Fourier-Laplace Analysis of Multigrid Waveform Relaxation Method for Hyperbolic Equations*

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Abstract

The multigrid waveform relaxation (WR) algorithm has been fairly studied and implemented for parabolic equations. It has been found that the performance of the multigrid WR method for a parabolic equation is practically the same as that of multigrid iteration for the associated steady state elliptic equation. However, the properties of the multigrid WR method for hyperbolic problems are relatively unknown. This paper studies the multigrid acceleration to the WR iteration for hyperbolic problems, with a focus on the convergence comparison between the multigrid WR iteration and the multigrid iteration for the corresponding steady state equations. Using a Fourier-Laplace analysis in two case studies, it is found that the multigrid performance on hyperbolic problems no longer shares the close resemblance in convergence factors between the WR iteration for parabolic equations and the iteration for the associated steady state equations.

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1 Introduction

The advent of a new generation of massively parallel computers, consisting of hundreds or thousands of processors, has caused previously unattractive numerical algorithms to be reexamined. For a numerical algorithm to fully exploit the power of such machines, it must be decomposable into largely independent pieces which can be distributed among the available computer processors. The waveform relaxation (WR) method (or dynamic iteration method), originally proposed for VLSI-simulation [8] [12], is such a method for solving systems of ordinary differential equations (ODEs) and time-dependent partial differential equations (PDEs). It decomposes a full system into smaller subsystems which can be solved concurrently. Additionally, it allows different integration step sizes to be used for different subsystems, resulting in substantial savings in computation for some applications [7] [10] [13] [14] [15] [19].

Many studies on the WR method and its acceleration techniques have been made on parabolic problems [4] [6] [9] [16] [17]. In those studies, an important approach has been to establish a quantitative comparison between the WR iteration for time-dependent PDEs and the associated iteration for the corresponding steady state problems, called *static iteration*, since the latter has been extensively investigated. The studies have found that, for parabolic problems, the convergence rates of the WR iteration and static iteration are quantitatively quite close [11]. When a multigrid technique (in space) is incorporated into the WR method, the typical multigrid acceleration can be achieved with a rate that is a small perturbation from the one for the associated static multigrid iteration [9] [16]. Therefore, the resulting multigrid WR method significantly increases the speed of convergence in solving parabolic problems, making it competitive with the traditional time-stepping methods.

For hyperbolic systems, the WR method has been shown to be conceptually similar to the semi-discrete subdomain iteration and share the same convergence properties as for parabolic problems [1]. Therefore, it is natural to consider a multigrid acceleration to the WR iteration because the technique has been effectively used for solving non-elliptic steady state problems [3]. For certain classes of non-elliptic steady state problems, the multigrid technique has substantially improved the speed of convergence. Since it can be adapted into the WR method for solving time-dependent hyperbolic PDEs in the exact same format, a question is raised about whether its performance is still analogous to its steady state counterpart as reported by

Brandt [3].

This paper addresses the issue of performance differences of the multigrid WR method on parabolic and hyperbolic problems. It studies the convergence factors of the multigrid WR method for problems whose space operator is not elliptic, or in which the elliptic principal part of its space operator is small. Many problems in fluid dynamics, and in other fields, are of these types. The Fourier method was shown to be a powerful tool for convergence analysis of numerical schemes for these types of problems [3]. In this paper, a Fourier-Laplace analysis, i.e., Fourier analysis in space and Laplace transform in time, is used in two case studies that involve hyperbolic equations. Interesting phenomena were observed. In particular, it was found that for a hyperbolic equation, the multigrid WR iteration could fail to converge while its static analogue works well on the associated steady state problem.

