REDUCED STRESS TENSOR AND DISSIPATION
AND THE TRANSPORT OF LAMB VECTOR *

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Abstract

We develop a methodology to ensure that the stress tensor, regardless of its number of independent components, can be reduced to an exactly equivalent one which has the same number of independent components as the surface force. It is applicable to the momentum balance if the shear viscosity is constant. A direct application of this method to the energy balance also leads to a reduction of the dissipation rate of kinetic energy. Following this procedure, significant saving in analysis and computation may be achieved. For turbulent flows, this strategy immediately implies that a given Reynolds stress model can always be replaced by a reduced one before putting it into computation. Furthermore, we show how the modeling of Reynolds stress tensor can be reduced to that of the mean turbulent Lamb vector alone, which is much simpler. As a first step of this alternative modeling development, we derive the governing equations for the Lamb vector and its square. These equations form a basis of new second-order closure schemes and, we believe, should be favorably compared to that of traditional Reynolds stress transport equation.

1. Introduction

Fluid dynamics needs very intensive analyses and computations, and a reduction of this work can be very useful. In this paper, we demonstrate that this can be achieved for the balance of momentum and energy, for both laminar and turbulent flows, and this reduction has roots in the very fundamental nature of fluid motion. Furthermore, our analysis suggests that the mean turbulent Lamb vector has advantages over the traditional Reynolds stress; its transport equation is derived as a foundation for an alternative approach in turbulence modeling.

2. Reduced Stress Tensor for Newtonian Fluid

Let $\mathbf{S}$ and $\mathbf{S}'$ be two arbitrary differentiable second-rank tensors. If their difference is a divergence-free tensor, there must be $\nabla \cdot \mathbf{S} = \nabla \cdot \mathbf{S}' = \boldsymbol{F}$, say. We call $\mathbf{S}$ and $\mathbf{S}'$ the tensor potential of the vector $\boldsymbol{F}$; clearly, the number of potentials for $\boldsymbol{F}$ is infinite. This gives us a chance to construct the simplest potential, whose number of independent components should be the same as that of $\boldsymbol{F}$. As a theorem, this construction is achievable through making the Stokes-Helmholtz (S-H) decomposition for $\boldsymbol{F}$:

$$\boldsymbol{F} = \nabla \phi + \nabla \times \boldsymbol{A}, \quad \nabla \cdot \boldsymbol{A} = 0, \tag{1}$$

where ϕ and $\boldsymbol{A}$ are the scalar and vector S-H potentials of $\boldsymbol{F}$, and have altogether three independent components. It is well known that decomposition (1) exists globally for any integrable $\boldsymbol{F}$. Now, let $\mathbf{I}$ be the unit tensor and write

$$\widehat{\mathbf{S}} = \phi \mathbf{I} + \mathbf{I} \times \boldsymbol{A}, \quad \text{or} \quad \widehat{S}_{ij} = \phi \delta_{ij} - \epsilon_{ijk} A_k, \tag{2}$$

then $\nabla \cdot \widehat{\mathbf{S}}$ is nothing but (1). This $\widehat{\mathbf{S}}$ is exactly the desired three-component tensor potential of $\boldsymbol{F}$, which we refer to as the *S-H tensor potential* of $\boldsymbol{F}$. Note that it consists of an anisotropic part and a skew-symmetric part, whereas itself is neither symmetric nor skew-symmetric.

Since the S-H decomposition stands at the center of our reasoning, we make some remarks on it. It is well known that for a given vector $\boldsymbol{F}$, finding its potentials ϕ and $\boldsymbol{A}$ amounts to solving a scalar and a vector Poisson equations, of which the integral representation is in terms of the Green's function of Laplace operator. However, this classical representation is often inconvenient in practice. If the flow is unbounded or with periodic boundary conditions, ϕ and $\boldsymbol{A}$ can of course be easily found in the Fourier space; but the best approach is using the helical-wave decomposition (HWD), with the eigenvectors of the curl operator forming a complete orthonormal basis. In fact, HWD is nothing but a further sharpening of the S-H decomposition.[1] Whereas the existing theory and application of HWD have been confined to unbounded flow, for which the basis vectors can be easily obtained in Fourier space or in terms of some special functions,[1,2,3] it has been proven by

Yoshida and Giga[4] that the complete orthonormal HWD basis also exists in an arbitrary bounded domain. In this paper we shall not go into these details (see Wu et al.[5]); rather, we simply assume that, in any flow domain and for any vector of our concern, the splitting of this vector into the longitudinal (curl-free) part and transverse (divergence-free) part can be obtained readily using the procedures discussed above.

