uses a transverse length scale; how does this compare to the longitudinal length scale of an isotropic turbulence. Should ϵ in the length scale definition include the compressible dilatation ϵ_c ? Clearly since the model is going to be used in sheared flows and near walls one must asses how accurate this relationship is for such situations. Sreenivasan (1984, 1994) has addressed the effect of shear on the relationship.

6. Summary and Conclusions

The mathematical consequences of a few assumptions about the size of the fluctuating pressure and density in a compressible turbulence are followed. A low turbulent Mach number singular perturbation has produced a *diagnostic* constitutive relationship relating the fluctuating dilatation to the fluctuating pressure and velocity fields. This constitutive relation is the lynchpin of the development allowing closure for the effects of compressibility in terms of the divergence-free portions of the fluctuating flow field. Moments of the constitutive relation produce analytically consistent representations for the dilatation variance and the pressure-dilatation covariance in a turbulence field with and without mean velocity gradients. Application of the methods of statistical fluid mechanics and the assumptions of quasi-homogeneity, quasi-normal behavior, and isotropy produces expressions for the covariances with the fluctuating dilatation. Except for the well-established empirical result, $\ell \sim (\frac{2}{3}k)^{3/2}/\epsilon_s$, used to close the expressions, no additional phenomenological assumptions are made. The analysis is, in the low M_t limit, exact and produces representations for the effects of compressibility in which there are no undefined constants. The constants that appear are known in terms of integrals of the longitudinal velocity correlation of an incompressible isotropic turbulence.

Both Lele (1994) and Blaisdell *et al.* (1994) have reflected that an algebraic closure for the effects of compressibility solely dependent on M_t^2 appears to be overly restrictive. The present analysis has quite naturally indicated the importance several additional parameters. The compressible dissipation is found to be a function of the local values of the turbulent Reynolds number, R_t, the turbulent Mach number, M_t, the two mean velocity gradient Mach numbers M_S and M_W, and the solenoidal dissipation. The pressure-dilatation is seen to be a nonequilibrium phenomena.

It is found to be a function of the *rate of change* of the turbulent Mach number, M_t, the mean density, $< \rho >$, the energy of the turbulence, k, and the two relative time scales Sk/ϵ_s and Wk/ϵ_s. Additional manipulations show that it can be expressed as a function of the production, the dissipation and the transport. There still remain the nonequilibrium effects associated with the adjustment time of the eddy turnover time to the time scale of the mean flow.

The analysis has produced a few simple and interesting metaphors for the effects of compressibility on a turbulent flows. For classes of turbulent flows for which this analysis is relevant, one example being the compressible mixing layer, the results suggest mechanisms that play a role in the reduction of the mixing layer growth rate. In short, the pressure-dilatation transfers turbulent kinetic energy to the internal energy field effectively reducing the relative excess of production over dissipation by a factor $\chi_{pd} M_t^2$. Further reduction of k occurs through the dilatational dissipation. These effects are dependent on the local mean flow gradients and as such suppress growth rates most in regions of high production.

It is hoped that these results will be of use in further understanding the complex effects of compressibility and stimulate additional new investigations including the assessment of the consequences and utility of the present pseudo-sound development. Some of the results worth noting are now summarized:

1) Noteworthy is the appearance of two mean flow Mach numbers based on the mean velocity gradients. One based on the mean deformation, $M_S = S\ell/c$, and another on the mean rotation, $M_W = W\ell/c$. These gradient Mach numbers have been identified as important parameters in assessing the effects of compressibility in the numerical experiments of Sarkar (1994).

2) The pressure-dilatation is essentially a reversible nonequilibrium phenomenon acting as a mechanism by which the fluctuating kinetic energy is transferred and stored in the internal energy field. The pressure-dilatation is shown to be interpretable as an added mass effect reducing the rate of change of turbulence quantities by the capacitance of the mean internal energy field. As such it may, in part, be responsible for some of the relaxational effects seen in compressible flows. This remains to be seen.

3) The rapid portion of the pressure-dilatation is seen to be a function of the rate of change of the relative times scales, Sk/ϵ_s and Wk/ϵ_s. These quantities may be interpreted as indicating how closely the eddy-turnover time tracks the mean velocity gradient. They occur because of the evolution of the length scale of the turbulence and its importance to the rapid pressure integral.