2 Fourier-Laplace Analysis

The Fourier method is an indispensable tool for analyzing both differential equations and discrete solution methods for time-dependent problems. It is especially powerful for the convergence analysis of multigrid iteration, because it provides insight into the details of the basic interaction between the coarse grid correction and fine grid relaxations. For steady state model problems, comprehensive Fourier analysis has been developed [2] [18]. For a parabolic model problem, the practical convergence estimates of the multigrid WR iteration were obtained by a Fourier-Laplace analysis [16]. In this paper, such analysis is used in two case studies that involve hyperbolic equations formulated as

$$\frac{\partial u}{\partial t} + Lu = f, \qquad t > 0, \qquad u(t=0) = u_0, \tag{1}$$

with periodic boundary conditions. L is an m-dimensional linear operator with constant coefficients defined in an infinite space. In order to employ a multigrid WR iteration to (1), the equation is first discretized in space

$$\frac{du_h}{dt} + L_h u_h = f_h, \qquad t > 0, \qquad u_h(0) = u_0, \tag{2}$$

where L_h, u_h, and f_h are discrete approximations to L, u, and f obtained by spatial finite differences. Then the multigrid WR iteration is applied to (2) with h as the finest grid size. Our analysis is restricted to a two grid WR cycle described as follows.

1. Let L_h be split as $L_h = M - N$. Perform v_1 pre-smoothing steps:

$$\frac{du_h^{(v)}}{dt} + Mu_h^{(v)} = Nu_h^{(v-1)} + f_h, \qquad t > 0, \quad u_h^{(v)}(0) = u_0, \quad v = 1, 2, \cdots, v_1;$$

   where the starting function $u_h^{(0)}$ is given.

2. Restrict the defect from grid h to grid H:

$$d_h := \frac{du_h^{(v_1)}}{dt} + L_h u_h^{(v_1)} - f_h, \qquad d_H := I_h^H d_h. \tag{3}$$

3. On the coarse grid, solve

$$\frac{dw_H}{dt} + L_H w_H = d_H, \qquad w_H(0) = 0.$$

4. Correct

$$\bar{u}_h = u_h^{(v_1)} - I_H^h w_H$$

   where I_H^h is a suitable interpolation from grid H to grid h.

5. Perform v_2 post-smoothing steps on $\bar{u}_h$.

The error of a complete two grid WR cycle described above satisfies

$$e_h^{(i)} = \mathcal{V} e_h^{(i-1)}, \qquad e_h^{(i)}(0) = 0, \tag{4}$$

where $\mathcal{V}$ is the two grid WR iteration operator. The Laplace transform of (4) is

$$\hat{e}_h^{(i)}(z) = V(z)\hat{e}_h^{(i-1)}(z), \quad Re(z) \geq 0, \tag{5}$$

with

$$V(z) = S^{v_2}(z) CG(z) S^{v_1}(z), \tag{6}$$

and

$$CG(z) = I - I_H^h (z + L_H)^{-1} I_h^H (z + L_h). \tag{7}$$

The matrix functions $CG(\cdot)$ and $S(\cdot)$ are the Laplace transforms of convolution kernels of the coarse grid corrector and the smoother respectively. Note that $CG(0)$, $S(0)$ and $V(0)$ are respectively, the coarse grid correction, smoothing and two grid iteration operators for the corresponding steady state problem $L_h u_h = f_h$. A detailed derivation of (6) and (7) can be found in [9].

Assuming that all the entries of $V(z)$ are rational functions of z vanishing at infinity with poles having negative real parts, and taking $\mathcal{V}$ as an operator on $L^p(R^+, C^n)$ $(1 \leq p \leq \infty)$, the spectral radius $\rho(\mathcal{V}) = \lim_{k\to\infty} ||\mathcal{V}^k||^{1/k}$ satisfies (see [9])

$$\rho(\mathcal{V}) = \max_{Rez\geq 0} \rho(V(z)) = \max_{Rez=0} \rho(V(z)). \tag{8}$$

Since the multigrid technique is used in space only, the convergence estimate of the multigrid WR iteration can be obtained by performing a Fourier analysis on each of equations (5).