Now, consider a Newtonian fluid, of which the viscous stress tensor is proportional to the strain-rate tensor $\mathbf{D} = \mathbf{D}^T$, where the superscript ^{T} means transpose. Let $\vartheta = \nabla \cdot \boldsymbol{u}$ and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ be the dilatation and vorticity, respectively, and $\boldsymbol{\Omega} = -\boldsymbol{\Omega}^T$ be the anti-symmetric spin tensor such that $\nabla \boldsymbol{u} = \mathbf{D} + \boldsymbol{\Omega}$. Moreover, let μ and λ be the first and second dynamic viscosities. Then, it is easily seen that there exists an identity[6−8]

$$\mathbf{D} = \vartheta \mathbf{I} + \boldsymbol{\Omega} - \mathbf{B},$$

where

$$\mathbf{B} \equiv \vartheta \mathbf{I} - \nabla \boldsymbol{u}^T \quad \text{with} \quad \nabla \cdot \mathbf{B} = 0 \qquad (3a, b)$$

is known as the surface-strain-rate tensor (Dishington[9], whose definition differs from (3a) by a transpose). Therefore, from the Cauchy-Poisson constitutive equation one obtains an intrinsic *triple decomposition of the stress tensor* $\mathbf{T}$:

$$\mathbf{T} = -\Pi \mathbf{I} + 2\mu \boldsymbol{\Omega} - 2\mu \mathbf{B}, \qquad (4a)$$

where

$$\Pi \equiv p - (\lambda + 2\mu)\vartheta \qquad (4b)$$

is the isotropic part of $\mathbf{T}$. Correspondingly, we have an intrinsic triple decomposition of the surface stress (the traction) at any surface element of unit area in the fluid or on its boundary:

$$\boldsymbol{t} \equiv \boldsymbol{n} \cdot \mathbf{T} = -\Pi \boldsymbol{n} + \boldsymbol{\tau} + \boldsymbol{t}_s, \qquad (5a)$$

where $-\Pi \boldsymbol{n}$ is the normal stress due to compression/expansion, and

$$\boldsymbol{\tau} \equiv \mu \boldsymbol{\omega} \times \boldsymbol{n}, \quad \boldsymbol{t}_s \equiv -2\mu \boldsymbol{n} \cdot \mathbf{B} \qquad (5b, c)$$

are the shear stress and stress due to the *resistance of the viscous fluid surface to its strain*, respectively.

It is now evident that, as long as $\mu = $ constant, then $\mu \mathbf{B}$ is divergence-less and can well be dropped off from the momentum equation. What left is precisely the three-component S-H tensor potential

$$\widehat{\mathbf{T}} = -\Pi \mathbf{I} + 2\mu \boldsymbol{\Omega}, \qquad (6a)$$

such that the Navier-Stokes equation with an external body force $\rho \boldsymbol{f}$,

$$\rho \frac{D\boldsymbol{u}}{Dt} - \rho \boldsymbol{f} = \nabla \cdot \widehat{\mathbf{T}} = -\nabla \Pi - \nabla \times (\mu \boldsymbol{\omega}) \qquad (6b)$$

represents a natural S-H decomposition of the total body force (inertial plus external).

Although equation (6b), in particular its incompressible version, has been known for long time,[10,11] *the concept of reduced stress tensor (6a) is not widely appreciated, and its application in practice has not been fully explored.* The only exception known to the authors is the work of Eraslan *et al.*[12], who, in developing a numerical method, noticed that some components of T_{ij} played no role in their scheme and simply ignored them. A cubic mesh box, originally having 27 control points, was thereby simplified to having 7 points only. This reduced the CPU time for estimating stress by 70%, and for the overall computation, by 40%. This example indicates that the saving due to *intentionally* replacing $\mathbf{T}$ by $\widehat{\mathbf{T}}$ can be very significant.

3. Self Balance of Surface Strain and Reduced Dissipation

In contrast to the momentum balance with constant μ, where the surface-strain tensor $\mathbf{B}$ simply plays no role, we now show that in the energy balance it does play a role but is always self-balanced. This interesting point has never been noticed before.

