# A Note on the Mathematical Modelling of Damped Second Order Systems

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**Abstract**

This note is concerned with the formulation of a damped second order system as a first order dynamical system on a product space. This problem comes from the desire to have explicit representations of the infinitesimal generator of the first order system and, in particular, of the domain of this operator. This analysis is motivated by the need to find specific representations for Riccati operators that can be used in the development of computational schemes for hyperbolic control problems. The approach we take here is based on a natural factorization of the differential operators that define the second order model.

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# 1 Introduction and Motivation

In [2]-[4], Burns and King consider feedback control problems for damped hyperbolic systems. Specifically, they are concerned with obtaining integral representations of the feedback control law for purposes of designing reduced order controllers and sensors/actuators. The kernels of these integral representations are called functional gains. In [4], Burns and King use information about the spatial support of functional gains to guide the construction of reduced order controllers for a nonlinear damped elastic system. In order to use this information, it is important to have both qualitative and quantitative information about these feedback gains. The very existence of these kernels is not always obvious and requires careful analysis of the system and the exact form of the damping model. Indeed, the damping model greatly impacts these gains (see [2]-[4]). One approach to modelling second order damped systems is to start with the undamped equation and then "add" a damping term of the form $\gamma D_0 \dot{x}(t)$ to the second order system. Very often the damping operator $D_0$ is assumed to have the form of a fractional power of the structural operator, i.e., $D_0 = A^\alpha$ for $0 \leq \alpha \leq 1$. This approach leads to formal models that mimic various damping models such as structural damping ($\alpha = 1/2$) and Kelvin-Voigt damping ($\alpha = 1$). However, in order to turn this formal second order system into a well-posed dynamical system on an appropriate state space, one is often faced with having to deal with fractional powers of differential operators leading to pseudo-differential operators. In this note we present several formulations of this problem, one of which makes use of physics based modelling. In many cases this approach can greatly simplify the analysis and the resulting first order system has an explicit representation that avoids pseudo-differential operators.

# 2 Abstract Second Order Damped Models

Let $H$ be a Hilbert space. We assume that $A$ is a self-adjoint, strictly positive operator on $H$ with domain $Dom(A)$ dense in $H$. Consider the undamped second order control system

$$\ddot{x}(t) + Ax(t) = Bu(t), \tag{2.1}$$

where $B$ is a compact linear operator. It is well known that in this case, (2.1) is not stabilizable (for example, see [6]). All elastic systems have some internal damping and the exact form of this damping is important in the analysis and solution of control problems for elastic systems. In this paper we concentrate on the development of explicit state space models for the uncontrolled systems. The application of these models to control design will appear in a future paper.

The general mathematical model often used as a prototype for describing controlled elastic systems with internal damping is obtained by adding a damping term of the form $\gamma A^\alpha \dot{x}(t)$ to (2.1) producing the abstract equation

$$\ddot{x}(t) + Ax(t) + \gamma A^\alpha \dot{x}(t) = Bu(t), \tag{2.2}$$

where

$$0 < \gamma, \quad 0 < \alpha \leq 1.$$

To write this second order system as a first order dynamical system one first defines the spaces

$$V = Dom(A^{1/2}) \tag{2.3}$$

and

$$E = Dom(A^{1/2}) \times H = V \times H, \tag{2.4}$$

with inner products

$$\langle u_1, u_2 \rangle_V = \left\langle A^{1/2} u_1, A^{1/2} u_2 \right\rangle_H \tag{2.5}$$

and

$$\left\langle \begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \right\rangle_E = \left\langle A^{1/2} u_1, A^{1/2} u_2 \right\rangle_H + \langle w_1, w_2 \rangle_H = \langle u_1, u_2 \rangle_V + \langle w_1, w_2 \rangle_H, \tag{2.6}$$

respectively.