Let $|\cdot|$ denote the max-norm in C^m (note, m is the dimension of the PDE (1)). Define the frequency

$$\theta = (\theta^1, \theta^2, \cdots, \theta^{2^m}), \qquad \theta^j = (\theta_1^j, \cdots, \theta_m^j) \in C^m,$$

where

$$|\theta^1| \leq \frac{\pi}{2}, \quad |\theta^j| \leq \pi \text{ and } \theta^j = \theta^1 + \Pi^j, \quad j = 2, \cdots, 2^m,$$

in which, $\Pi^j := \pi(i_1, i_2, \cdots, i_m)$, $i_k = 0$ or 1, and at least one of i_k's equals 1. The exponential Fourier mode on h grid

$$exp(i\theta^j x/h), \qquad x \in C^m,$$

is an infinite dimensional vector determined by the grid points, and appears as the mode $exp(i2\theta^j x/H)$ on $H = 2h$ grid. Therefore, on H grid, it coincides with all Fourier modes $exp(i\theta^{j'} x/h)$, $j' = 1, \cdots, 2^m$, $j' \neq j$. Thus, restriction operators introduce coupling between each lower mode θ^1 and its $(2^m - 1)$ high frequency harmonics θ^j, $j = 2, \cdots, 2^m$. Interpolation operators introduce coupling among the same modes. This can be represented as

$$I_h^H exp(i\theta^j x/h) = \hat{I}_h^H(\theta^j) exp(i2\theta^1 x/H), \qquad j = 1, \cdots, 2^m, \tag{9}$$

and

$$I_H^h exp(i2\theta^1 x/H) = \sum_{j=1}^{2^m} \hat{I}_H^h(\theta^j) exp(i\theta^j x/h). \tag{10}$$

If the set of Fourier modes in h and H grid are denoted by

$$X_h(\theta) = [exp(i\theta^1 x/h), \cdots, exp(i\theta^{2^m} x/h)] \quad \text{and} \quad X_H(\theta^1) = [exp(i2\theta^1 x/H)],$$

the equation (9)-(10) can be written as

$$I_h^H X_h(\theta) = X_H(\theta^1)\tilde{I}_h^H(\theta), \quad \tilde{I}_h^H(\theta) = [\hat{I}_h^H(\theta^1), \cdots, \hat{I}_h^H(\theta^{2^m})],$$

and

$$I_H^h X_H(\theta^1) = X_h(\theta)\tilde{I}_H^h(\theta), \quad \tilde{I}_H^h(\theta) = \begin{bmatrix} \hat{I}_H^h(\theta^1) \\ \vdots \\ \hat{I}_H^h(\theta^{2^m}) \end{bmatrix}.$$

The matrices $\tilde{I}_h^H(\theta)$ and $\tilde{I}_H^h(\theta)$ are called the *matrix symbols* of I_h^H and I_H^h respectively. In this paper, the symbol of a matrix A will be denoted by $\tilde{A}(\theta)$.

If the operators L_h and L_H, and the smoother in Laplace domain $S(z)$ do not introduce coupling of more Fourier modes, then the set of 2^m harmonic modes $X_h(\theta)$ is an invariant subspace of iteration operator $V(z)$ satisfying

$$V(z)X_h(\theta) = X_h(\theta)\tilde{V}(\theta, z) \qquad \text{for all } \theta,$$

where the matrix symbol of $V(z)$ has the form

$$\tilde{V}(\theta, z) = \tilde{S}^{v_2}(\theta, z)[I - \tilde{I}_H^h(\theta)(z + \tilde{L}_H(2\theta))^{-1}\tilde{I}_h^H(\theta)(z + \tilde{L}_h(\theta))]\tilde{S}^{v_1}(\theta, z), \quad Rez \geq 0.$$

The spectral radius of $V(z)$ can then be obtained by collecting its value on each set of 2^m Fourier modes

$$\rho(V(z)) = \sup_{\theta} \rho(\tilde{V}(\theta, z)) = \max_{|\theta^1| \leq \frac{\pi}{2}} \rho(\tilde{V}(\theta, z)).$$

Combining (8), the asymptotic convergence rate of the two grid WR iteration can be calculated by

$$\rho(\mathcal{V}) = \max_{Rez=0} \max_{|\theta^1| \leq \frac{\pi}{2}} \rho(\tilde{V}(\theta, z)). \tag{11}$$

3 Case Studies

In order to assess the multigrid performance on hyperbolic equations, it is useful to start from the case study on the model hyperbolic equation

$$u_t + au_x + bu_y = f. \tag{12}$$

A wide class of discretization methods for equations of the above form, even in the context of a system of PDEs, involves central differencing of the original equation with an additional term which is used for stabilization. This additional term, called *artificial viscosity*, may be of different order and usually corresponds to a discretization of elliptic operator times a small coefficient. Here, two such discretization schemes are chosen for the case studies. The analysis of these schemes are done using First Differential Approximation (FDA) [3].