The conventional energy balance reads

$$\rho \frac{D}{Dt} \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) = \rho \boldsymbol{f} \cdot \boldsymbol{u} + \vartheta p + \nabla \cdot (\mathbf{T} \cdot \boldsymbol{u}) - \Phi, \tag{7}$$

where

$$\Phi = 2\mu D_{ij}(T_{ij} + p\delta_{ij}) = \lambda \vartheta^2 + 2\mu D_{ij} D_{ij} \tag{8a}$$

is the dissipation function. Using an identity of Truesdell,[10] we found

$$\Phi = (\lambda + 2\mu)\vartheta^2 + \mu \omega^2 - \nabla \cdot (2\mu \mathbf{B} \cdot \boldsymbol{u}). \tag{8b}$$

Therefore, substituting $\mathbf{T} = \widehat{\mathbf{T}} - 2\mu \mathbf{B}$ into (7), the terms containing $\mathbf{B}$ are precisely canceled. This result implies:

Theorem. *For Newtonian fluid with constant shear viscosity, the work rate done by the viscous resistance to surface strain is locally balanced by its own dissipation rate.*

Like the skin-friction, the viscous resistance of a fluid-surface element to its strain usually does negative work, and hence its dissipation, $-\nabla \cdot (2\mu \mathbf{B} \cdot \boldsymbol{u}) \equiv \Phi_s$, say, is positive. But, this may not be always so.

Now, due to the theorem as well as (6), for constant μ, the balance of both momentum and energy of a Newtonian fluid can always be replaced by that of a *fictitious fluid* which has no surface strain at all. Indeed, the energy balance now reads

$$\rho \frac{D}{Dt} \left(\frac{1}{2} |\boldsymbol{u}|^2 \right) = \rho \boldsymbol{f} \cdot \boldsymbol{u} + \vartheta p - \nabla \cdot (\boldsymbol{u} \Pi + \mu \boldsymbol{\omega} \times \boldsymbol{u}) - \widehat{\Phi}, \tag{9a}$$

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or, for any material volume V,

$$\frac{D}{Dt}\int_V \frac{1}{2}\rho|\boldsymbol{u}|^2 dV = \int_V (\rho\boldsymbol{f}\cdot\boldsymbol{u} + \vartheta p)dV + \int_{\partial V} \boldsymbol{u}\cdot\widehat{\boldsymbol{t}}dS - \int_V \widehat{\Phi}dV, \qquad (9b)$$

where

$$\widehat{\Phi} \equiv (\lambda + 2\mu)\vartheta^2 + \mu\omega^2 \geq 0 \qquad (10)$$

is the *reduced dissipation*, and

$$\widehat{\boldsymbol{t}} = \boldsymbol{n}\cdot\widehat{\mathbf{T}} = -\Pi\boldsymbol{n} + \mu\boldsymbol{\omega}\times\boldsymbol{n}$$

is the *reduced stress* due to compressing and shearing processes only. These equations can also be directly derived from the inner product of (6b) and $\boldsymbol{u}$.

Care must be taken when providing interpretations for Eqs. (9)-(10). We find that there is an exact cancelation between the portion of the dissipation and that of the work rate done by surface stress. Although computation efforts can be reduced by using the reduced stress, it would be misleading to take the "reduced dissipation" as the "total dissipation". Otherwise, for example, even a solid-like rotation (say, occurred near the axis of a vortex) would have a dissipation. This is of course not the real physics. Like the case of stress tensor, the true dissipation for incompressible fluid is still given by

$$\Phi = \widehat{\Phi} + \Phi_s, \quad \text{with} \quad \widehat{\Phi} = \mu\omega^2, \quad \Phi_s = 2\mu\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}).$$

With the above explanation, our results (9) and (10) imply another significant simplification and some further physical insight. While it is known that for unbounded flow or under periodic boundary conditions the total dissipation in V can be written as

$$\int_V \Phi dV = \int_V [(\lambda + 2\mu)\vartheta^2 + \mu\omega^2]dV = \int_V \widehat{\Phi}dV, \qquad (11)$$

so that the integration volume can be reduced to the regions with nonzero $\boldsymbol{\omega}$ and ϑ. For example, for a Rankine vortex, the total dissipation can be computed based on (7a), which requires integration of $2\mu D_{ij}D_{ij}$ over the whole irrotational domain, but, by (10) it becomes an integration of the constant $\mu\omega^2$ over the solid vortex core. Now we have shown that, with constant μ and the reduced dissipation function, the boundary-condition dependence of (11) can be removed.

