

# Renormalization Group (RG) in Turbulence: Historical and Comparative Perspective<sup>1</sup>

Ye Zhou<sup>1,2</sup>, W. David McComb<sup>3</sup>, and George Vahala<sup>4</sup>

<sup>1</sup>Institute for Computer Applications in Science and Engineering  
NASA Langley Research Center, Hampton, VA 23681

<sup>2</sup>IBM Research Division, T.J. Watson Research Center  
P.O. Box, 218, Yorktown Heights, NY 10598

<sup>3</sup>University of Edinburgh, Edinburgh, United Kingdom

<sup>4</sup>College of William and Mary, Williamsburg, VA 23185

## Abstract

The terms *renormalization* and *renormalization group* are explained by reference to various physical systems. The extension of renormalization group to turbulence is then discussed; first as a comprehensive review and second concentrating on the technical details of a few selected approaches. We conclude with a discussion of the relevance and application of renormalization group to turbulence modelling.

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# Part I. Introduction

## Part I

# Introduction

## 1 Preliminary

The term “RG” (or “RNG”) has now become familiar to the turbulence modeling and computational fluid dynamics (CFD) community. However, understanding just what RG is actually about is often viewed as an obscure mathematical problem. Accordingly, we feel that there is a need for an appropriate article for this diverse community. The present review article, which surveys many aspects of applications of RG to turbulence, is intended to fill this gap.

Turbulence in fluids has presented a notoriously difficult problem for more than a century. As discussed in standard text books on fluid motion, the major difficulty is the simultaneous existence of many different space and time scales at high Reynolds number. All these scales are of equal importance, hence the theoretical difficulty.

Now, RG was a tool originally developed for dealing with such problems. Indeed, the work of Wilson in developing the RG method for critical phenomena led to his being awarded the Nobel prize in Physics in 1982. In *The Nobel Prize Winners: Physics* (Magill 1989), we read:

“Wilson clarified a natural phenomenon that has puzzled scientists throughout the centuries: the behavior of substances at critical points, or phase transitions. With his renormalization group theory, Wilson divided this seemingly insoluble problem into a number of small ones.”

### 1.1 General remarks

As we have said, at high Reynolds numbers ( $R_e$ ), a wide range of length and time scales in turbulent flows become excited. Direct numerical simulations (DNS) – which are three-dimensional, time-dependent numerical solutions in which all significant scales of motion are computed without any modeling – are currently restricted to low turbulence Reynolds numbers and simple geometries (Reynolds 1990). Some recent DNS of homogeneous and isotropic turbulence at  $512^3$  resolutions are Chen *et al.* (1993), Wang *et al.* (1996), and Yeung and Zhou (1997).

Although DNS can be extremely useful in many areas related to the study of turbulence physics and the assessment of theories (*ibid*), virtually all scientific and engineering calculations of nontrivial turbulent flows, at high  $R_e$ , are based on some

type of modeling (Speziale 1991). Most modeling efforts have been focused on subgrid scale models for large-eddy simulations and Reynolds stress models – the two forms of modeling that will be discussed in this article.

Large-eddy simulations (LES) are a logical ‘modeling extension’ of DNS. They are based on the observation that the small scales are more universal in character than the large, energy containing scales of motion. In LES, the three-dimensional time-dependent motion of these large scales are computed directly while the effects of the small scales on the large scales are modeled. This procedure leads to the so called ‘subgrid scale modeling’ problem; see Rogallo and Moin (1984), (Reynolds 1990), Lesieur and Metaif (1996), and Moin (1997). The traditional concept of eddy viscosity (Smagorinsky, 1963) is that the subgrid turbulence simply enhances the viscous term in the momentum equation, and that the value of the enhanced eddy viscosity can be determined by equating the resulting increase in the energy dissipation rate to the rate at which energy is transferred from resolved to subgrid scales. LES is useful in the study of turbulence physics at high Reynolds numbers that are unattainable by DNS.

LES is also intended to be useful in the development of turbulence models for the prediction of the complex flows of technical interest where simpler modeling approaches fail (Reynolds 1990). However, all applications of LES have been in simple geometries where Van Driest damping could be used – an empirical approach that generally does not work well when there is flow separation (Speziale 1991).

We note that LES and DNS are recent developments, being direct products of the rapid increase in computer capabilities. The Reynolds stress modeling concept, however, was introduced by Osborne Reynolds as long ago as 1895. Reynolds stress models remain the most important tool for the highly repetitive engineering calculations associated with design (Reynolds 1990). The goal of Reynolds stress modeling is to provide reliable information about first and second one-point moments (e.g., the mean velocity, mean pressure, and turbulence intensity), which is usually all that is needed for design purposes (Speziale 1991). In principle, this approach attempts to model all of the fluctuating scales.

Recently, the methodology of RG has attracted considerable attention as a systematic approach to subgrid scale modeling in LES and Reynolds stress modeling. In principle, the RG removal of only the smallest scales generates a subgrid scale model while Reynolds stress models are obtained in the limit when all fluctuating scales are removed. This point of view is due to Yaghoti and Orszag (1986).

## 1.2 Overview

Because we are addressing such a diverse community, some innovation in the structure of the review is indicated. Part II is intended for those readers who wish to obtain a basic knowledge of “RG” and what are the main results from this approach to studying turbulence; and in that sense it is complete within itself.

Part III is for those readers who would like to go more deeply into RG theory. Here we present the detailed mathematical procedures of two RG approaches. In particular, we clarify their major approximations for the benefit of the reader who is willing to pursue this line of research. In this part, we have included several related pieces of work and have also attempted to compile a comprehensive reference list.

Some turbulence modelers or CFD workers may wish to skip Part III and proceed directly to Part IV. Here we present recent efforts to develop turbulent Reynolds stress models based on RG.

## Part II

# General description of RG

## 2 What Is Renormalization Group?

In this section we shall explain something of the origins and meaning of RG. In the process we shall define terms and introduce concepts which will be needed when we consider the question of how RG can be applied to fluid turbulence.

We should begin by noting the difference between the concept of renormalization and renormalization group. The renormalization procedure was introduced originally as a method of removing divergences in quantum field theory. The renormalization group theory, on the other hand, was used for improving perturbation theory (Gell-Mann and Low, 1953) by exploiting the non-uniqueness in the renormalization procedure (Stuechelberg and Petermann, 1953). We can illustrate this idea by considering a problem which may be discussed entirely in terms of classical physics. This is the electron gas, which consists of a gas of electrons in a uniform background of positive charge, so that the system is, overall, electrically neutral. Such a gas can be a model for the conduction electrons in a metal, for a plasma, or for an electrolyte. This allows us to introduce the general idea of renormalization, and is the subject of Section 2.1. In order to introduce the more specific topic of renormalization group, we consider as an example the phase transition from para- to ferromagnetism. This is dealt with in Section 2.2. We will review the applications of RG to critical phenomena in Sections 2.3.

### 2.1 Renormalization

It is a cardinal assumption of physics that the macroscopic properties of a material, such as its dielectric constant or elastic modulus, will not in general depend on the size of the sample of material under consideration. So, for a microscopic theory, this implies that we must be able to take limits, corresponding to the distance between constituent microscopic particles shrinking to zero, or to the overall size of the system going to infinity, without upsetting our theory. However, if we take our example of an electron gas, it is easily seen that problems may arise in both these limits.

Let us suppose, for instance, that we wished to work out the total electrostatic potential energy of all the electrons in the system and that we did that naively by adding up the contributions from individual electrons, each electron contributing the usual Coulomb ' $1/r$ '. It is obvious that a continuum limit would require  $r \rightarrow 0$ , with each potential diverging in that limit. Less obviously, the limit of infinite system size also leads to a divergence. Suppose we take the test charge to be at the origin and assume that the surrounding charges have a uniform density  $\rho$ , say. Then, as the

number of electrons in a spherical shell between  $r$  and  $r + dr$  goes as  $\rho r^2 dr$ , it is easily seen that this overpowers the Coulomb potential to give a total potential which varies as  $2\pi\rho eR^2$ , where  $e$  is the electronic charge and  $R$  is the system size.

The first of these divergences is usually referred to as being ‘ultra-violet’, because short distances correspond to high spatial frequencies. Similarly, the long-range divergence in the second case, is referred to as being an ‘infra-red divergence’. From our point of view, the second case is the more interesting. In a charge neutral plasma, the energy at a point does not diverge with system size because a cloud of ions around each individual electron acts to screen the long-range Coulomb potential and turns it into an exponentially decaying potential (for stationary particles). Theoretically, this effect shows up when we take the interactions between individual electrons into account. In perturbation theory it would correspond to summing a certain class of terms to all orders. In fact, the potential at lowest nontrivial order, the potential becomes  $e \exp[-r/l_D]/r$ , where  $l_D$  is the Debye length and depends on the density of electrons.

This is known as a screened potential. But, an alternative interpretation is to say that we replace the electron charge  $e$  by the renormalized electron charge  $e \exp[-r/l_D]$ . This replacement of a ‘bare’ particle by a ‘dressed’ particle, in order to take account of the effect of interactions, is an important technique in many-body physics and nowadays is widely referred to as *renormalization*. It should be noted that a characteristic feature of the process is that the renormalized properties (as in our example) are scale-dependent.

This discussion serves to introduce the idea of renormalization. Our next step is to consider what is meant by renormalization *group*, and it turns out that this is just one way of achieving renormalization.