4) Unlike the solenoidal dissipation, the theory predicts that the compressible dilatation is a function of the Reynolds number and vanishes, for fixed turbulent Mach number, as the Reynolds number increase. The compressible dilatation, in the low M_t limit, doesn't appear to be interpretable as a spectral flux.

A brief overview of the more specific results of the theory can be found in the introductory section. Limitations and extensions of the theory have been indicated in the previous section. A summary of the representations is given in the appendix.

This article is a documentation of the physical implications of a pseudo-sound analysis for for the covariances of the fluctuating dilatation. The uniqueness of the investigation is the small number of phenomenological assumptions made. The results are a mathematical consequence of the initial assumptions. As an analytical investigation the article is complete. The assessment and utility of these results as models for engineering computations is the subject of several works planned or in progress, Ristorcelli (1995), Ristorcelli *et al.* (1995). Preliminary computations in a few simple benchmark flows have been successful.

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Appendix 1: Synopsis of the dilatational covariance representations

The analytical results, and associated mathematical formulae are, as an aid to the user, briefly summarized. It should be emphasized that, at this point, the representations do not constitute a fully developed or tested turbulence model. Testing, verifying and evolving the present analytical results into a working turbulence model suitable for engineering calculations is the subject of work now in progress.

The kinetic energy equation for the turbulence is

$$< \rho > \frac{D}{Dt}\, k = P_k - < \rho > \epsilon_s + < pd > - \; < \rho > \epsilon_c \; + \; T_k \tag{105}$$

where P_k represents the production and T_k represents the transport terms. The effects of compressibility are given by the second two terms. The dissipation $\epsilon_s = \nu < \omega_j \omega_j >$ is the usual dissipation associated with the vortical motions of the incompressible turbulence. The dilatational dissipation or the compressible dissipation and denoted $\epsilon_c = \frac{4}{3}\nu < dd >$. There are additional terms representing the the contraction of the dyad of the mass flux on the mean flow acceleration. T_k will be used to represent all such terms as well. Local isotropy has been assumed for the dissipation. The usual modeled dissipation equation

$$D/Dt\, \varepsilon = - \left(C_{\varepsilon 1} < u_i u_j > U_{i,j} + C_{\varepsilon 2} \varepsilon\right) \varepsilon / k + \; T_\epsilon \tag{106}$$

is carried to describe its evolution. Note that no corrections for compressibility have been made in this equation. The mean temperature equations , with the assumption of constant c_v, is

$$< \rho > c_v \frac{D}{Dt}\, T \quad = \quad P_T - < pd > + < \rho > \epsilon_s + < \rho > \epsilon_c + \; T_T$$

Where T_T is the transport of the mean temperature including such effects as the mean heat flux and the turbulent or pressure transport. The production for a homogeneous flow with homogeneous mean velocity gradients is $P_T = -PD + 2 < \mu > (U_{1,2})^2$. Here P is the mean pressure and D is the mean dilatation. Depending on the particular application the temperature equation may have more or fewer production terms than indicated here. Note that if one carries the total energy equation in a simulation, rather than the mean temperature, that that the pressure-dilatation only needs to be carried in the k equation.

The representations for the effects of the compressible dissipation are given by the sum of the slow and rapid portions, $\epsilon_c = \epsilon_c^r + \epsilon_c^s$.

$$\begin{aligned} \epsilon_c^s &= \quad \frac{16}{3\alpha^2} \frac{M_t^4}{R_t}\, \epsilon_s \; [I_2^s + 6 I_1^s I_3^s]. \\ \epsilon_c^r &= \quad (\frac{2}{3})^5 \; \frac{M_t^4}{R_t}\, \epsilon_s \; [3\hat{S}^2 + 5\hat{W}^2][\frac{3}{5} I_3^r \; + \; (\frac{1}{15})^2 [13\hat{S}^2 + 15\hat{W}^2]\, \alpha^2 \; I_1^r]. \end{aligned} \tag{107}$$