Case 1. Consider the equation

$$u_t + au_x + bu_y - h\beta\Delta u = f$$

with given initial condition and periodic boundary condition. This is the FDA for the following discretization of (12):

$$L_h := \frac{1}{h}\begin{bmatrix} & b/2-\beta & \\ -a/2-\beta & 4\beta & a/2-\beta \\ & -b/2-\beta & \end{bmatrix}.$$

Note, L_h is a first order discretization to $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$. The matrix symbol of L_h is

$$\tilde{L}_h(\theta) = \begin{bmatrix} \hat{L}_h(\theta^1) & & \\ & \ddots & \\ & & \hat{L}_h(\theta^4) \end{bmatrix}, \quad \theta^j = (\theta_1^j, \theta_2^j), \quad j = 1, \cdots, 4, \tag{13}$$

with

$$\hat{L}_h(\theta^j) = \frac{4\beta(sin^2(\theta_1^j/2) + sin^2(\theta_2^j/2)) + i(asin\theta_1^j + bsin\theta_2^j)}{h}.$$

The matrix symbol of the coarse grid operator is

$$\tilde{L}_H(2\theta) = [\hat{L}_H(2\theta^1)]. \tag{14}$$

Let the bi-linear interpolation be chosen for I_H^h and the restriction operator be $I_h^H = (I_H^h)^T$. Their matrix symbols are given by

$$\tilde{I}_H^h(\theta) = [\hat{I}_h^H(\theta^1), \cdots, \hat{I}_h^H(\theta^4)]^T \quad \text{and} \quad \tilde{I}_h^H(\theta) = (\tilde{I}_H^h(\theta))^T,$$

with

$$\hat{I}_h^H(\theta^j) = (\frac{1+cos\theta_1^j}{2})(\frac{1+cos\theta_2^j}{2}), \qquad j = 1, \cdots, 4.$$

Symbols for the damped Jacobi and red-black Gauss-Seidel WR smoother are found to be

$$\tilde{S}_J(\theta, z) = \begin{bmatrix} \hat{S}(\theta^1, z) & & \\ & \ddots & \\ & & \hat{S}(\theta^4, z) \end{bmatrix}, \tag{15}$$

$$\hat{S}(\theta^j, z) = \frac{1}{z\omega + d}(d - \omega \hat{L}_h(\theta^j)), \quad d = \frac{4\beta}{h}, \quad 0 < \omega \leq 1, \quad j = 1, \cdots, 4;$$

$$\tilde{S}_{GS}(\theta, z) = \begin{bmatrix} \hat{S}_A(\theta, z) & 0 \\ 0 & \hat{S}_B(\theta, z) \end{bmatrix},$$

$$\hat{S}_A(\theta, z) = \frac{A}{2}\begin{bmatrix} 1 + A & -(1 + A) \\ 1 - A & -(1 - A) \end{bmatrix}, \quad \hat{S}_B(\theta, z) = \frac{B}{2}\begin{bmatrix} 1 + B & -(1 + B) \\ 1 - B & -(1 - B) \end{bmatrix},$$

$$A = \frac{2\beta(cos\theta_1^1 + cos\theta_2^1) - i(asin\theta_1^1 + bsin\theta_2^1)}{4\beta + zh},$$

$$B = \frac{2\beta(-cos\theta_1^1 + cos\theta_2^1) - i(-asin\theta_1^1 + bsin\theta_2^1)}{4\beta + zh}.$$

Tables 1 and 2 list the computed spectral radii of the two grid WR operators for the damped Jacobi relaxation and Red-Black Gauss-Seidel relaxation based on (11). The number of smoothing steps is chosen as $v = v_1 + v_2 = 1$. For a comparison, using the same number of sample points for the space variables, the computed spectral radii of the related two grid static iteration operators are listed inside parentheses. These data show that the performance of the two grid WR iteration on this problem is quantitatively quite close to its static analogue, a phenomenon that has been observed and theoretically justified for parabolic problems.