## 2.2 Renormalization group

RG was based on the obvious equivalence between taking the continuum limit and making a scaling transformation of the basic variables such as position or momentum. However, this theory was regarded as rather opaque in terms of its underlying physics, and from our point of view, it is better to consider its more modern incarnation in the theory of critical phenomena, in which it is seen as a way of dealing with problems involving a large range of length or time scales. Its possible relevance to turbulence might now begin to emerge; but we shall first consider another simple physical system: the lattice-spin model of a ferromagnet.

Magnetism arises because spins at different lattice sites tend to become aligned. This tendency is opposed by thermal effects, which promote random orientation of the spins. At any fixed temperature, alignments occur on length scales ranging from the lattice spacing  $L_0$  up to some correlation length  $\xi$ . Take a temperature  $T_0$  which



is sufficiently high that  $\xi(T_0)$  is of order  $L_0$ . It is evident that  $\xi$  will increase as  $T$  decreases below  $T_0$ . In fact,  $\xi$  actually becomes infinite as  $T$  approaches the *Curie temperature*, which defines the critical point for this problem. At this critical point, fluctuations on all scales from  $L_0$  up to the size of the material specimen can occur, and therefore a net overall magnetization can appear.

The theoretical problem is to calculate the partition function and thence the thermodynamic properties of the material. The partition function is a sum over all possible spin configurations; although this sum is easily evaluated if the spins at different sites are independent, coupling between spins makes this problem difficult. The fact that all length-scales are (in principle) equally important makes it difficult to know what to do.

The problem is eased to some extent by considering models, such as the Ising model, in which spin vectors are Boolean in character and are either ‘up’ or ‘down’, with no intermediate states permitted. Another helpful feature of this Ising model is that the spins only have nearest-neighbour interactions. Then the application of RG to this problem can be understood as a form of coarse-graining, in the following sense. The version of the RG method, called the block-spin transformation, consists in breaking down an intractable problem with multiple scales of length into a sequence of smaller problems, each of which is confined to a single length scale. This procedure consists of three steps. First, the lattice is divided into blocks of a few spins each ( $N^2$ , say, for a two-dimensional lattice). Then each block is replaced by a single spin whose value is the average of all the spins in the block; there the average is determined by the majority rule. Therefore, a new lattice is created with  $N$  times the original spacing and  $1/N$  the density of the spins. Finally, the original scale is restored by reducing all dimensions by a factor of  $N$ .

We now provide a brief description on the RG transformation with the block spin method. Our starting point is the Hamiltonian  $H_0$ , associated with two spins separated by a distance  $L_0$  (i.e. the lattice spacing). Then we calculate an effective Hamiltonian  $H_1$ , associated with regions of size  $2L_0$ , by averaging out the effects of scales  $L_0$ . Next we calculate  $H_2$ , associated with regions of size  $4L_0$ , with the effects of scales less than or equal to  $2L_0$  averaged out. This well known block spin method is due to Kadanoff (1977).

The above process can be expressed in terms of a transformation  $\mathcal{T}$ , which is applied iteratively:  $\mathcal{T}H_0 \rightarrow H_1, \mathcal{T}H_1 \rightarrow H_2, \mathcal{T}H_2 \rightarrow H_3 \dots$ . At each stage, the length scales are changed:  $L_0 \rightarrow 2L_0, 2L_0 \rightarrow 4L_0 \dots$ , and in order to compensate, the spin variables are also scaled in an appropriate fashion (see next subsection). It is this rescaling which leads to renormalization, and the set of transformations  $\{\mathcal{T}\}$  defines a semi- group: hence *renormalization group*. If iterating this transformation leads to the result that an integer  $N$  exists such that

$$\mathcal{T}H_n = H_{n+1} = H_n, \quad \text{for all } n \geq N$$

then  $H_n = H_N$  (say) is referred to as a *fixed point*. In the case of critical phenomena,

this corresponds to a critical point. We conclude this brief discussion of the magnetic case by indicating how its methodology may be made to come closer to a continuum problem (as in fluid motion), which is what concerns us.

Suppose we make the Fourier transformation from  $\mathbf{x}$ -space to wavevector  $\mathbf{k}$ -space. Further suppose that we have eliminated (in some way) the smallest length scales, less than or equal to  $\Lambda^{-1}$ , say, so that we are left with a Hamiltonian which is a function of continuous variables in  $\mathbf{k}$ -space,  $H(\Lambda)$ . Then we may formally eliminate further modes in a band  $\Lambda/b \leq k \leq \Lambda$ , as follows ( $1 < b < \infty$ ). Denote the three-dimensional spin field by  $S_\alpha(\mathbf{k})$ , where  $\alpha = 1, 2, \text{ or } 3$ , and take the following steps:

1. Integrate out all  $S_\alpha(\mathbf{k})$  for which  $\Lambda/b \leq k \leq \Lambda$ .
2. Rescale the remaining modes of the spin field by enlarging wavevectors by a factor  $b$ .
3. Multiply each  $S_\alpha(\mathbf{k})$  by a constant factor  $\zeta_b$ .

The parameter  $b$  is known as the *spatial rescaling factor*; while  $\zeta_b$  is the spin rescaling factor, and involves technicalities of the magnetic case, which we shall not pursue here. If we write the above three-step process as

$$H' = \mathcal{T}_b H,$$

then the set  $\mathcal{T}_b$ ,  $1 < b < \infty$  is called the renormalization group. It is often remarked that, since  $\mathcal{T}_b^{-1}$  does not exist, this is strictly a semi-group.

We shall now consider RG as applied to the 2D Ising model.

## 2.3 Renormalization group theory in critical phenomena

### 2.3.1 Iterative Hamiltonian structure

As we discussed already, an important and early application of RG was to critical phenomena. Here we shall provide a more detailed review of the calculation of Wilson (1975; Wilson and Kogut, 1974) for the 2D Ising model. In the 2D Ising model of ferromagnetism, one considers a lattice of points and on these lattice sites are particles (called 'spin' particles) which (i) can take on only two values [ i.e., 'spin up' or 'spin down'], and (ii) can only interact with its nearest neighbors. The system is in thermal equilibrium, with a lattice temperature  $k_B T$ . For simplicity, we shall consider a square lattice with many spin particles. From equilibrium statistical mechanics (Huang, 1963), it is known that all thermodynamic properties of this system can be determined from the partition function

$$Z = \sum \exp H_o[s] \tag{1}$$

where the Hamiltonian  $H_o$ , for the 2D Ising model,

$$H_o[s] = K \sum \sum s_n s_{n+i} \quad (2)$$

$K = -J/k_B T$ , where  $J$  is the spin-spin interaction coupling constant [with  $J > 0$  for ferromagnetism]. The sum in (1) is evaluated over all possible spin configurations. The outer sum in (2) is over all lattice points, while the inner sum, for the representative lattice site vector  $n = (n_1, n_2)$ , is a sum over the four nearest neighbors of the square lattice

$$\sum_i s_n s_{n+i} = s_n s_{(n_1+1, n_2)} + s_n s_{(n_1-1, n_2)} + s_n s_{(n_1, n_2+1)} + s_n s_{(n_1, n_2-1)}. \quad (3)$$

The difficulty in determining the thermodynamic properties of this microscopic Ising model is in the evaluation of the summation in (1) : there are just too many degrees of freedom<sup>2</sup>. The important things to note are (i) the original problem involved only nearest-neighbor interactions, (ii) the coupling strength  $K$  is related to the strength of the interactions,  $J$ , and the lattice temperature  $k_B T$ .

### 2.3.2 Iterative decimation of the number of degrees of freedom

Suppose that the degrees of freedom are decimated by restricting the summation in Eq. (1) to run only over half the lattice sites. Thus the partition function  $Z$  must now be determined from

$$Z = \sum_1 \exp[H_1[\sigma]] \quad (4)$$

where the new lattice sites are denoted as  $\sigma$ , the sum is over these sites only, and  $H_1$  is a Hamiltonian to be determined so that (4) yields the identical partition function as Eq. (1).

It can be shown that the new Hamiltonian  $H_1[\sigma]$  takes the form

$$H_1[\sigma] = (N^2/2)A + 2B \sum_{nn} \mu_i \mu_k + B \sum_{nnn} \mu_i \mu_j \mu_k + C \sum_p \mu_i \mu_j \mu_k \mu_l \quad (5)$$

where ‘‘nn’’ denotes nearest-neighbor coupling, ‘‘nnn’’ next-nearest-neighbor coupling and ‘p’ the plaquette four-spin interactions (Hu, 1982) with

$$A(k) = \ln 2 + (1/8)[\ln(\cosh 4K) + 4\ln(\cosh 2K)] \quad (6)$$

$$B(k) = (1/8)\ln(\cosh 4K) \quad (7)$$

$$C(k) = (1/8)[\ln(\cosh 4K) - 4\ln(\cosh 2K)] \quad (8)$$

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<sup>2</sup>We are actually using this 2D Ising model as an example of how to treat problems that have so many degrees of freedom that new techniques must be introduced to solve them. The 2D Ising model is actually exactly soluble, but that is not of direct concern to us.

Here  $A$  is simply a normalization. Eq. (5) should be compared to the original Ising Hamiltonian  $H_0$ , eq. (2). First we see the expected changes: (a) the nearest neighbor interactions term has strength  $2B(K)$ , rather than  $K$ , and (b) there is a normalization constant  $A(K)$  introduced due to the change in the correlation length of the decimated system, What is of significant interest is the appearance of new, non-Ising, interactions: diagonal nearest neighbor interactions [strength  $B(k)$ ], and four spin interactions [with strength  $C(k)$ ]. Therefore, although the original Hamiltonian has only nearest-neighbor coupling, other types of interactions are generated by the renormalization group to compensate for the reduction in the number of degrees of freedom. This is a general feature of the renormalization group (Hu, 1982).