M_t is the turbulent Mach number, $M_t^2 = \frac{2}{3}k/c^2$, where $c_\infty^2 = \gamma < p > / < \rho >$ is the local sound speed. Note that its definition follows that used in the acoustic literature where such a Mach number is traditionally used to describe the sound generation problem by turbulence. It is the parameter that arises naturally in nondimensionalization of the equations for a small parameter expansion. The turbulent Reynolds number is given by $R_t = \frac{\tilde{u}\ell}{\nu} = \frac{4k^2}{9\epsilon\nu}$ using the facts that $\tilde{u} = 2k/3$ and $\epsilon_s = \alpha\tilde{u}^3/\ell$ which is used to express the length scale as, $\ell = \alpha(2k/3)^{3/2}/\epsilon_s$. For simple incompressible shear flows, the constant, α, varies between 1 - 4 depending on flow, Sreenivasan (1995). Note that in the definition the characteristic velocity $\frac{2}{3}k$ is used; not $< u_1 u_1 >$ as is sometimes the case. The nondimensional strain and rotation rates are given by: $\hat{S}^2 = (Sk/\epsilon_s)^2$, $\hat{W}^2 = (Wk/\epsilon_s)^2$ where of course, $S = \sqrt{S_{ij}S_{ij}}$ and $W = \sqrt{W_{ij}W_{ij}}$. The strain and rotation tensors are defined in analogy with the incompressible case, *ie.* traceless $S_{ij} = \frac{1}{2}[U_{i,j} + U_{j,i} - \frac{2}{3}D\delta_{ij}]$, $W_{ij} = \frac{1}{2}[U_{i,j} - U_{j,i}]$. Note that $S_{jj} = 0$ since $D = U_{j,j}$. In a simple planar shear flow, $U_{i,j} = U_{1,2}\,\delta_{i1}\delta_{2j}$ that $S^2 = W^2 = \frac{1}{2}U_{1,2}^2$. A quick of order of magnitude estimate for the integrals can be made using $f = e^{-\xi^2\pi/4}$. The following values are found: $I_1^s = \frac{1}{2}$, $I_2^s = \frac{41\pi}{27} = 4.77$, $I_3^s = \frac{\pi}{4} = 0.785$, $I_1^r = \frac{4}{\pi} = 1.273$, $I_3^r = 3$. The values found from high Reynolds number wind tunnel data are different: $I_1^s = 0.300$, $I_2^s = 13.768$, $I_3^s = 2.623$, $I_1^r = 1.392$, $I_3^r = 3$, Zhou (1995). The values given for the integrals reflect the assumption of an equilibrium isotropic turbulence and are to be understood as suggestive of the order of magnitude that they may have in more complex anisotropic and inhomogeneous situations.

The full pressure-dilatation covariance is a sum of two terms, $< pd > = < pd >^r + < pd >^s$,

$$
\begin{aligned}
< pd >^s = & \quad -\frac{2}{3}I_1^s\,\frac{D}{Dt}\,[< \rho > M_t^2 k] \\
< pd >^r = & \quad -\frac{1}{30}\,(\frac{2}{3})^3 I_1^r\,\alpha^2\frac{D}{Dt}\,[< \rho > k M_t^2[3\hat{S}^2 + 5\hat{W}^2]\,]
\end{aligned}
\tag{108}
$$

Summing and using the evolution equations for T and k produces the following quasi-algebraic representation for the full pressure-dilatation covariance,

$$
\begin{aligned}
< pd > = \quad & -\chi_{pd} M_t^2\,[P_k - < \rho > \epsilon + T_k - \tfrac{3}{4}M_t^2\gamma(\gamma-1)(P_T + < \rho > \epsilon + T_T)] \\
& - < \rho > k \;\; M_t^2\;\chi_{pd}^r\;\tfrac{D}{Dt}\mathcal{T}
\end{aligned}
$$

$$
\begin{aligned}
\chi_{pd} &= \frac{2I_{pd}}{1 + 2I_{pd}M_t^2 + \frac{3}{2}I_{pd}^r M_t^4\gamma(\gamma-1)} \\
\chi_{pd}^r &= \frac{I_{pd}^r}{1 + 2I_{pd}M_t^2 + \frac{3}{2}I_{pd}^r M_t^4\gamma(\gamma-1)} \\
I_{pd} &= \frac{2}{3}I_1^s + I_{pd}^r\,[3\hat{S}^2 + 5\hat{W}^2] \\
I_{pd}^r &= \frac{1}{30}(\frac{2}{3})^3\,\alpha^2 I_1^r.
\end{aligned}
\tag{109}
$$

Note that $\epsilon = \epsilon_s + \epsilon_c$ and $\mathcal{T} = [3\hat{S}^2 + 5\hat{W}^2]$. The term inside the inner brackets is the right hand side of the mean temperature equation.