Let $\hat{A}_\alpha$ denote the operator defined on $E$ by

$$\hat{A}_\alpha = \begin{bmatrix} 0 & I \\ -A & -\gamma A^\alpha \end{bmatrix}$$

with domain $Dom(\hat{A}_\alpha)$ defined by

$$Dom(\hat{A}_\alpha) = Dom(A) \times \left[ Dom(A^{1/2}) \cap Dom(A^\alpha) \right].$$

It is well known (see, [1] and [5]) that the operator $\hat{A}_\alpha$ is densely defined and dissipative on $E$ and hence closable. The closure of $\hat{A}_\alpha$, denoted by $A_\alpha$, generates a strongly continuous semigroup on $E$ and the domain, $Dom(A_\alpha)$, is given by

$$Dom(A_\alpha) = \left\{ \begin{bmatrix} u \\ w \end{bmatrix} \in E \mid w \in V, \ \{Au + \gamma A^\alpha w\} \in H \right\}. \tag{2.7}$$

Although (2.7) provides one representation of the domain of $A_\alpha$, other (more explicit) representations can also be obtained. We proceed to describe two formulations of this closure based on factorizations of $A$ which we denote as $A_{\alpha,1}$ and $A_{\alpha,2}$. In order to keep this note at a minimal length and yet present the basic ideas, we restrict ourselves to the case where $1/2 \leq \alpha \leq 1$. In this case the following result provides a representation of $Dom(A_\alpha)$ in terms of the fractional powers of $A$. The first formulation, $A_{\alpha,1}$, is a special case of Theorem 1.1 in [5].

**Theorem 2.1** *If $\frac{1}{2} \le \alpha \le 1$, then $Dom(A_\alpha) = Dom(A_{\alpha,1})$ where*

$$Dom(A_{\alpha,1}) = \left\{ \begin{bmatrix} u \\ w \end{bmatrix} \in E \mid u \in Dom(A^{3/2-\alpha}), w \in V, \right. \\ \left. \{A^{1-\alpha}u + \gamma w\} \in Dom(A^\alpha) \right\}, \tag{2.8}$$

*and if $z \in Dom(A_{\alpha,1})$, then*

$$A_{\alpha,1}z = A_{\alpha,1} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} w \\ -A^\alpha\{A^{1-\alpha}u + \gamma w\} \end{bmatrix}. \tag{2.9}$$

*Moreover, $A_{\alpha,1}$ generates an analytic semigroup on $E$.*

One way to view the representation (2.8)-(2.9) is to think of "factoring" (2.2) so that it is written in the form

$$\ddot{x}(t) + A^\alpha\{A^{1-\alpha}x(t) + \gamma\dot{x}(t)\} = Bu(t), \tag{2.10}$$

and then constructing the first order system based on this model. However, this approach does not always capture the true physics of the problem. In the next section we consider a second factorization and compare the corresponding first order model to the system with $A_{\alpha,1}$ given by (2.8)-(2.9).

# 3 A Symmetric Factorization

Another factorization of (2.2), that seems equally justified, is based on factoring $A$ as $A = A^{1/2} \cdot A^{1/2}$ and writing (2.2) in the form

$$\ddot{x}(t) + A^{1/2}\{A^{1/2}x(t) + \gamma A^{\alpha-1/2}\dot{x}(t)\} = Bu(t). \tag{3.1}$$

This is a very natural factorization and it leads to a "physics based" formulation of the first order model. This form of (2.2) leads us to consider the operator $A_{\alpha,2}$ defined on $E$ by

$$Dom(A_{\alpha,2}) = \left\{ z = \begin{bmatrix} u \\ w \end{bmatrix} \in E \mid u, w \in V,\ \{A^{1/2}u + \gamma A^{\alpha-1/2}w\} \in V \right\} \tag{3.2}$$

and for $z \in Dom(A_{\alpha,2})$,

$$A_{\alpha,2}z = A_{\alpha,2} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} w \\ -A^{1/2}\{A^{1/2}u + \gamma A^{\alpha-1/2}w\} \end{bmatrix}. \tag{3.3}$$

At first glance it might appear that $A_{\alpha,1}$ and $A_{\alpha,2}$ have little in common. However, the following result establishes the equivalence between $A_{\alpha,1}$ and $A_{\alpha,2}$ for the case where $1/2 \le \alpha \le 1$. Hence, it follows that (3.2)-(3.3) provides another representation of the closure of $\hat{A}_\alpha$.