Table 1: Spectral Radius of Two Grid Damped Jacobi WR ($\omega = 2/3$, $b = 1$)

a	$\beta = .4$	$\beta = .5$	$\beta = .75$	$\beta = 1.$
1.0	.8416 (.8357)	.7086 (.7001)	.6072 (.5816)	.5887 (.5644)
0.5	.7644 (.7480)	.7509 (.7151)	.7815 (.7298)	.7992 (.7789)
0.0	1.0 (1.0)	.9999 (.9999)	.9999 (.9999)	.9999 (.9999)

Case 2. Next, consider a third order approximation of (12), for which the FDA is

$$u_t + au_x + bu_y + h^3\beta\Delta^2 u = f,$$

i.e., the operator $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ is approximated by

$$L_h := \frac{1}{h}\begin{bmatrix} & & \beta & & \\ & 2\beta & -8\beta + b/2 & 2\beta & \\ \beta & -8\beta - a/2 & 20\beta & -8\beta + a/2 & \beta \\ & 2\beta & -8\beta - b/2 & 2\beta & \\ & & \beta & & \end{bmatrix}.$$

The matrix symbols for L_h and L_H are again represented by (13) and (14), but with the elements

$$\hat{L}_h(\theta^j) = \frac{16\beta(sin^2(\theta_1^j/2) + sin^2(\theta_2^j/2))^2 + i(asin\theta_1^j + bsin\theta_2^j)}{h}, \qquad j = 1, \cdots, 4.$$

The damped Jacobi WR smoother has the matrix symbol (15) with

$$\hat{S}(\theta^j, z) = \frac{1}{z\omega + d}(d - \omega\hat{L}_h(\theta^j)), \quad d = \frac{20\beta}{h}, \quad 0 < \omega \le 1, \qquad j = 1, \cdots, 4.$$

Thus, it has exactly the same Fourier smoothing factor (see [18]) as its static analogue

$$\rho_r = \max_{Rez\ge 0}\max_{2\le j\le 4} |\hat{S}(\theta^j, z)| = \max_{2\le j\le 4} |\hat{S}(\theta^j, 0)|$$

$$= \max_{2\le j\le 4} |1 - \omega(\frac{4}{5}(sin^2(\theta_1^j/2) + sin^2(\theta_2^j/2))^2 + \frac{i}{20\beta}(asin\theta_1^j + bsin\theta_2^j))|.$$

Let $\omega = 1/2$. Since $\pi/2 \le |\theta^j| \le \pi$, $j = 2,3,4$, the Fourier Jacobi WR smoothing factor satisfies

$$\rho_r \le \sqrt{(\frac{3}{5})^2 + (\frac{|a| + |b|}{40\beta})^2} < 1 \qquad \text{when } \beta > \frac{|a| + |b|}{32}.$$

This implies that the Jacobi WR smoother is as effective as the associated static smoother in eliminating the error with oscillatory modes.

Using the same intergrid operators as in Case 1, the $(1,1)$-component of the matrix symbol of the coarse grid corrector CG (see (7)) for low frequencies, denoted as

$$\theta^1 = \alpha = (\alpha_1, \alpha_2) \approx 0,$$

can be approximated by

$$\tilde{CG}(\theta,z)_{11} \approx 1 - \frac{z+\hat{L}_h(\alpha)}{z+\hat{L}_H(2\alpha)}$$

$$\approx \frac{7\beta(\alpha_1^2+\alpha_2^2)^2 + i((asin(2\alpha_1)+bsin(2\alpha_2))/2-(asin\alpha_1+bsin\alpha_2))}{h(z+\hat{L}_H(2\alpha))}$$

$$\approx \frac{7\beta(\alpha_1^2+\alpha_2^2)^2 - i(a\alpha_1^3+b\alpha_2^3)/2}{h(z+\hat{L}_H(2\alpha))}.$$