We now proceed to the second iteration to determine the Hamiltonian  $H_2$  for  $N^2/2^2$  lattice sites.  $H_2$  is to be determined so that the partition function  $Z$  remains invariant under decimation of spins. If all the 4 terms are kept in (5) in determining  $H_2$ , then one cannot calculate the sum over the decimated spins exactly. To proceed, one can start with the nearest-neighbor interaction and treat the other types of interactions perturbatively. Wilson proceeds under the assumption that the 4-spin interaction (and any higher order spin interaction) is weak and can be neglected, but it is critical that the “ $uuu$ ” interactions be retained. The self-consistency of these assumptions can be checked at the end of the calculation.

Hence, if at the  $i - 1^{th}$  stage, the Hamiltonian is  $H_{i-1}[s]$ , with its dependence on the Ising (original) nearest-neighbor interaction strength  $K_{i-1}$  and the (non-Ising) diagonal-nearest-neighbor interaction  $L_{i-1}$ , then at the  $i^{th}$  iteration, the Hamiltonian  $H_i[s]$  is a function of  $K_i$  and  $L_i$ . It is important to note that *only* in degenerate cases is the Hamiltonian form invariant under RG transformations.

### 2.3.3 RG transformations

Now for sufficiently weak interactions, one can expand the interaction strength coefficients in a Taylor series in  $K = -J/K_B T$ :

$$B(K) = \ln(\cosh 4K)/8 \approx K^2 + O(K^4) \quad (9)$$

$$C(K) = (1/8)[\ln(\cosh 4K) - 4\ln(\cosh 2K)] = O(K^4), \quad (10)$$

allowing us to drop the 4-spin interaction term in (5). Note further that the coefficients for “ $uu$ ” and “ $uuu$ ” interactions are of the same order in  $K$ , requiring us to keep these newly created diagonal “ $uuu$ ” interactions. To determine the RG transformation, we must note the following: (a) the diagonal nearest-neighbor interaction at the  $i^{th}$  stage,  $L_i$ , arises from the coupling coefficient  $B(K_{i-1}) = K_{i-1}^2 + O(K_{i-1}^4)$ , on using (7): i.e.,

$$L_i = K_{i-1}^2 \quad (11)$$

(b) the nearest-neighbor interaction at the  $i^{th}$  stage arises from 2 terms: (i) the nearest-neighbor interactions at the  $i - 1^{th}$  stage, with coupling coefficient  $B(K_{i-1}) =$

$K_{i-1}^2$ , and (ii) the diagonal-nearest-neighbor interactions at the  $i - 1^{th}$  stage which yields a coupling coefficient  $L_{i-1}$  to  $K_i$ . This is because the diagonal- nearest-neighbor interaction at the  $i - 1^{th}$  stage will become the nearest- neighbor interaction at the  $i^{th}$  stage due to the decimation of spins.

Hence the RG transformations to proceed from the  $i - 1^{th}$  to the  $i^{th}$  stage are

$$K_i = 2K_{i-1}^2 + L_{i-1} \quad (12)$$

$$L_i = K_{i-1}^2 \quad (13)$$

with

$$L_0 = 0 \quad (14)$$

since initially we are dealing with an Ising model which, by definition only has nearest-neighbor interactions.

The fixed points,  $K_*$  and  $L_*$ , of (12)- (14) are immediately determined from:

$$K_* = 2K_*^2 + L_* \quad (15)$$

$$L_* = K_*^2 \quad (16)$$

There are 3 possible fixed points:

$$K_{*1} = 0, L_{*1} = 0 \quad (17)$$

$$K_{*2} = \infty, L_{*2} = \infty. \quad (18)$$

$$K_{*3} = 1/3, L_{*3} = 1/9 \quad (19)$$

and the initial domain of attraction in the K-L plane for these fixed points are readily determined (with  $L_0 = 0$ ): (a) for  $0 < K_0 < 0.39209$ ,  $L_0 = 0$  : stable fixed point is (0,0); (b) for  $K_0 > 0.3921$ ,  $L_0 = 0$  : stable fixed point is  $(\infty, \infty)$ ; (c) for  $K_0 = 0.39209$ ,  $L_0 = 0$  : unstable fixed point is (1/3,1/9)

In the Ising problem, the relevant fixed point is the saddle point (c). The fixed point (a) corresponds to  $K_* = 0 = J/k_B T_*$ , i.e.,  $T_* \rightarrow \infty$ . Physically, this corresponds to a perfectly disordered system, with only short scale fluctuations that tend to zero as the number of RG iterations. The fixed point (b) corresponds to the temperature  $T_* = 0$ , and this corresponds to a perfectly ordered system, again with negligible fluctuations. Hence, as far as critical phenomena are of interest, these fixed points are unimportant. However, the fixed point (c) is important since even after all the RG transformations, there are still many length scales present. Critical exponents prove to be properties of this fixed point.

## 3 Application of RG To Turbulence

### 3.1 Similarity and differences between fluids and critical phenomena

The macroscopic motion of a fluid may be thought of as exhibiting two ‘phase transitions’. That is, as we increase the Reynolds number of a given flow, we may expect first a transition to turbulence and second a transition to scaling behaviour of fully developed turbulence. In the latter case, this means the existence of an intermediate range of wavenumbers in which the energy spectrum takes the form of a power-law. It is this transition which concerns us here. Specifically, we will assume the inertial range ‘5/3’ law for the energy spectrum developed by Kolmogorov [1941]. As an example, consider a stationary homogeneous isotropic turbulence where energy is injected with a forcing spectrum peaked a wavenumber  $k_0(= 1/l_0)$ , where  $l_0$  is the integral length scale. The energetics of eddies with wavenumbers smaller than a given wavenumber  $k$  are determined by an equilibrium between the injection, the losses due to viscous dissipation and the energy flux function  $\Pi(k)$  that moves energy to higher wavenumbers (i.e., to smaller scales). Here,  $\Pi(k)$  depends upon the inertial terms in the Navier-Stokes equation, and is expressible in terms of the triple velocity correlation (see for example, Rose and Sulem, 1978). The problem is defined by three parameters plus the underlying Navier-Stokes equation. These parameters are the kinematic viscosity, the total rate of dissipation,  $\varepsilon$ , which equals the energy flux function, and the characteristic length  $l_0$ . To make contact with the critical phenomena, the external length scale  $l_0$  might be considered analogous to a lattice constant (Nelkin, 1974).

It is of course difficult to identify the onset of scaling behaviour as such and a convenient way to think about this problem is to consider the case where the Reynolds number (based on the turbulent microscale, say) is large enough for a power-law spectrum to exist over an appreciable range of wavenumbers. Now define a *local* Reynolds number, using the inverse of wavenumber as the length scale. Then as we scan through the wavenumbers, from the maximum towards smaller values, we are, in effect, increasing the local Reynolds number to the point where power-law behaviour begins (In other words, the boundary between the inertial and viscous ranges). Conceptually, therefore, the simplest application of RG to turbulence involves the progressive elimination of high- $k$  modes, with the fixed point corresponding to a renormalized viscosity in the inertial range (The equivalent, in terms of CFD, would be a subgrid effective viscosity, in large-eddy simulations). We shall carry out these operations on the Navier-Stokes equations.

### 3.2 RG applied to the Navier-Stokes equations

We introduce the Fourier components of the velocity field  $u_\alpha(\mathbf{k}, t)$  in the usual way,

$$u_\alpha(\mathbf{x}, t) = \sum_{\mathbf{k}} u_\alpha(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (20)$$

Then the solenoidal Navier-Stokes equation for incompressible fluid motion takes the form (e.g. see McComb 1990)

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha(\mathbf{k}, t) = \lambda M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j}, t) u_\gamma(\mathbf{k} - \mathbf{j}, t), \quad (21)$$

where

$$M_{\alpha\beta\gamma}(\mathbf{k}) = (2i)^{-1} \{k_\beta D_{\alpha\gamma}(\mathbf{k}) + k_\gamma D_{\alpha\beta}(\mathbf{k})\}, \quad (22)$$

and

$$D_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|\mathbf{k}|^2}, \quad (23)$$

$\lambda$  is a book-keeping parameter, which can be used to facilitate the iterative solution of the Navier-Stokes equation by expanding in powers of  $\lambda$ . One then sets  $\lambda = 1$  at the end of the calculation. This procedure is well known in statistical and many-body physics and is usually referred to as the  $\lambda$ -expansion. It is believed, however, that this expansion is divergent for the Navier-Stokes equation (Orszag 1977) – nevertheless it is utilized in all RG theories.

Turbulence almost invariably occurs because there is a mean rate of shear in a fluid. If we wish to study stationary, isotropic turbulence, then the artificial nature of this problem requires us to add hypothetical stirring forces to the right-hand side of the Navier-Stokes equations, in order to maintain the turbulence against viscous dissipation. Denoting the random force by  $f_\alpha(\mathbf{k}, t)$ , it is usual to specify its distribution as multivariate normal and to choose the force-force covariance to be

$$\langle f_\alpha(\mathbf{k}, t) f_\beta(-\mathbf{k}, t') \rangle = D_{\alpha\beta}(\mathbf{k}) W(k) \delta(t - t'), \quad (24)$$

where  $W(k)$  has dimensions of *velocity*<sup>2</sup>/*time*, and remains to be specified. Stationarity requires that the rate at which the stirring forces do work is the same as the rate of viscous dissipation  $\varepsilon$ , or

$$\int_0^\infty 4\pi k^2 W(k) dk = \varepsilon, \quad (25)$$

where

$$\varepsilon = \int_0^\infty 2\nu_0 k^2 E(k) dk. \quad (26)$$

It has always been normal practice in turbulence theory, to choose  $W(k)$  to be peaked near the origin, so that its arbitrary nature is only of importance at low

wavenumbers, and a universal energy spectrum can develop at large wavenumbers. More recently, the application of RG to stirred fluid motion (Forster et al., 1977; Yakhot and Orszag. 1986) has introduced theories which depend strongly on the choice of  $W(k)$ . Accordingly, questions of how  $W(k)$  is chosen and justified must then be considered. In particular, in this context, the question of whether or not  $W(k)$  should depend on some characteristic length scale (such as the ultra-violet cutoff  $\Lambda$ ) will surface later as a major issue.