**Theorem 3.1** *If* $\frac{1}{2} \le \alpha \le 1$, *then* $A_\alpha = A_{\alpha,1} = A_{\alpha,2}$.

**Proof.** If $\frac{1}{2} \le \alpha \le 1$, then $0 \le \alpha - \frac{1}{2} \le \frac{1}{2}$ and $\frac{1}{2} \le \frac{3}{2} - \alpha \le 1$. It follows that

$$Dom(A^{3/2-\alpha}) \subseteq Dom(A^{1/2}) \subseteq Dom(A^{\alpha-1/2}). \tag{3.4}$$

Let $z = [u, w]^T \in Dom(A_{\alpha,1})$. We show that $z \in Dom(A_{\alpha,2})$ and $A_{\alpha,1}z = A_{\alpha,2}z$.
Since $z = [u, w]^T \in Dom(A_{\alpha,1})$, it follows that

$$u \in Dom(A^{3/2-\alpha}) \subseteq Dom(A^{1/2}), \qquad w \in Dom(A^{1/2}) \subseteq Dom(A^{\alpha-1/2})$$

and

$$\{A^{1-\alpha}u + \gamma w\} \in Dom(A^\alpha).$$

If $y = -A^\alpha\{A^{1-\alpha}u + \gamma w\} \in H$, then $A^{-\alpha}y \in Dom(A^\alpha)$, where

$$A^{-\alpha}y = -\{A^{1-\alpha}u + \gamma w\} = -A^{1/2}\{A^{1/2-\alpha}u + \gamma A^{-1/2}w\}. \tag{3.5}$$

Also,

$$-\{A^{1/2-\alpha}u + \gamma A^{-1/2}w\} = A^{-1/2-\alpha}y = A^{-\alpha}(A^{-1/2}y) \in Dom(A^\alpha). \tag{3.6}$$

Since

$$A^{1/2-\alpha}u = A^{-\alpha}(A^{1/2}u) \in Dom(A^\alpha),$$

it follows from (3.6) that $\gamma A^{-1/2}w \in Dom(A^\alpha)$. Hence,

$$-\{A^{1/2}u + \gamma A^{\alpha-1/2}w\} = -A^\alpha\{A^{1/2-\alpha}u + \gamma A^{-1/2}w\} = (A^{-1/2}y) \in Dom(A^{1/2}), \tag{3.7}$$

and

$$-A^{1/2}\{A^{1/2}u + \gamma A^{\alpha-1/2}w\} = y \in H. \tag{3.8}$$

Combining (3.4), (3.7) and (3.8), it follows that $z \in Dom(A_{\alpha,2})$ and $A_{\alpha,2}z = A_{\alpha,1}z$.
Conversely, assume that $z = [u, w]^T \in Dom(A_{\alpha,2})$. Let

$$y = -A^{1/2}\{A^{1/2}u + \gamma A^{\alpha-1/2}w\} \in H$$

and observe that

$$A^{-1/2}y = -\{A^{1/2}u + \gamma A^{\alpha-1/2}w\} = -A^\alpha\{A^{1/2-\alpha}u + \gamma A^{-1/2}w\} \in Dom(A^{1/2}), \tag{3.9}$$

implies

$$-\{A^{1/2-\alpha}u + \gamma A^{-1/2}w\} = A^{-1/2-\alpha}y = A^{-1/2}(A^{-\alpha}y) \in Dom(A^{1/2}). \tag{3.10}$$

However, $\gamma A^{-1/2}w \in Dom(A^{1/2})$, so it follows from (3.10) that

$$A^{1/2-\alpha}u \in Dom(A^{1/2}).$$

Hence,

$$-A^{1/2}\{A^{1/2-\alpha}u+\gamma A^{-1/2}w\}=-\{A^{1-\alpha}u+\gamma w\}=(A^{-\alpha}y)\in Dom(A^{\alpha}), \quad (3.11)$$

and

$$-A^{\alpha}\{A^{1-\alpha}u+\gamma w\}=y\in H. \quad (3.12)$$

Note that $w\in V=Dom(A^{1/2})$ and $\{A^{1-\alpha}u+\gamma w\}=A^{-\alpha}y\in Dom(A^{\alpha})$. Hence, $A^{1-\alpha}u=A^{-\alpha}y-\gamma w\in Dom(A^{1/2})$. If $\hat{y}=A^{1/2}(A^{1-\alpha}u)$, then

$$u=A^{\alpha-1}(A^{-1/2}\hat{y})=A^{\alpha-3/2}\hat{y}=A^{-(3/2-\alpha)}\hat{y}\in Dom(A^{3/2-\alpha}).$$

Combining (3.4) with (3.11) and (3.12), it follows that $z\in Dom(A_{\alpha,1})$ and that $A_{\alpha,1}z=A_{\alpha,2}z$. This completes the proof.