Based on the fact that for incompressible flow the viscous force depends on the skew-symmetric part of $\nabla\boldsymbol{u}$ but the dissipation rate (8a) depends on its symmetric part, Largeström[13] noticed the existence of potential flow with viscous dissipation. He illustrated the situation by a steady axisymmetric flow driven by a rotating cylinder of radius a and tangent velocity V (equivalent to the solid core of a Rankine vortex). The flow

is irrotational but with a local dissipation rate $\Phi = 4\mu V^2 a^2 / r^4$ and a total dissipation $4\pi\mu V^2$. We now further see that this dissipation is exclusively from the surface strain, and is solely balance by the work rate done by the resistance to that strain, which is in turn provided by the rotating cylinder with a total power input $4\pi\mu V^2 = \mu\omega^2 \cdot \pi a^2$ to keep the flow steady. Note that on a rotating solid boundary with angular velocity $\boldsymbol{W}$, like the above cylinder surface, the stress $\boldsymbol{t}_s$ due to surface strain can always be absorbed into the shear stress $\boldsymbol{\tau}$, as long as the vorticity $\boldsymbol{\omega}$ in (5b) is replaced by a relative vorticity $\boldsymbol{\omega}_r = \boldsymbol{\omega} - 2\boldsymbol{W}$.[7] Therefore, the work down by the cylinder surface to retain the flow can be explained as from either the surface-stain stress $\boldsymbol{t}_s$ or the shear stress $\boldsymbol{\tau}$.

It should be emphasized that the stress $\boldsymbol{t}_s$ due to surface strain may appear should we go beyond the balance of momentum and energy.[6,14] Its most important appearance is in the stress balance on an *open fluid surface element*. Specifically, on a fluid-fluid interface or free surface $\boldsymbol{t}_s$ plays a crucial role:[8] its normal component joins the balance with pressure and surface tension, while its tangent components are solely responsible for balancing the interfacial tangent vorticity. Note, however, *this appearance of $\boldsymbol{t}_s$ is relevant only to the dynamic boundary condition on such an interface only; thus, Eqs. (6) and (9a,b) still hold.* As a result, saving is still achieved even with an interface as the flow boundary.

As noted already, eqs (6) and (9a,b) do not apply to flow with variable shear viscosity, where a direct coupling between momentum balance and thermodynamics adds additional terms to the S-H tensor potential (6), which, although can always be obtained, do not have the above neat form.

4. Turbulent Stress Tensor versus Turbulent Force

We now turn to turbulent flow. We first show how to achieve a computational saving for a *given* model of turbulent stress tensor, say $\tau_{ij} = -\overline{u_i u_j} = \tau_{ji}$. Here and below, unless stated differently, $\boldsymbol{u}$ is referred to fluctuating velocity, and so are its derivatives. The overline means ensemble average. Once again, in the mean turbulent momentum equation and energy equation [or filtered equation if large-eddy simulations (LES) is considered], what counts is only the turbulent force $\tau_{ij,i} = f_j$, say; so we may similarly replace τ_{ij} by a three-component S-H tensor potential $\widehat{\tau}_{ij}$, which amounts to finding the S-H decomposition of $\boldsymbol{f}$.

As an illustration, consider an unbounded or spatially periodic flow so that in the Fourier space spanned by the wave vector $\boldsymbol{k}$ (where we use $\alpha, \beta, \gamma, ... = 1, 2, 3$ to denote Cartesian components), there is

$$f_\beta(\boldsymbol{k}) = ik_\alpha \tau_{\alpha\beta}(\boldsymbol{k}), \quad i = \sqrt{-1}, \tag{12}$$

and (1) becomes

$$f_\beta(\boldsymbol{k}) = i[k_\beta \phi(\boldsymbol{k}) + \epsilon_{\beta\alpha\gamma} k_\alpha A_\gamma(\boldsymbol{k})]. \tag{13}$$

To obtain $\phi(\boldsymbol{k})$ and $A_\gamma(\boldsymbol{k})$ from the given $\tau_{\alpha\beta}(\boldsymbol{k})$, note that the vector identity in physical space

$$\nabla^2 \boldsymbol{f} = \nabla(\nabla \cdot \boldsymbol{f}) - \nabla \times (\nabla \times \boldsymbol{f})$$

implies

$$k^2 f_\beta(\boldsymbol{k}) = k_\beta k_\alpha f_\alpha(\boldsymbol{k}) + \epsilon_{\beta\alpha\gamma}\epsilon_{\gamma\delta\rho}k_\alpha k_\rho f_\delta(\boldsymbol{k}).$$