In order to try to carry out the RG algorithm given above in Section 2.2, we consider the Fourier components to be defined on the interval  $0 \leq k \leq k_0$ , where  $k_0$  is the largest wavenumber present and is of the order of the Kolmogorov dissipation wavenumber (e.g. see McComb 1990). Then, for some  $k_1$ , such that  $k_1 < k_0$ , we filter the velocity field at  $k = k_1$ . Here,  $k_1 = (1 - \eta)k_0$  with  $0 \leq \eta < 1$ . This may be expressed in terms of the unit step functions,

$$\theta^-(k) = \begin{cases} 1 & \text{if } 0 \leq k \leq k_1 \\ 0 & \text{if } k_1 \leq k \leq k_0; \end{cases} \quad (27)$$

$$\theta^+(k) = \begin{cases} 0 & \text{if } 0 \leq k \leq k_1 \\ 1 & \text{if } k_1 \leq k \leq k_0, \end{cases} \quad (28)$$

allowing us to define the following useful filtered forms:

$$u_\alpha^-(\mathbf{k}, t) = \theta^-(k)u_\alpha(\mathbf{k}, t),$$

$$u_\alpha^+(\mathbf{k}, t) = \theta^+(k)u_\alpha(\mathbf{k}, t),$$

$$M_{\alpha\beta\gamma}^-(\mathbf{k}) = \theta^-(k)M_{\alpha\beta\gamma}(\mathbf{k}),$$

$$M_{\alpha\beta\gamma}^+(\mathbf{k}) = \theta^+(k)M_{\alpha\beta\gamma}(\mathbf{k}).$$

Then substitution of these forms into equation (24) allows us to decompose the Navier-Stokes equation into separate low- $k$  and high- $k$  forms, viz.,

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu_0 k^2\right)u_\alpha^-(\mathbf{k}, t) &= \lambda M_{\alpha\beta\gamma}^-(\mathbf{k}) \int d^3j \{u_\beta^-(\mathbf{j}, t)u_\gamma^-(\mathbf{k} - \mathbf{j}, t) \\ &\quad + 2u_\beta^-(\mathbf{j}, t)u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \\ &\quad + u_\beta^+(\mathbf{j}, t)u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\}, \quad 0 \leq k \leq k_1 \end{aligned} \quad (29)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu_0 k^2\right)u_\alpha^+(\mathbf{k}, t) &= \lambda M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \{u_\beta^-(\mathbf{j}, t)u_\gamma^-(\mathbf{k} - \mathbf{j}, t) \\ &\quad + 2u_\beta^-(\mathbf{j}, t)u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \\ &\quad + u_\beta^+(\mathbf{j}, t)u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\}. \quad k_1 \leq k \leq k_0 \end{aligned} \quad (30)$$

If we now try to carry out the first of the three steps outlined at the end of Section 2.2, for the lattice-spin Hamiltonian, we can adapt that procedure to the present (Navier-Stokes) case, as follows:



1. Solve equation (30) for  $u^+$ .
2. Substitute this solution into equation (29) for the  $u^-$  and do a partial average over the  $u^+$ .
3. Terms resulting from this procedure which are linear in  $u^-$  can be interpreted as contributing an increment to the turbulent viscosity.

The main problems which arise due to mode-mode coupling, are easily understood:

- First, the solution of (30) for  $u^+$  contains terms in  $u^-$ . When these are substituted into the right-hand side of (29), the result is a triple nonlinearity in the  $u^-$ , which can then generate even higher-order nonlinearities in subsequent iterations.
- Second, averaging out the high-wavenumber modes requires the hypothesis

$$\langle u^- u^+ u^+ \rangle = u^- \langle u^+ u^+ \rangle,$$

which cannot be strictly true, as  $u^-$  and  $u^+$  are just parts of the same velocity field and are not statistically independent.  $\langle \dots \rangle$  represents the partial average over the high-wavenumber modes.

It cannot be emphasized too strongly that these two problems are fundamental stumbling blocks in applying RG to the Navier-Stokes equation. It is the way in which they are tackled which distinguishes one RG theory of turbulence from another. We will discuss in detail two methods for dealing with these problems: the recursive mode elimination and the conditional averaging method. In the approach to RG of Forster *et al* (1977), Fournier and Frisch (1993), which culminated in the theory of Yakhot and Orszag (1986), suitable assumptions are made so that these problems are avoided. This approach is described in the Sec. 6 of this review.

### 3.3 Iterative conditional averaging: a summary

We now introduce a method of eliminating turbulent modes which is based on the use of a conditional average to distinguish between amplitude and phase correlation effects. It has its roots in the method of iterative averaging, which was developed over a number of years as a possible method of applying the renormalization group approach to real fluid turbulence (McComb 1982, 1986, 1990). However, an essential feature of the more recent work is the formal treatment of the conditional average and the development of methods of approximating its relationship to the usual ensemble average (McComb and Watt 1990, 1992; McComb, Roberts and Watt 1992).

The basic idea is that the turbulent velocity field in wavenumber space may be decomposed into two distinct fields. One is a purely *chaotic field*; while the other is a

*correction field*, and carries all the phase information. Application of this decomposition to a thin shell of wavenumbers in the dissipation range allows the elimination of modes in that shell; with the usual mode-coupling problems being circumvented by the use of a conditional average. The (conditional) mean effect of the eliminated modes appears as an increment to the viscosity, with terms of order  $\eta^2$  being neglected, where  $\eta$  is a dimensionless measure of bandwidth thickness, such that  $0 \leq \eta \leq 1$ . An iteration (with appropriate rescaling) to successively lower shells, reaches a fixed point, corresponding to a renormalized turbulent viscosity.

### 3.4 Recursive renormalization group: a summary

If we return to the equilibrium Ising model, in Sec. 2.3, in which the Hamiltonian  $H_0$  involves only nearest-neighbor interactions, it was shown by Wilson (1975) that the effective Hamiltonians  $H_n$  ( $n \geq 1$ ) now involve both nearest-neighbor and diagonal nearest-neighbor interactions. Form invariance of  $H_n$  is then imposed by assuming that the interactions of still higher order are small. Similarly, in our form of recursive RG (Zhou et al., 1988, 1989; Zhou and Vahala, 1992, 1993a,b), as applied to the Navier-Stokes equations, form invariance can be imposed, when both the quadratic and the RG-induced cubic nonlinearities are retained, by truncating the  $\lambda$ -expansion.

The recursive RG theory extends Rose's (1977) treatment of the linear problem of passive scalar convection to Navier-Stokes turbulence and eliminates the small scales recursively. Navier-Stokes equation include (1) a renormalized eddy viscosity and (2) a triple product in the fluid velocity. The eddy viscosity is calculated by means of a difference equation (Zhou *et al.*, 1988, 1989) or, in the limit  $\eta \rightarrow 0$ , by means of a differential equation (Zhou and Vahala, 1992). The resulting eddy viscosity term exhibits a mild cusp behavior in the renormalized momentum equation (Zhou et al., 1988, 1989). The triple velocity products are absent only when there is a spectral gap between the subgrid and resolvable scales (Zhou et al., 1988). Beyond the second iteration, the RG-induced triple products also contribute to the eddy viscosity in the renormalized Navier-Stokes equation. The recursive RG, therefore, has potential for capturing a variety of dynamical features that depends on the interplay between local and nonlocal triad interactions (Zhou, 1993a,b; Zhou *et al.*, 1995).

It is of particular interest to determine the effect of the new triple velocity product on the resolvable scale energy transfer and the difference, if any, from that of the usual Navier-Stokes quadratic velocity product, and we do this by considering their individual contributions to the eddy viscosity. Within recursive RG theory (Zhou and Vahala, 1993a), it has been shown that the triple velocity product in the renormalized momentum equation, which produces a fourth order velocity product in the energy equation, removes energy from the resolved scales. The averaged fourth-order product can be decomposed into a product of averaged second-order products and formally takes the form of a gradient diffusion process with eddy viscosity  $\nu_T$ . The triple term contribution to the eddy viscosity is zero as  $k \rightarrow 0$  and increases rapidly as

$k \rightarrow k_c$ . Here  $k_c$  is the cutoff wavenumber which separates the resolvable and subgrid scales. The spectral eddy viscosity is simply the sum of the contributions from the momentum equation and the triple nonlinear term and it appears to be in qualitative agreement with the closure theory (Kraichnan, 1976; Chollet and Lesieur, 1981) and direct numerical measurement (Domaradzki et al., 1987; Lesieur and Rogallo, 1989; Zhou and Vahala, 1993a). In particular, it predicts the correct asymptotic behaviors of the eddy viscosity as  $k \rightarrow 0$  and  $k \rightarrow k_c$ . The methodology has been applied to the passive scalar being advected by incompressible turbulence (Zhou and Vahala, 1993b).

## Part III

# Technical Aspects of selected RG approaches

## 4 Iterative conditional averaging: details

In this section we amplify our earlier summary of the method and show in some more detail how the iterative conditional averaging technique is used to implement the general strategy of the renormalization group for Navier-Stokes fluid turbulence.