This result shows that there are several equivalent representations of the closure of $\hat{A}_{\alpha}$. Although (2.8)-(2.9) and (3.2)-(3.3) both provide explicit characterizations of the domain of this closure, both characterizations are in terms of fractional powers of $A$. However, the representation (3.2)-(3.3) can be very useful. For example the following theorem follows from a direct calculation.

**Theorem 3.2** *If $\frac{1}{2}\leq\alpha\leq 1$, then the Hilbert adjoint $[A_{\alpha,2}]^*$ is defined on the domain*

$$Dom([A_{\alpha,2}]^*)=\left\{z=\begin{bmatrix}u\\w\end{bmatrix}\in E\mid u,w\in V,\ \{A^{1/2}u-\gamma A^{\alpha-1/2}w\}\in V\right\} \quad (3.13)$$

*by*

$$[A_{\alpha,2}]^*z=[A_{\alpha,2}]^*\begin{bmatrix}u\\w\end{bmatrix}=\begin{bmatrix}-w\\A^{1/2}\{A^{1/2}u-\gamma A^{\alpha-1/2}w\}\end{bmatrix}. \quad (3.14)$$

Observe that although $A_{\alpha,2}$ and $[A_{\alpha,2}]^*$ have a similar structure, they have different domains. In particular, $Dom([A_{\alpha,2}]^*)\neq Dom(A_{\alpha,2})$ and $Dom([A_{\alpha,2}]^*)\cap Dom(A_{\alpha,2})\subseteq Dom(A)\times Dom(A^{\alpha})$. Again, the adjoint is given in terms of fractional powers of $A$, and in many cases, these operators are pseudo-differential operators without simple explicit representations. A third approach to this problem is based on returning to fundamental physics.

# 4 A Physics Based Factorization

The factorization (3.1) is a special case of a more general form. If we define the operators

$$S=A^{1/2} \text{ and } D=A^{\alpha-1/2},$$

then (3.1) can be written as

$$\ddot{x}(t)+S^*\{Sx(t)+\gamma D\dot{x}(t)\}=Bu(t). \quad (4.1)$$

Observe that $S^* = S = A^{1/2}$ since $A$ is assumed to be self-adjoint and positive definite. Moreover, the basic spaces given in (2.3) - (2.6) are defined in terms of $Dom(S)$ by

$$V = Dom(A^{1/2}) = Dom(S), \tag{4.2}$$

and

$$E = Dom(S) \times H = V \times H, \tag{4.3}$$

with inner products

$$\langle u_1, u_2 \rangle_V = \langle Su_1, Su_2 \rangle_H \tag{4.4}$$

and

$$\left\langle \begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \right\rangle_E = \langle Su_1, Su_2 \rangle_H + \langle w_1, w_2 \rangle_H = \langle u_1, u_2 \rangle_V + \langle w_1, w_2 \rangle_H, \tag{4.5}$$

respectively.

However, it is sometimes more useful to use a different factorization. We illustrate the basic idea by restricting attention to a simple 1D wave equation. Although the presentation here is focused on this example problem, the approach can be extended to a wide variety of 2D and 3D problems in elasticity. Consider the problem of an undamped vibrating string on the interval $0 \le s \le 1$ with fixed left end and free right end. If $w(t,s)$ denotes the displacement of the string and $\sigma(t,s)$ denotes the stress, then the wave equation becomes

$$\frac{\partial^2}{\partial t^2} w(t,s) - \frac{\partial}{\partial s} \sigma(t,s) = 0, \tag{4.6}$$

with displacement boundary condition at $s = 0$

$$w(t,0) = 0, \tag{4.7}$$

and "natural" boundary condition at $s = 1$

$$\sigma(t,1) = 0. \tag{4.8}$$

The strain is defined by $\varepsilon(t,s) = \frac{\partial}{\partial s} w(t,s)$ and if one uses the stress-strain law

$$\sigma(t,s) = \tau \varepsilon(t,s), \tag{4.9}$$

then the equation (4.6) becomes

$$\frac{\partial^2}{\partial t^2} w(t,s) - \frac{\partial}{\partial s} \left\{ \tau \frac{\partial}{\partial s} w(t,s) \right\} = 0. \tag{4.10}$$