Comparing this and (13), it follows that

$$i\phi(\boldsymbol{k}) = \frac{k_\alpha f_\alpha(\boldsymbol{k})}{k^2}, \quad iA_\gamma = \epsilon_{\gamma\delta\rho}\frac{k_\rho f_\delta(\boldsymbol{k})}{k^2}.$$

Therefore, from (12) we obtain the reduced stress tensor

$$\widehat{\tau}_{\alpha\beta}(\boldsymbol{k}) = k^{-2}[k_\gamma k_\delta \tau_{\delta\gamma}(\boldsymbol{k})\delta_{\alpha\beta} + k_\alpha k_\gamma \tau_{\gamma\beta}(\boldsymbol{k}) - k_\beta k_\gamma \tau_{\gamma\alpha}(\boldsymbol{k})]. \tag{14}$$

It can be easily checked that

$$ik_\alpha\widehat{\tau}_{\alpha\beta}(\boldsymbol{k}) = ik_\alpha\tau_{\alpha\beta}(\boldsymbol{k}) = f_\beta(\boldsymbol{k}).$$

By using (14), any models of the turbulent stress tensor $\tau_{\alpha\beta}(\boldsymbol{k})$, linear[15,16] or nonlinear[17−21], can be reduced to a S-H tensor potential before being put into numerical computation. As noted earlier, in principle this procedure can be applied to any bounded domain as long as the HWD basis therein has been established.

Although (14) can already bring great computational saving, however, it does not simplify theoretical analysis since one still has to model the full Reynolds stress $\tau_{\alpha\beta}$. We thus propose a more thorough approach, which would lead to a significant theoretical simplification in turbulence modeling as well. The basic idea is that as long as an S-H decomposition of $\boldsymbol{f}$ is obtained, it is sufficient to directly model the three-component turbulent force, rather than the much more complicated six-component turbulent stress tensor.

We assume the flow is incompressible for simplicity. The turbulent force can be expressed in terms of the "vorticity form":[22]

$$\boldsymbol{f} = -\nabla \cdot (\overline{\boldsymbol{u}\boldsymbol{u}}) = -\nabla K - \overline{\boldsymbol{l}}, \tag{15}$$

where

$$K \equiv \frac{1}{2}\overline{|\boldsymbol{u}|^2} \quad \text{and} \quad \overline{\boldsymbol{l}} \equiv \overline{\boldsymbol{\omega} \times \boldsymbol{u}} \tag{16a, b}$$

are the mean turbulent kinetic energy and mean turbulent Lamb vector, respectively. Obviously, K is a part of the scalar potential of $\boldsymbol{f}$, and the other part comes from the curl-free longitudinal part of $\overline{\boldsymbol{l}}$. Therefore, as long as we have split $\overline{\boldsymbol{l}}$ into a longitudinal

part, $\bar{\boldsymbol{l}}^0 = \nabla\chi$, and a transverse part, $\bar{\boldsymbol{l}}_\perp$, then we immediately arrive at the desired decomposition of turbulent force:

$$\boldsymbol{f} = -\nabla(K + \chi) - \bar{\boldsymbol{l}}_\perp, \tag{17}$$

which sharpens (15). Moreover, since for incompressible flow there is

$$\nabla \cdot \bar{\boldsymbol{l}} = -\nabla^2 \overline{h_0} = -\nabla^2 K,$$

where h_0 is the stagnation enthalpy, it follows that if we define

$$\psi \equiv \chi + K, \tag{18}$$

then

$$\nabla^2 \psi = 0, \tag{19}$$

of which the solution depends on boundary values of χ and K only. Once these values are given, the internal values of K in the flow domain can be inferred from that of χ. Therefore, *the problem of modeling turbulent force exclusively amounts to modeling the mean turbulent Lamb vector*. This observation further confirms that the Lamb vector is the key in nonlinear fluid dynamics including anisotropic turbulence; a rational mathematical reduction is often associated with a sharper physical insight.