Taking $-z$ to be the imaginary part of $\hat{L}_H(2\alpha)$,

$$\tilde{CG}(\theta,z)_{11} \approx \frac{7\beta(\alpha_1^2+\alpha_2^2)^2 - i(a\alpha_1^3+b\alpha_2^3)/2}{8\beta(\alpha_1^2+\alpha_2^2)^2},$$

which diverges to ∞ as $\alpha \to 0$ when $a\alpha_1^3+b\alpha_2^3 \neq 0$. However, for the related static two grid iteration, $z \equiv 0$, one has

$$\tilde{CG}(\theta,z)_{11} \approx \frac{7\beta(\alpha_1^2+\alpha_2^2)^2 - i(a\alpha_1^3+b\alpha_2^3)/2}{8\beta(\alpha_1^2+\alpha_2^2)^2 + i((a\alpha_1+b\alpha_2)-2(a\alpha_1^3+b\alpha_2^3)/3)}.$$

As $\alpha \to 0$, it converges to 0 except in the direction of $a\alpha_1+b\alpha_2=0$. In this direction, $\tilde{CG}(\theta,z)_{11}$ converges to either 7/8 or 6/8 depending upon whether $a\alpha_1^3+b\alpha_2^3$ vanishes.

Since the effectiveness of a coarse grid corrector is strongly influenced by its action on the error that has very low frequencies, the above observation indicates that, instead of reducing the error in smooth modes, the coarse grid corrector in the two grid WR iteration could magnify the error, causing a divergence of the iterative process. Meanwhile, its static analogue works well [5].

4 Numerical Results and Conclusions

The numerical experiments were carried out on two case studies discussed in the previous section. The problems were solved by the two grid Jacobi WR method over the unit square along $0 \leq t \leq tf$, $tf = 1$ (it has been found that the dependency of the measured convergence rate on the length of time interval can be ignored [16]). The space derivatives were discretized as described in Section 3 with the uniform fine grid size $h = 1/64$. The initial guess $u^{(0)}$ was randomly generated to excite all possible Fourier modes. The trapezoidal rule was used with step size .01 for the time integration. In both cases, the asymptotic ratios of the defect (see (3))

$$r^{(i)} = \frac{\max_{t\in[0,tf]} ||d_h^{(i)}(t)||}{\max_{t\in[0,tf]} ||d_h^{(i-1)}(t)||} \qquad i = 1, 2, \cdots, \tag{16}$$

were collected and their mean value, denoted by $\bar{r}$, was used as an approximation to the spectral radius $\rho(\mathcal{V})$.

Tables 3 and 4 list the range of the ratios for Case 1 and 2 respectively. Table 5 lists the defects collected from the experiments in Case 2 with $a = b = 1$ and $\beta = .125$, using and without using the multigrid technique. The results show that, in Case 1, the multigrid technique accelerates the convergence of the WR iteration in a similar way as it does on static iteration for the corresponding steady state problem. However, in Case 2, the multigrid WR process quickly diverges. The results in Table 5 clearly indicate that it was the coarse grid correction that caused the divergence, as shown by the previous analysis.

All the numerical data confirm the results obtained by the Fourier-Laplace analysis. This demonstrates that, for a converging iterative process, the analysis provided practical estimates to the actual convergence rates; while for a diverging iterative process, it was able to locate the source of divergence, which can be very helpful in searching new directions and developing better methods for accelerating the WR iteration.

References

[1] M. Bjφrhus, *Semi-discrete subdomain iteration for hyperbolic systems*, Technical Report Numerics no.4/1995, Department of Mathematics, University of Trondheim, Trondheim, Norway, 1995.

[2] A. Brandt, *Multi-level adaptive solutions to boundary-value problems*, Math. Comp., 31 (1977), pp. 333–390.

[3] ——, *Multigrid solvers for non-elliptic and singular-perturbation steady-state problems*, technical report, Weizmann Institute of Science, Rehovot, Israel, 1981.

[4] G. Horton and S. Vandewalle, *A space-time multigrid method for parabolic P.D.E.s*, SIAM J. Sci. Comput., 16 (1995), pp. 848–864.

[5] A. Jameson, *Solution of the Euler equations for two dimensional transonic flow by multigrid method*, Appled Mathematics and Computation, 13 (1983), pp. 327–355.

[6] R. Jeltsch and B. Pohl, *Waveform relaxation with overlapping splittings*, SIAM J. Sci. Comput., 16 (1995), pp. 40–49.

[7] B. Leimkuhler, U. Miekkala, and O. Nevanlinna, *Waveform relaxation for linear RC circuits*, Impact of Computing in Science and Engineering, 3 (1991), pp. 123–145.