The appropriate boundary conditions are

$$w(t,0) = 0, \quad \tau \frac{\partial}{\partial s} w(t,1) = 0. \tag{4.11}$$

On the other hand if one uses a *dynamic* stress-strain law such as

$$\sigma(t,s) = \tau\varepsilon(t,s) + \gamma\frac{\partial}{\partial t}\varepsilon(t,s), \tag{4.12}$$

then the equation (4.6) becomes

$$\frac{\partial^2}{\partial t^2}w(t,s) - \frac{\partial}{\partial s}\left\{\tau\frac{\partial}{\partial s}w(t,s) + \gamma\frac{\partial^2}{\partial t\partial s}w(t,s)\right\} = 0. \tag{4.13}$$

In this case the appropriate boundary conditions are now

$$w(t,0) = 0, \quad \sigma(t,1) = \left\{\tau\frac{\partial}{\partial s}w(t,1) + \gamma\frac{\partial^2}{\partial t\partial s}w(t,1)\right\} = 0. \tag{4.14}$$

The partial differential equation in (4.13) with boundary conditions (4.14) can be written as a second order system in the space of *virtual displacements* $H = L^2(0,1)$. We also introduce the space of *deformations* (or strains) $\Sigma = L^2(0,1)$ and define the operator $S$ from $H$ into $\Sigma$ on the domain

$$Dom(S) = H^1_L(0,1) = \left\{w(\cdot) \in H^1(0,1) \mid w(0) = 0\right\} \tag{4.15}$$

by

$$[Sw(\cdot)](s) = \frac{d}{ds}w(s). \tag{4.16}$$

Then the adjoint of $S$ is defined on $\Sigma$ into $H$ by

$$Dom(S^*) = H^1_R(0,1) = \left\{\sigma(\cdot) \in H^1(0,1) \mid \sigma(1) = 0\right\} \tag{4.17}$$

and for $\sigma(\cdot) \in Dom(S^*)$

$$[S^*\sigma(\cdot)](s) = -\frac{d}{ds}\sigma(s). \tag{4.18}$$

If the damping operator $D : H^1_L(0,1) \to L^2(0,1)$ is defined by $D = S$, then the wave equation (4.13) with boundary conditions (4.14) can be written as

$$\ddot{x}(t) + S^*\{\tau Sx(t) + \gamma D\dot{x}(t)\} = 0. \tag{4.19}$$

Observe that we do not distribute $S^*$ through the brackets. In fact, (4.19) is the proper form of the physics based second order model for Kelvin-Voigt damping. Moreover, the first order form of (4.19) is easily expressed in terms of the basic operators $S$ and $D$. To construct the first order model we define the spaces

$$V = Dom(S) = H^1_L(0,1) \tag{4.20}$$

and

$$E = V \times H = Dom(S) \times H = H^1_L(0,1) \times L^2(0,1), \tag{4.21}$$

with inner products

$$\langle u_1, u_2 \rangle_V = \langle \tau S u_1, S u_2 \rangle_H \tag{4.22}$$

and

$$\left\langle \begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \right\rangle_E = \langle \tau S u_1, S u_2 \rangle_H + \langle w_1, w_2 \rangle_H = \langle u_1, u_2 \rangle_V + \langle w_1, w_2 \rangle_H \,, \tag{4.23}$$

respectively. Now define $\mathcal{A}$ on $E = H_L^1(0,1) \times L^2(0,1)$ by

$$Dom(\mathcal{A}) = \left\{ z = \begin{bmatrix} u \\ w \end{bmatrix} \in E \mid u, w \in V, \; \{\tau S u + \gamma D w\} \in Dom(S^*) \right\}, \tag{4.24}$$

where for $z \in Dom(\mathcal{A})$,

$$\mathcal{A} z = \mathcal{A} \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} w \\ -S^*\{\tau S u + \gamma D w\} \end{bmatrix}. \tag{4.25}$$

Note that (4.24) - (4.25) is similar to (3.2) - (3.3) where $\alpha = 1$. However, there are two primary differences between the operators $\mathcal{A}$ and $A_{1,2}$. The operator $S$ is not a fractional power of $A = S^*S$ and $S^* \neq S$. Therefore, the operator defined by (4.24) - (4.25) is (in general) less "complicated" than $A_{1,2}$ and yet we still have the following easily established result.