### 4.1 The conditional average

First of all, we formulate the operation of taking a conditional average. Then we use this average in such a way that we can eliminate the triple nonlinearity referred to earlier.

The idea is quite simple. Putting it at its most basic level, we select from the full ensemble of turbulence realizations a sub-ensemble, the members of which have their low- $k$  modes equal to  $u_{\alpha}^{-}(\mathbf{k}, t)$ . Then we perform averages of functions of the  $u_{\alpha}^{+}(\mathbf{k}, t)$  over this sub-ensemble.

However, there is more to it than this. If  $\mathbf{u}(\mathbf{k}, t)$  is the solution of the Navier-Stokes equation, corresponding to prescribed boundary conditions, then we are faced with (in principle) a deterministic process and to prescribe  $\mathbf{u}^{-}$  is to prescribe  $\mathbf{u}^{+}$ . That is to say, if  $\mathbf{u}^{-}$  is invariant under our conditional average, then so also is  $\mathbf{u}^{+}$ . In order to get round this problem, we invoke the defining characteristic of deterministic chaos. This is to the effect that any uncertainty in the specification of the system will be amplified exponentially; so that as time goes on the difference between almost identical solutions will increase to the point of unpredictability. In the present case, we replace the concept of time going on, by the number of steps of the cascade in wavenumber. That is, our ideas about the turbulent cascade, and particularly ideas about localness of energy transfer, suggest that, if we prescribe conditions at wavenumber  $k_1$ , then  $\mathbf{u}^{+}(\mathbf{k}_0, t)$  will be unaffected, provided that  $k_0$  is much larger than  $k_1$ . In other words, as the bandwidth becomes large ( $\eta \rightarrow 1$ ), the conditional average of  $u^{+}$  becomes free of constraint and we can expect that

$$\langle u^{+}(\mathbf{k}_0, t) \rangle_c \rightarrow \langle u^{+}(\mathbf{k}_0, t) \rangle.$$

On the other hand, for  $k_0 \rightarrow k_1$ , it is intuitively clear that the conditional average must tend to become effectively deterministic, with

$$\langle u^{+}(\mathbf{k}_0, t) \rangle_c \rightarrow u^{+}(\mathbf{k}_0, t).$$

We shall return to this at a later stage, but for the present it should be appreciated that the point at issue here is one of phase correlation. If two realizations are slightly out of phase at wavenumber  $k_1$ , then we can (given the nature of turbulence) expect that this phase difference will amplify exponentially, so that phase correlation will decline throughout the band as one moves from  $k_1$  to  $k_0$ . Hence, for  $u^+(\mathbf{k}_0, t)$  to be chaotic, despite the prescription  $u^+(\mathbf{k}_1, t) = u^-(\mathbf{k}_1, t)$ , it is a requirement that the bandwidth be large enough. This imposes a lower bound on possible values of the bandwidth parameter  $\eta$ .

We may take account of these various aspects by introducing a fuzzy criterion for our conditional average. We now choose as our sub-ensemble, the subset of realizations for which the low- $k$  modes differ from  $\mathbf{u}^-(\mathbf{k}, t)$  by a small amount  $\Phi^-(\mathbf{k}, t)$ . Obviously  $\Phi^-$  is an arbitrary criterion (apart from the constraint that  $u^- + \Phi^-$  must be a possible solution of the Navier-Stokes equation) and should be chosen to satisfy the conditions which we wish to impose upon our average.

We shall not go into this procedure any further here, and in order to simplify the algebra, we shall omit the  $\Phi^-$  from the equations. The interested reader who wishes to pursue the matter will find a full account in the papers by McComb and Watt (1992) and McComb *et al* (1992).

## 4.2 Conditionally-averaged equations for high and low wavenumbers

Let us now denote the operation of taking a conditional average over the modes in the band  $k_1 \leq k \leq k_0$  by angle brackets, with a subscript 0. This notation permits the subsequent generalization to subscripts 1, 2,  $\dots$ ,  $n$ , as we remove shells of wavenumbers progressively. Then, we list the ideal defining properties of the conditional average as

$$\langle \mathbf{u}^-(\mathbf{k}, t) \rangle_0 = \mathbf{u}^-(\mathbf{k}, t), \quad (31)$$

$$\langle \mathbf{u}^-(\mathbf{k}, t) \mathbf{u}^-(\mathbf{k}', t) \rangle_0 = \mathbf{u}^-(\mathbf{k}, t) \mathbf{u}^-(\mathbf{k}', t), \quad (32)$$

and so on, for products of the low-wavenumber modes of any order.

We now conditionally average both equations (29) and (30) which are, respectively, the low- $k$  and high- $k$  filtered Navier-Stokes equations.

First, we obtain the conditional-averaged NSE on the interval  $0 \leq k \leq k_1$ , by averaging according to equations (31) and (32), to obtain

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \nu_0 k^2 \right) u_\alpha^-(\mathbf{k}, t) &= M_{\alpha\beta\gamma}^-(\mathbf{k}) \int d^3j \{ u_\beta^-(\mathbf{j}, t) u_\gamma^-(\mathbf{k} - \mathbf{j}, t) \\ &\quad + 2u_\beta^-(\mathbf{j}, t) \langle u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_0 \\ &\quad + \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_0 \}. \end{aligned} \quad (33)$$

We now repeat the steps just taken, but this time we apply them to equation (33) for the high- $k$  modes. Thus, we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u_\alpha^+(\mathbf{k}, t) &= M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \{u_\beta^-(\mathbf{j}, t) u_\gamma^-(\mathbf{k} - \mathbf{j}, t) \\ &\quad + 2u_\beta^-(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t) + u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t)\}, \end{aligned} \quad (34)$$

and taking the conditional average of each term according to equations (32) and (33), gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \nu_0 k^2\right) \langle u_\alpha^+(\mathbf{k}, t) \rangle_0 &= M_{\alpha\beta\gamma}^+(\mathbf{k}) \int d^3j \{u_\beta^-(\mathbf{j}, t) u_\gamma^-(\mathbf{k} - \mathbf{j}, t) \\ &\quad + 2u_\beta^-(\mathbf{j}, t) \langle u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_0 \\ &\quad + \langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_0\}. \end{aligned} \quad (35)$$

### 4.3 First-shell elimination using the two-field decomposition

Our objective now is to solve equation (35) for the conditional average  $\langle u_\beta^+(\mathbf{j}, t) u_\gamma^+(\mathbf{k} - \mathbf{j}, t) \rangle_0$  and substitute the result back into (33), in order to have a closed equation for the low-wavenumber modes. In order to do this, we shall ultimately have to reckon with the need to relate conditional averages to full ensemble averages. Accordingly, we begin this section with the two-field decomposition which is our basis for this procedure.

Let us write the exact decomposition:

$$u_\alpha^+(\mathbf{k}, t) = v_\alpha^+(\mathbf{k}, t) + \Delta_\alpha^+(\mathbf{k}, t), \quad (36)$$

where  $v_\alpha^+(\mathbf{k}, t)$  is any other realisation of our turbulent ensemble. In other words,  $v_\alpha^+(\mathbf{k}, t)$  has exactly the same statistical properties as  $u_\alpha^+(\mathbf{k}, t)$ , but has no phase relationship to  $u_\alpha^-(\mathbf{k}, t)$ . It follows, by definition, that  $\Delta_\alpha^+(\mathbf{k}, t)$  is simply a measure of the phase difference (in the band of modes to be eliminated) between the two realizations. It also follows that, from the point of view of the realization under study,  $v_\alpha^+(\mathbf{k}, t)$  is the purely chaotic part of the field and  $\Delta_\alpha^+(\mathbf{k}, t)$  is the correction field which carries all the phase information.

We are now in a position to write down an expression relating the conditional average of the high-wavenumber part of the velocity field to its full ensemble average. Taking the conditional average of both sides of equation (33), it may be shown McComb *et al* (1992), or it is intuitively obvious, that

$$\langle u_\alpha^+(\mathbf{k}, t) \rangle_0 = \langle v_\alpha^+(\mathbf{k}, t) \rangle + \langle \Delta_\alpha^+(\mathbf{k}, t) \rangle_0. \quad (37)$$

Now we seek a relationship between  $\mathbf{v}^+$  and  $\mathbf{u}^+$ , which is such that the conditionally averaged correction term  $\langle \Delta^+ \rangle_0$  may be neglected as small. In other words, we need

an *ansatz* for the correction term and naturally this will depend on the physical nature of the system that we are studying.

In the case of macroscopic fluid turbulence, we are guided by the well established idea that turbulent energy transfer in wavenumber takes the form of a cascade and is therefore to some extent local in wavenumber. In terms of our present approach, we take this to mean that, in any particular realisation, the effect on phase correlation of the mode-mode coupling is short-range in  $k$ -space. Thus, on a statistical picture, based on many such realisations, modes which are widely separated may be taken to be independent of each other. Hence, providing that the bandwidth parameter  $\eta$  is not too small, we can assume that  $\mathbf{u}^+(\mathbf{k}_0, t)$  is independent of  $\mathbf{u}^+(\mathbf{k}_1, t)$ , in the sense that we can write

$$\langle \mathbf{u}^+(\mathbf{k}_0, t) \rangle_0 = \langle \mathbf{u}^+(\mathbf{k}_0, t) \rangle = \langle \mathbf{v}^+(\mathbf{k}_0, t) \rangle, \quad (38)$$

where the last step follows from the definition of  $v^+$ , as another realization of the turbulence ensemble with the same statistical properties as  $u^+$ , but with no phase relationship to  $u^-$ .