[8] E. Lelarasmee, A. Ruehli, and A. Sangiovanni-Vincentelli, *The waveform relaxation method for time-domain analysis of large scale integrated circuits*, IEEE Trans. Computer-Aided Design, 1 (1982), pp. 131–145.

[9] C. Lubich and A. Ostermann, *Multigrid dynamic iteration for parabolic equations*, BIT, 27 (1987), pp. 216–234.

[10] A. Lumsdaine, M. Reichelt, and J. White, *Conjugate direction waveform methods for transient two-dimensional simulation of MOS devices*, Proceedings of the International Conference on Computer Aided Design, Santa Clara, California, (1991), pp. 116–119.

[11] U. Miekkala and O. Nevanlinna, *Convergence of dynamic iteration methods for initial value problems*, SIAM J. Sci. Stat. Comput., 8 (1987), pp. 459–482.

[12] A. Newton and A. Sangiovanni-Vincentelli, *Relaxation-based electrical simulation*, IEEE Trans. Computer-Aided Design, 3 (1984), pp. 308–331.

[13] P. Odent, L. Claesen, and H. De Man, *Acceleration of relaxation based circuit simulation using a multiprocessor system*, IEEE Trans. CAD, 9 (1990), pp. 1063–1072.

[14] C. Oosterlee and P. Wesseling, *Multigrid schemes for time-dependent incompressible Navier-Stokes equations*, Impact of Computing in Science and Engineering, 5 (1993), pp. 153–175.

[15] A. Ruehli and C. Zukowski, *Convergence of waveform relaxation for RC circuits*, In F. Odeh, J. Cole, W. Coughran, P. Loyd, and J. White, editors, Semiconductors (part I and part II), IMA Volumes in Mathematics and its Applications, Springer Verlag, 1992.

[16] S. Taásan and H. Zhang, *On the multigrid waveform relaxation method*, SIAM J. Sci. Comput., 16 (1995), pp. 1092–1104.

[17] S. Vandewalle, *Parallel multigrid waveform relaxation for parabolic problems*, B.G. Teubner Verlag, Stuttgart, 1993.

[18] P. Wesseling, *An Introduction to Multigrid Methods*, John Wiley & Sons, Ltd. Chichester, England, 1992.

[19] E. Xia and R. Saleh, *Parallel waveform-Newton algorithms for circuit simulation*, IEEE Trans. CAD, 11 (1992), pp. 432–442.

Table 2: Spectral Radius of Two Grid Red-Black Gauss-Seidel WR ($b = 1$)

a	$\beta = .4$	$\beta = .5$	$\beta = .75$	$\beta = 1.$
1.0	1.0 (1.0)	1.0 (1.0)	.5189 (.5189)	.4999 (.4999)
0.5	.8789 (.8789)	.5960 (.5960)	.4938 (.4882)	.4914 (.4886)
0.0	.6574 (.6339)	.5373 (.5133)	.4967 (.4257)	.4947 (.4119)

Table 3: Case 1 ($\omega = 2/3$, $a = b = 1$)

β	$r^{(i)}$	$\bar{r}$	$\rho(\mathcal{V})$
0.50	.5062 - - .8465	.6583	.7086
0.75	.5349 - - .7090	.6354	.6072

Table 4: Case 2 ($\omega = 1/2$, $a = b = 1$)

β	$r^{(1)}$	$r^{(2)}$	$r^{(3)}$	$r^{(4)}$	$r^{(5)}$	$r^{(6)}$	$r^{(7)}$	$r^{(8)}$	$r^{(9)}$	$r^{(10)}$
0.125	0.9	0.5	1.3	4.1	3.6	4.4	3.7	3.0	2.8	2.3
0.5	0.7	0.4	0.7	1.3	2.8	3.0	2.8	2.4	2.0	2.2

Table 5: Case 2: Measured Defect

Method	$\max_{t\in[0,tf]} \|\|d_h^{(i)}(t)\|\|$				
	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$
Two Grid WR	70.2	286.5	1017.7	4521.0	16512.6
WR	62.6	47.9	41.3	38.7	37.7