**Theorem 4.1** *The operator $\mathcal{A}$ defined by (4.24) - (4.25) generates an analytic semigroup on $E = H_L^1(0,1) \times L^2(0,1)$.*

The above theorem has been extended to other PDE based models of elastic systems. Currently, we are working on a framework that applies to general abstract second order systems of the form (4.19). This framework has the advantage that the underlying spaces and operators are basic differential operators defined on standard Sobolev spaces. In addition, the physics based factorization is the natural choice when developing approximations (see [1]).

It is interesting to observe that if one starts with the undamped equation (4.10) with physical boundary conditions (4.11) and simply "adds a damping term", then it is possible to lose the correct physical boundary conditions. For example, define $A$ on $H = L^2(0,1)$ with the domain

$$Dom(A) = \left\{ w(\cdot) \in H^2(0,1) : w(0) = 0, \frac{d}{ds} w(1) = 0 \right\} \tag{4.26}$$

by

$$[Aw(\cdot)](s) = -\frac{d^2}{ds^2} w(s). \tag{4.27}$$

The wave equation (4.6) becomes

$$\ddot{x}(t) + Ax(t) = 0, \tag{4.28}$$

and if one adds a damping term with $D_0 = A^1$ (i.e. $\alpha = 1$), one obtains the second order system

$$\ddot{x}(t) + \gamma A\dot{x}(t) + Ax(t) = 0. \tag{4.29}$$

Note that (4.29) is actually the abstract form of the damped wave equation

$$\frac{\partial^2}{\partial t^2}w(t,s) - \frac{\partial}{\partial s}\left\{\tau\frac{\partial}{\partial s}w(t,s) + \gamma\frac{\partial^2}{\partial t\partial s}w(t,s)\right\} = 0, \tag{4.30}$$

with boundary conditions

$$w(t,0) = 0, \quad \tau\frac{\partial}{\partial s}w(t,1) = 0.$$

However, the correct physical boundary conditions (given by (4.7) and (4.8)) should be

$$w(t,0) = 0, \quad \left\{\tau\frac{\partial}{\partial s}w(t,1) + \gamma\frac{\partial^2}{\partial t\partial s}w(t,1)\right\} = 0. \tag{4.31}$$

Therefore, the system (4.29) is not the abstract form of the physics based model defined by the partial differential equation (4.30) with boundary conditions (4.31). We close by remarking that although (4.29) does not "capture" the correct physical boundary conditions, that is not to say that $A_{\alpha,1}$ defined by (2.8)- (2.9) is not important in the study of such systems. However, it is crucial to understand that this system may not be the abstract form of the physical problem that is under control.

# 5 Conclusions

In this paper, we present three formulations of the abstract form of damped second order systems based upon different factorizations of the structural operator. One form which is based upon the physics is especially useful in that the underlying operators are differential operators with simple explicit representations. Further, this form captures the physics, specifically, the correct boundary conditions. This formulation has been extended to other PDE based models of elastic systems. Additionally, we are currently working on a framework that applies to general abstract second order systems of the form (4.19). This framework has the advantage that the underlying spaces and operators are basic differential operators defined on standard Sobolev spaces. Moreover, this framework is a natural choice when developing approximations (see [1]).

# References

[1] H.T. Banks, R.C. Smith and Y. Wang, *Smart Material Structures: Modeling, Estimation and Control*, Masson/John Wiley, Paris/Chichester, 1996.

[2] J.A. Burns and B.B. King, "Representation of Feedback Operators for Hyperbolic Systems," *Computation and Control IV*, Eds. K. Bowers and J. Lund, Birkhäuser, Boston, 1995, pp. 57-73.

[3] J.A. Burns and B.B. King, "Optimal Sensor Location for Robust Control of Distributed Parameter Systems," *Proc. $33^{rd}$ IEEE Conf. on Decision and Control*, Orlando, FL, Dec. 1994, pp. 3967-3972.

[4] J.A. Burns and B.B. King, "A Reduced Basis Approach to the Design of Low Order Controllers for Nonlinear Continuous Systems," *J. Vibration and Control,* to appear.

[5] S. Chen and R. Triggiani, "Characterization of Domains of Fractional Powers of Certain Operators Arising in Elastic Systems and Applications," *J. Diff. Equations*, v. 31, 1993, pp. 279-293.

[6] J.S. Gibson, "A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Linear Feedback," *SIAM J. Contr. and Opt.*, v. 18, 1980, pp. 311-316.