5. Reynolds Stress Transport versus Lamb Vector Transport

The second-order closure models (full Reynolds stress transport) represents the highest level of closure currently feasible in practical Reynolds-average computations. In principle, these models account for more turbulence physics than lower-level models, e.g., two-equation models. As reviewed by Speziale[16], full Reynolds-stress transport may be used as a starting point to deduce improved two-equations models under the equilibrium limit of homogeneous turbulence.

The previous section clearly demonstrated that the problem of Reynolds-stress modeling can be cast to the modeling of turbulent Lamb vector. Thus, the first step towards directly modeling turbulent force at the level of second-order closure is to derive the transport equation for $\bar{\boldsymbol{l}}$, which is much simpler than that of the full Reynolds stress.

From the exact Navier-Stokes equation and vorticity equation (the notations now stand for instantaneous quantities), it follows that the *Lamb-vector transport equation* reads

$$\frac{\partial \boldsymbol{l}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{l} + \boldsymbol{l} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{l} = \boldsymbol{Q}, \tag{20a}$$

where

$$\boldsymbol{Q} \equiv \nabla h_0 \times \boldsymbol{\omega} - 2\nu \boldsymbol{\omega}_{,i} \times \boldsymbol{u}_{,i} \tag{20b}$$

is the source. The first term of (20b) is an inviscid coupling between longitudinal and transverse parts, but the second does not represent the dissipation of $\boldsymbol{l}$ at all. We found that

$$-2\nu\boldsymbol{\omega}_{,l} \times \boldsymbol{u}_{,l} = -2\nu\nabla \cdot (\boldsymbol{\omega} \times \mathbf{D}) + \frac{1}{2}(\rho^{-1}\nabla\widehat{\Phi} + \nu\boldsymbol{l}_\omega), \quad \boldsymbol{l}_\omega \equiv (\nabla \times \boldsymbol{\omega}) \times \boldsymbol{\omega}, \qquad (21)$$

where $\widehat{\Phi} = \mu\omega^2$ is the reduced dissipation.

In order to gain more understanding of the evolution of Lamb vector, let us consider its square, characterized by $J \equiv l^2/2$. It is easily shown that the J-equation reads

$$\frac{\partial J}{\partial t} + \boldsymbol{u} \cdot \nabla J + \boldsymbol{l} \cdot \mathbf{D} \cdot \boldsymbol{l} - \nu\nabla^2 J = -\nu\nabla\boldsymbol{l} : \nabla\boldsymbol{l} + Q_J, \qquad (22a)$$

where : means twice contraction so that $\nu\nabla\boldsymbol{l} : \nabla\boldsymbol{l}$ is the J-dissipation, and

$$Q_J \equiv \boldsymbol{l} \cdot \boldsymbol{Q} = (\mathcal{H}\boldsymbol{\omega} - \omega^2\boldsymbol{u}) \cdot \nabla h_0 - \frac{\nu}{2}(\nabla\omega^2 \cdot \nabla u^2 + 4\boldsymbol{\omega} \cdot \mathbf{D} \cdot \mathbf{D} \cdot \boldsymbol{\omega} - 4\boldsymbol{\omega} \cdot \mathbf{D} \cdot \nabla\mathcal{H}), \quad (22b)$$

where $\mathcal{H} \equiv \boldsymbol{\omega} \cdot \boldsymbol{u}$ is the helicity. In (22b), all viscous terms are from the second term of (20b). For two-dimensional flow Q_J reduces to

$$Q_J = -\omega^2\boldsymbol{u} \cdot \nabla h_0 - \frac{\nu}{2}\nabla\omega^2 \cdot \nabla u^2. \qquad (23)$$

Note that for potential flow or Beltrami flow both (20a,b) and (22a,b) become the trivial identity $0 = 0$.

It is now clear that the structure of (20) and (22) is precisely the same as that of *the two-dimensional equations for vorticity gradient and its square*, but subjected to the forcing terms. Therefore, the existing studies on the latter (e.g., Novikov[23]) can well be utilized to explore the behavior of (20) and (22). In particular, the increase of J is due to the *shrinking* of the fluid element rather than stretching. If the stretching is dominating in a flow field, the integrated "Lamb-enstrophy" J must exponentially decrease by this mechanism. This is consistent with the fact that as the turbulent eddies becomes smaller and smaller due to stretching, the fluctuating Lamb vector is also reduced — eventually we have $\bar{\boldsymbol{l}} \simeq 0$ for fine-scale turbulence, which then becomes approximately homogeneous and isotropic. Note that in (22b), the term $-2\nu\omega_i D_{ij} D_{jk} \omega_k$ is associated with vortex stretching, which also provides a negative source (a sink) of J in a highly stretched vortical flow, thus further reduces the Lamb vector in fine-scale turbulence. Consequently, in the turbulent force there remains the direct balance between the turbulent kinetic energy K and reduced dissipation $\widehat{\epsilon} \equiv \mu\overline{\omega^2}$, which leads to the Kolmogorov law.