This now leads us towards a natural *ansatz* for the relationship between  $\mathbf{v}^+$  and  $\mathbf{u}^+$ . Relying on the fact that we are dealing with a problem in continuum mechanics, we take  $v_\alpha^+(\mathbf{k}, t)$  to be given by a first order truncation of the expansion of  $u_\alpha^+(\mathbf{k}, t)$  in Taylor series about  $\mathbf{k} = \mathbf{k}_0$ , thus:

$$v_\alpha^+(\mathbf{k}, t) = u_\alpha^+(\mathbf{k}_0, t) + (\mathbf{k} - \mathbf{k}_0) \cdot \nabla_k u_\alpha^+(\mathbf{k}, t) |_{k=k_0} + O(\eta^2). \quad (39)$$

Note that we conclude that terms of order  $\eta^2$  have been neglected because the maximum value of  $|\mathbf{k} - \mathbf{k}_0|$  is  $\eta k_0$ . Hence it follows that we have

$$\langle \Delta^+(\mathbf{k}, \mathbf{t}) \rangle_0 = O(\eta^2). \quad (40)$$

It also follows, therefore, that we are simultaneously imposing both upper and lower bounds on acceptable values of  $\eta$ . On the one hand,  $\eta$  must be large enough for us to assume that  $u^+(\mathbf{k}_0, t)$  is independent of  $u^+(\mathbf{k}_1, t)$ ; while, on the other hand,  $\eta$  must be small enough for us to neglect terms which are of order  $\eta^2$  in equation (39).

Then, with all these points in mind, above equations yield for the viscosity acting on the explicit scales:

$$\nu_1 = \nu_0 + \delta\nu_0, \quad (41)$$

where the formula for the increment to viscosity is

$$\delta\nu_0(k) = \frac{1}{k^2} \int d^3j \frac{L(\mathbf{k}, \mathbf{j}) Q_v^+(|\mathbf{k} - \mathbf{j}|)}{\nu_0 j^2 + \nu_0 |\mathbf{k} - \mathbf{j}|^2} + O(\eta^m), \quad (42)$$

with  $0 \leq k \leq k_1$ ,  $k_1 \leq j$ ,  $|\mathbf{k} - \mathbf{j}| \leq k_0$  and  $Q_v^+$  is merely an extension of the spectral density to the  $\mathbf{v}^+$  field. The coefficient  $L(\mathbf{k}, \mathbf{j})$  is given by

$$\begin{aligned} L(\mathbf{k}, \mathbf{j}) &= -2M_{\rho\beta\gamma}(\mathbf{k})M_{\beta\rho\delta}(\mathbf{j})D_{\delta\gamma}(|\mathbf{k} - \mathbf{j}|) \\ &= -\frac{[\mu(k^2 + j^2) - kj(1 + 2\mu^2)](1 - \mu^2)kj}{k^2 + j^2 - 2kj\mu}, \end{aligned} \quad (43)$$

where  $\mu$  is the cosine of the angle between the vectors  $\mathbf{k}$  and  $\mathbf{j}$ .

We extend the procedure to further wavenumber shells, as follows:

(a) Set  $u_\alpha^-(\mathbf{k}, t) = u_\alpha(\mathbf{k}, t)$  in equation (19), so that we now have a new NSE with effective viscosity  $\nu_1(k)$  for Fourier modes on the interval  $0 < k < k_1$ .

(b) Make the decomposition of (61), but this time at  $k = k_2$ , such that  $u_\alpha^+(\mathbf{k}, t)$  is now defined in the band  $k_2 \leq k \leq k_1$ .

(c) Repeat the procedures used to eliminate the first shell of modes in order now to eliminate modes in the band  $k_2 \leq k \leq k_1$ .

In this way, we can progressively eliminate the effect of high wavenumbers in a series of bands  $k_{n+1} < k < k_n$ , where

$$k_n = (1 - \eta)^n k_0 ; 0 \leq \eta \leq 1, \quad (44)$$

with, by induction, the recursion relation for the effective viscosity given by

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k), \quad (45)$$

where the increment of order  $n$  takes the form

$$\delta\nu_n(k) = \frac{1}{k^2} \int d^3j \frac{L(\mathbf{k}, \mathbf{j}) \{Q(l)|_{l=k_n} + (l - k_n) \frac{\partial Q(l)}{\partial l} |_{l=k_n} + O(\eta^2)\}}{\nu_n(j)j^2 + \nu_n(|\mathbf{k} - \mathbf{j}|)|\mathbf{k} - \mathbf{j}|^2}, \quad (46)$$

Also, we may form an energy equation for the explicit scales, hence obtaining the renormalized dissipation relation, viz:

$$\int_0^{k_n} 2\nu_n(k)E(k)dk = \varepsilon, \quad (47)$$

which may be compared with the unrenormalized where  $k_n$  and  $\nu_n$  should be replaced by  $k_0$  and  $\nu_0$ , respectively.

If we now assume that the energy spectrum in the band is given by a power law and make the scaling transformation

$$k_{n+1} = hk_n, \quad (48)$$

where, for compactness, we define  $h = (1 - \eta)$ , it follows from equations (45) and (46) that the effective viscosity may be written

$$\nu(k_n k') = \alpha^{1/2} \epsilon^{1/3} k_n^{-4/3} \tilde{\nu}_n(k') \quad (49)$$

where  $\alpha$  is the constant of proportionality in the assumed spectrum. Now the recursion relation becomes

$$\tilde{\nu}_{n+1}(k') = h^{4/3} \tilde{\nu}_n(hk') + h^{-4/3} \delta \tilde{\nu}_n(k') \quad (50)$$

with

$$\delta \tilde{\nu}_n(k') = \frac{1}{4\pi k'^2} \int d^3j' \frac{L(\mathbf{k}', \mathbf{j}') Q'}{\tilde{\nu}_n(hj')j'^2 + \tilde{\nu}_n(hl')l'^2} \quad (51)$$



for the wavenumber bands  $0 \leq k' \leq 1; 1 \leq j', l' \leq h^{-1}$  where  $l' = |\mathbf{k}' - \mathbf{j}'|$ , and

$$Q' = h^{11/3} - \frac{11}{3}h^{14/3}(l' - h^{-1}) + h.o.t. \quad (52)$$

Iteration of equations (50) and (51) reaches a fixed point and once this fixed point is found we can calculate the Kolmogorov constant by solving equations (47) and (49) simultaneously. One merit of taking  $\alpha$  as a test is that it does have known experimental values, albeit scattered in the range  $1.2 < \alpha < 2.2$ . Our calculated value of the Kolmogorov spectral constant is  $\alpha = 1.60 \pm 0.01$  independent of bandwidth in the range  $0.25 \leq \eta \leq 0.45$ , and in good agreement with experiment. For values of  $\eta$  outside this range, the calculated  $\alpha$  diverges from the experimental value. At large values of  $\eta$ , this is due to the breakdown of the first-order Taylor series approximation, while at small values, one is seeing the effects of mode coupling, which would invalidate the assumption that  $\mathbf{u}(\mathbf{k}_0)$  is independent of  $\mathbf{u}(\mathbf{k}_1)$ .

## 5 Recursive RG

### 5.1 Introduction

We shall apply recursive renormalization group (RG) procedures to the problem of subgrid modeling. Subgrid modeling is necessary for the high-Reynolds number turbulent flows of interest because of the limitations of current and foreseeable supercomputers. A motivation is that the spectral transport coefficients (such as the eddy viscosity) determined from recursive RG theory can be compared to those arising from closure-based theories (Kraichnan, 1976; Chollet and Lesieur, 1981). It should be noted that the transport coefficients in these closure theories are determined over the entire resolvable scales, and are a function of  $k$  in the resolvable scales.

In particular, we point out here that in  $\epsilon$ -RG, a small parameter  $\epsilon$  is introduced through the forcing correlation function. Yakhot & Orszag (1986) have to extrapolate from  $\epsilon \ll 1$  to  $\epsilon \rightarrow 4$  in order to reproduce the Kolmogorov energy spectrum. Furthermore, it is also necessary to take the distant interaction limit,  $k \rightarrow 0$ . Thus, it is difficult to compare the wave-number dependent transport coefficients (Kraichnan, 1976; Leslie and Quarini, 1979; Chollet and Lesieur, 1981), with that determined from  $\epsilon$ -RG.

In recursive RG, no attempt is made to introduce a special form of overlapping as in conditional averaging, but one proceeds directly with standard averaging and handles the triple nonlinearity directly. The basic differences between the recursive and  $\epsilon$  RG procedures are that in recursive RG:

- (i) The  $\epsilon$ -expansion is *not* applied.
- (ii) The turbulent transport coefficients are determined for the whole resolvable wavenumber scales,

(iii) Higher order nonlinearities are generated in the renormalized momentum equation and play a critical role in determining the transport coefficients.

(iv) RG rescaling, as in conditional averaged RG, is performed.

It should be emphasized that there are two singular limits:  $h \rightarrow 1$  and  $k \rightarrow 0$ . A careful analysis must be done regarding these two limits and the associated averaging operations. We will address this issue here in the present review.

## 5.2 Renormalized momentum equation

For notation consistency, we shall use “+ , −” instead of “> , <” which are typically employed in recursive RG.

The first and third terms on the RHS of (29) are symmetric in  $j$  and  $|\mathbf{k} - \mathbf{j}|$  in terms of their respective wavenumber constraints in wavenumbers. As a result, the distant interaction limit  $k \rightarrow 0$  has no effect on the existence of these terms, and these terms will give rise to the standard quadratic nonlinearity (first term of Eq. 29) and eddy viscosity (third term of Eq. 29). However, the second term on the RHS of (29) has the following constraint:  $j$  is in the subgrid while  $|\mathbf{k} - \mathbf{j}|$  is in the resolvable scales. Specifically, the consistency condition requires that, for small  $k$ ,  $j$  satisfies

$$j > k_1 \quad \text{and} \quad j < k_1 + kz$$

where  $\mathbf{k} \cdot \mathbf{j} = kjz$ . Since  $|z| \leq 1$ , the range of integration must be  $O(k)$ .