To investigate the effects of the mean flow on turbulence, we perform a Reynolds average. The mean-flow part is now denoted by capital letters and fluctuating part is still

presented by lower-case letters. After subtracting mean-flow terms, the desired transport equation for the mean fluctuating Lamb vector reads

$$\frac{\partial \bar{l}}{\partial t} + U \cdot \nabla \bar{l} + \bar{l} \cdot \nabla U - \nu \nabla^2 \bar{l} = q_1 + q_2 + q_3 \tag{24}$$

On the right of (24), the first source is

$$q_1 \equiv \overline{\nabla h_0 \times \omega} - 2\nu \overline{\omega_{,i} \times u_{,i}}, \tag{25}$$

which has the same structure as (20b) and involves second moments at the same point. The second source reads

$$q_2 \equiv -\overline{(u \cdot \nabla L^\dagger + L^\dagger \cdot \nabla u)}, \tag{26}$$

which contains the interaction between the mean flow and some second moments. Here,

$$L^\dagger \equiv \omega \times U + \Omega \times u$$

is a "quasi-Lamb vector" in the sense that its first term is the analogy of Coriolis force acted on the mean flow caused by fluctuating vorticity, and the second term is that acted on fluctuating flow caused by the mean vorticity. Finally, the third source

$$q_3 \equiv -\overline{(u \cdot \nabla l + l \cdot \nabla u)} \tag{27}$$

is a triple correlation and surely needs modeling. Equation (24) indicates that $\bar{l}$ is advected and stretched-turned by the mean flow field.

The structure of q_1 clearly implies that *the pressure-strain term in common modeling is now reduced to the correlation of ∇h_0 and ω*. Because the variation of h_0 is expected to be milder than h, and since only the regions with $\omega \neq 0$ needs be modeled, it is expected that some troublesome aspects of pressure-strain modeling[16] could be bypassed. The second term of (25), having been identified not as the dissipation, needs some attention. Although it is a viscous effect, it might not be completely negligible since the derivatives of fluctuating vorticity and velocity are involved. A further analysis of q_1 could be made by looking at (22).

On the other hand, we see in (24) that q_2, the triple correlation, and the advecting-stretching-turning of $\bar{l}$ by the mean flow, have the same theoretical structure since they come from the single root; but their roles are very different. A further study of this type of structure is desirable.

Obviously, various second moments are not equally important, and one can only select the most relevant one to derive its transport equation and close the turbulence modeling thereon. This is the basic spirit of second-order closure. In the usual approach, one takes the transport equation of Reynolds stress, in which some other second moments, such as

the pressure-stain correlation and dissipation-rate correlation, have to be modeled. Much efforts have been paid to deal with some redundant quantities and complexity because the full Reynolds stress is reducible. Now the tensorial Reynolds-stress equation is reduced to the vectorial mean Lamb-vector equation, in which the remaining quantities can no longer be further reduced. In this sense, we may call the above approach as an *irreducible second-order closure*. An obvious feature of this irreducibility is that the terms to be modeled are nonzero only in those regions where $\boldsymbol{\omega} \neq 0$; or, approximately, the regions with high peak of fluctuating vorticity. This is precisely the essence of turbulence as randomly stretched vortices.

6. Conclusion

In this theoretical paper we systematically explored the concept and application of reduced stress tensor and dissipation function, showing that they may bring significant savings in analysis and computation. The development is solely based on the classic Stokes-Helmholtz decomposition, or its modern sharpening, the helical-wave decomposition.

In particular, for turbulent flows we propose that the study and modeling of Reynolds stress tensor could be reduced to that of the mean turbulent Lamb vector alone, of which the governing equation is closely similar to that for the two-dimensional vorticity gradient. This equation would be a new basis of irreducible second-order closure schemes. A numerical examination of the budget and spectra of terms in this equation, including those that need be modeled, is being undertaken and will be reported separately.

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