Thus, the second term on the RHS of Eq. (29) can not contribute in the limit  $k \rightarrow 0$  since the integrand is bounded. Now it is well known that the higher order nonlinearities are induced by this second term under discussion. Since this term is absent in the  $k \rightarrow 0$  limit, we conclude that the higher order nonlinearities will not contribute to the renormalized momentum equations and recursion relation for the transport coefficients *in the distant interaction limit*,  $k \rightarrow 0$ . However, they will contribute to the renormalized Navier-Stokes for  $0 < k \leq k_1$ . After obtaining the final renormalized Navier-Stokes equation

$$\begin{aligned} & [\partial/\partial t + \nu(k)k^2]u_\alpha(\mathbf{k}, t) = f_\alpha(\mathbf{k}, t) + M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j u_\beta(\mathbf{j}, t)u_\gamma(\mathbf{k} - \mathbf{j}, t) \\ & + 2 \frac{1}{\nu(k_c)k_c^{\frac{y+1}{3}}} M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j d^3j' \frac{1}{j^{\frac{5-y}{3}}} M_{\beta\beta'\gamma'}(j) u_{\beta'}(\mathbf{j}', t) u_{\gamma'}(\mathbf{j} - \mathbf{j}', t) u_\gamma(\mathbf{k} - \mathbf{j}, t). \end{aligned} \quad (53)$$

by removing other subgrid shells iteratively, one can write down the corresponding wavenumber restrictions and perform the same analysis on the  $k \rightarrow 0$  limit.

## 5.3 Galilean invariance of the renormalized Navier-Stokes

Here, we turn our attention to the question of the Galilean invariance of the renormalized Navier-Stokes equations. The importance of Galilean invariance in turbulence

modelling has been emphasized by Speziale (1985). To be consistent with the basic physics, it is required that the description of the turbulence be the same in all inertial frames of reference. The appearance of the triple nonlinear term, which is a function of the resolvable scales velocity fields, makes the property of the Galilean invariance of our recursive RG procedure not transparent. We now show that the renormalized Navier-Stokes equation is Galilean invariant (Zhou and Vahala, 1993b).

The Galilean transformation is

$$\mathbf{x} \rightarrow \mathbf{x}^* - \mathbf{U}_0 t^* \quad t \rightarrow t^*.$$

Here  $\mathbf{U}_0$  is an uniform velocity field. Thus, one has

$$\mathbf{u} = \mathbf{u}^* - \mathbf{U}_0, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} + \mathbf{U}_0 \cdot \frac{\partial}{\partial \mathbf{x}^*}.$$

While the Galilean transformation for the Navier-Stokes equation in physical space is trivial, the Galilean transformation in wavenumber space is less obvious, due to the lack of differential operations. For convenience, we first review how Galilean invariance is preserved for the Navier-Stokes equation in the wavenumber space.

Under the Galilean transformation, the LHS of the Navier-Stokes equation [cf. Eq. (21)] becomes

$$\begin{aligned} & \frac{\partial u_\alpha^*(\mathbf{k}^*, t)}{\partial t^*} + U_{0\beta} i k_\beta^* u_\alpha^*(\mathbf{k}^*, t) + \nu_0 k^{*2} [-U_{0\alpha} \delta(\mathbf{k}^*) + u_\alpha^*(\mathbf{k}^*, t)] \\ & = \frac{\partial u_\alpha^*(\mathbf{k}^*, t)}{\partial t^*} + U_{0\beta} i k_\beta^* u_\alpha^*(\mathbf{k}^*, t) + \nu_0 k^{*2} u_\alpha^*(\mathbf{k}^*, t) \end{aligned}$$

where in the last step, we have used the the  $\delta$  function property  $k^{*2} \delta(\mathbf{k}^*) = 0$ .

Also, under the Galilean transformation, the RHS of the Navier-Stokes equation becomes

$$\begin{aligned} & M_{\alpha\beta\gamma}(k^*) \int d^3 j [u_\beta^*(\mathbf{j}^*, t) - U_{0\beta} \delta(\mathbf{j}^*)] [u_\beta^*(\mathbf{k}^* - \mathbf{j}^*, t) - U_{0\gamma} \delta(\mathbf{k}^* - \mathbf{j}^*)] \\ & = M_{\alpha\beta\gamma}(k^*) \int d^3 j^* u_\beta^*(\mathbf{j}^*, t) u_\gamma^*(\mathbf{k}^* - \mathbf{j}^*, t) + i U_{0\beta} k_\beta^* u_\alpha^*(\mathbf{k}^*, t) \end{aligned}$$

where we have used the property of the  $\delta$  function, the incompressible condition, and

$$M_{\alpha\beta\gamma}(k^*) U_{0\beta} u_\gamma^*(\mathbf{k}^*, t) = U_{0\beta} k_\beta^* u_\alpha^*(\mathbf{k}^*, t) / (2i)$$

Thus, as expected, the Navier-Stokes equation is invariant under a Galilean transformation due to the cancellation of the second term on the RHS.

To show that the renormalized Navier-Stokes equation is invariant under a Galilean transformation, we need only consider the recursive RG induced triple nonlinear term, denoted by  $NS_T$ :

$$NS_T \equiv 2M_{\alpha\beta\gamma}(k) \int d^3j d^3j' \left(\frac{j}{k_c}\right)^{4/3} \frac{M_{\beta\beta'\gamma'}(j)}{\nu(k_c)j^2} u_{\beta'}(\mathbf{j} - \mathbf{j}', t) u_{\gamma'}(\mathbf{j}', t) u_{\gamma}(\mathbf{k} - \mathbf{j}, t) \quad (54)$$

It is important to note that  $j$  is in the subgrid.

Under a Galilean transformation, Eq. (54) becomes

$$NS_T^* = 2M_{\alpha\beta\gamma}(k^*) \int d^3j^* d^3j'^* \left(\frac{j^*}{k_c}\right)^{4/3} \frac{M_{\alpha\beta\gamma}(j^*)}{\nu(k_c^*)j^{*2}} [u_{\beta'}^*(\mathbf{j}^* - \mathbf{j}'^*, t) - U_{0\beta'}\delta(\mathbf{j}^* - \mathbf{j}'^*)] \\ [u_{\gamma'}^*(\mathbf{j}'^*, t) - U_{0\gamma'}\delta(\mathbf{j}'^*)] [u_{\gamma}^*(\mathbf{k}^* - \mathbf{j}^*, t) - U_{0\gamma}\delta(\mathbf{k}^* - \mathbf{j}^*)]$$

Since  $j^*$  is in the subgrid scale, while  $j'^*$  and  $k^*$  are in the supergrid,  $\delta(\mathbf{k}^* - \mathbf{j}^*)$  and  $\delta(\mathbf{j}^* - \mathbf{j}'^*)$  can never be simultaneously satisfied. As a result,

$$NS_T^* = 2M_{\alpha\beta\gamma}(k^*) \int d^3j^* d^3j'^* \left(\frac{j^*}{k_c}\right)^{4/3} \frac{M_{\alpha\beta\gamma}(j^*)}{\nu(k_c^*)j^{*2}} u_{\beta'}^*(\mathbf{j}^* - \mathbf{j}'^*, t) [u_{\gamma'}^*(\mathbf{j}'^*, t) - U_{0\gamma'}\delta(\mathbf{j}'^*)] u_{\gamma}^*(\mathbf{k}^* - \mathbf{j}^*, t) \quad (55)$$

Now only one term in Eq. (55) could violate the Galilean invariance of the renormalized Navier-Stokes equation. However,

$$\int d^3j'^* \delta(\mathbf{j}'^*) u_{\beta'}^*(\mathbf{j}^* - \mathbf{j}'^*, t) \rightarrow u_{\beta'}^*(\mathbf{j}^*, t)$$

This is not permissible since  $u_{\beta'}^* \equiv u_{-\beta'}$  and  $j^*$  is restricted to the subgrid. Thus  $NS_T = NS_T^*$ . Hence the triple term is Galilean invariant.

## 5.4 The effect of cubic nonlinearity

We consider the contribution of the triple nonlinear term in the renormalized momentum equation to the eddy viscosity. The second moment for the velocity field is defined as

$$U_{\alpha\beta}(\mathbf{k}, t) \equiv \langle u_{\alpha}(\mathbf{k}, t) u_{\beta}(-\mathbf{k}, t) \rangle. \quad (56)$$

The time evolution of  $U_{\alpha\beta}(\mathbf{k}, t)$  is

$$\frac{\partial U_{\alpha\beta}(\mathbf{k}, t)}{\partial t} = -2\nu(k)k^2 U_{\alpha\beta}(\mathbf{k}, t) + T_{\alpha\beta}^D(\mathbf{k}, t) + T_{\alpha\beta}^T(\mathbf{k}, t). \quad (57)$$

In this equation,  $T_{\alpha\alpha}^D(\mathbf{k}, t)$  is the standard energy transfer from the quadratic nonlinearity. In contrast,  $T_{\alpha\alpha}^T(\mathbf{k}, t) = -2\nu_T(k)k^2 E(k)$  is the energy transfer arising from the RG induced triple nonlinearity. It is readily shown that

$$\nu_T(k) \equiv -\frac{T_{\alpha\alpha}^T(\mathbf{k}, t)}{2E(k)k^2} = \frac{1}{2\nu(k_c)} \frac{1}{k^2} \int_{k_c}^{k+k_c} dj dz \frac{L(k, j, q) |\mathbf{k} - \mathbf{j}|^{-y-2} j^{y+1/3}}{\nu(\mathbf{k} - \mathbf{j})}. \quad (58)$$

It has been shown that that  $\nu_T(k)$  is the major contributor to the cusp-like behavior of the spectral eddy viscosity as  $k \rightarrow 0$  (Zhou and Vahala, 1993a). This type of term was found to be the major contribution to the strong cusp in the spectral eddy viscosity found from the closure models (Kraichnan, 1976; Leslie and Quarini, 1979; Chollet and Lesieur, 1981).

Rose (1977) discussed the role of the triple nonlinear terms in physical space. He pointed out that it represents the possibility of an exchange of scalar eddies between the resolvable and subgrid scales. This effect is an inherent property of measurements made on the passive scalar system with instruments which have a spatial resolution limited to an eddy width size greater than  $1/k_c$ .

## 5.5 Difference equations for the renormalized eddy viscosity

After the removal of the  $(n + 1)^{th}$  subgrid shell, the spectral eddy viscosity in the renormalized momentum equation is determined by the recursion relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k) \quad (59)$$

where

$$\delta\nu_n(k) = \frac{D_0}{k^2} \sum_{h=0}^n \int d^3j \frac{L(k, j, q) |\mathbf{k} - \mathbf{j}|^{-y}}{\nu_n(j) j^2 \nu(|\mathbf{k} - \mathbf{j}|) |\mathbf{k} - \mathbf{j}|^2}, \quad (60)$$

and

$$L(k, j, q) = -\frac{kj(1 - z^2)[zq^2 - kj]}{q^2}, \quad (61)$$

with  $\mathbf{k} \cdot \mathbf{j} = k j z$  and  $q = |\mathbf{k} - \mathbf{j}|$ . This difference equation, after rescaling, has been solved by Zhou et al. (1988, 1989) and fixed points were readily determined for finite  $h \leq 0.7$ . Note that the spectral eddy viscosity shows a mild cusp as  $k \rightarrow k_c$ , in qualitative agreement that that of closure theory. However, it was very difficult to determine fixed points for finer subgrid partition factor  $h > 0.7$ . In the next subsection we shall pass to the differential subgrid limit  $h \rightarrow 1$  and determine an ordinary differential equation (o.d.e) for the renormalized eddy viscosity over the entire resolvable scale which can be readily integrated.

## 5.6 Differential equations for the renormalized eddy viscosity

The differential limit,  $h \rightarrow 1$ , is a singular one and this point has been discussed recently (Zhou and Vahala, 1992). In particular, it is related to the assumption of

local versus non-local interactions in  $k$ . In this section we will calculate the eddy viscosity under the differential equation limit for recursive RG.

For recursive RG we will find that the differential equations hold throughout the resolvable wavenumber range  $0 < k \leq k_c$ . This should be contrasted with the  $\epsilon$  - RG eddy viscosity differential equation which is valid only in the  $k \rightarrow 0$  limit.

### 5.6.1 The differential equation limit, $h \rightarrow 1$

We now derive the differential equation from which the transport coefficients for finite  $k$ ,  $0 < k \leq 1$ , are determined. The o.d.e. in the distant interaction limit ( $k = 0$ ) will be derived in the next subsection. After the rescaling, we rewrite the recursion relation in the form

$$\nu_{n+1}(k) - h^{(y+1)/3}\nu_n(hk) = h^{(y+1)/3}\delta\nu_n(hk). \quad (62)$$

For  $h \rightarrow 1$ , the number of iteration  $n \rightarrow \infty$ . Similarity consideration leads to

$$\nu_{n+1}(k) \rightarrow \nu(k), \quad n \rightarrow \infty. \quad (63)$$

The LHS of Eq. (62) becomes

$$\nu(k) - [1 - \eta]^{(y+1)/3}\nu[k(1 - \eta)] \rightarrow \eta \left[ \frac{d\nu(k)}{dk} + \frac{y+1}{2}\nu(k) + O(\eta) \right] \quad (64)$$

As noted earlier, the partial average of Rose must be employed in order to ensure the existence of the differential limit. The partial average is introduced since the distinction between the resolvable and subgrid scales becomes fuzzy in the limit of a differential subgrid partitioning,  $h \rightarrow 1$ .

Following Rose, we first change the variable from  $j, z$  to  $j, q$ , with  $djdz = (q/kj)djq$ , so that the RHS of Eq. (62) becomes

$$\begin{aligned} \delta\nu(k) &= \int djdq \left( \frac{q}{kj} \right) \frac{L(k, j, q)}{\nu(j)k^2\nu(|\mathbf{k} - \mathbf{j}|)|\mathbf{k} - \mathbf{j}|^2|\mathbf{k} - \mathbf{j}|^y} \\ &+ \int djdq \left( \frac{q}{kj} \right) \frac{L(k, j, q)}{\nu(j)k^2\nu(|\mathbf{k} - \mathbf{j}|)|\mathbf{k} - \mathbf{j}|^2|\mathbf{k} - \mathbf{j}|^y} \left( \frac{j}{k_c} \right)^{(y+1)/3} \\ &\rightarrow \eta \int_{1 < q < 1+k} dq \frac{L(k, 1, q)}{k^3\nu^2(1)q^{y-1}} + \eta \int_{1 < j < 1+k} dj \frac{L(k, j, 1)}{k^3\nu^2(1)} j^{(y-2)/3} \end{aligned} \quad (65)$$

where

$$L(k, 1, q) = k(1 - z^2)(k - zq^2)/q^2 \quad (66)$$

and

$$L(k, j, 1) = kj(1 - z^2)(kj - z). \quad (67)$$

As a result, the fixed point renormalized eddy viscosity  $\nu(k)$  is determined from the o.d.e. at  $O(\Delta)$

$$k \frac{d\nu(k)}{dk} + \frac{y+1}{3} \nu(k) = \frac{1}{\nu^2(1)} [A_\nu(k) + B_\nu(k)] \quad (68)$$

where

$$A_\nu(k) = \frac{1}{k^3} \int_1^{1+k} dq \frac{L(k, 1, q)}{q^{y-1}} \quad (69)$$

$$B_\nu(k) = \frac{1}{k^3} \int_1^{1+k} dj L(k, j, 1) j^{(y-2)/3} \quad (70)$$

Here,  $z$  is evaluated at  $j = 1$  and  $q = 1$ , respectively in the  $L(k, j, q)$  expression. The o.d.e. for the momentum equation eddy viscosity is readily solved (Zhou and Vahala, 1993a).

### 5.6.2 Differential equations in the $k \rightarrow 0$ limit

In the limit  $k \rightarrow 0$ , we have seen that the triple nonlinearities induced by RG do not contribute to the eddy viscosity. As a result, the recursion relation will now contain only the usual quadratic contribution. We further simplify the analysis by taking the standard subgrid linear propagator  $G_h^{-1}(|\mathbf{k} - \mathbf{j}|) = [\frac{\partial}{\partial t} + \nu_h(|\mathbf{k} - \mathbf{j}|)] \sim G_h^{-1}(|\mathbf{j}|)$  as  $k \rightarrow 0$ .

The limits of the integration are given by

$$1 < j - kz < 1/f, \quad 1 + kz < j < 1/f + kz \quad (71)$$

Thus, one has

$$\delta\nu_n(k) \equiv S - F - G \quad (72)$$

where the integral limits for these terms are

$$\int_1^{1/f} dj \int_{-1}^1 dz \quad \text{for } S \quad (73)$$

$$\int_0^1 dz \int_1^{1+kz} dj \quad \text{for } F \quad (74)$$

$$\int_{-1}^0 dz \int_{1/f+kz}^{1/f} dj \quad \text{for } G. \quad (75)$$

Terms  $F$  and  $G$  are the corrections to the symmetric term  $S$ . They are important for a finite bandwidth  $f$ . However, it is easy to show that  $F + G = 0$  for  $f \rightarrow 1$  in the  $k \rightarrow 0$  limit. Hence

$$\delta\nu_n^D(k) \rightarrow -\eta \frac{1}{k\nu^2(1)} \left[ \frac{1}{j^{y+1}} \right]_{j=1} \int_{-1}^1 dz [1 - z^2] \left[ z + \frac{k}{j} (yz^2 - 1) \right] = \eta \frac{8}{15} \frac{1}{\nu^2(1)} \quad (76)$$

while the LHS of the same equation yields

$$k \frac{d\nu(k)}{dk} + \frac{y+1}{3} \nu(k) \rightarrow \frac{y+1}{3} \nu(k), \quad \text{as } k \rightarrow 0$$

since  $\frac{d\nu(k)}{dk}$  is bounded as  $k \rightarrow 0$ .

Thus, as  $k \rightarrow 0$ ,

$$\nu(k \rightarrow 0) = \frac{3}{y+1} \frac{8}{15} \frac{1}{\nu^2(1)}. \quad (77)$$

Again, the o.d.e. for the momentum equation eddy viscosity is readily solved. We observe that the eddy viscosity has a similar plateau structure as  $k \rightarrow 0$ . As  $k \rightarrow k_c$ , eddy viscosity displays a weak cusp like behavior as  $k \rightarrow k_c$ . In this case, those curves are similar to that of Zhou *et al* (1988; 1989).

## 5.7 Recursive RG Spectral eddy viscosity

The spectral eddy viscosity is simply the sum of the contributions from the momentum equation and that of the effect of the RG induced triple nonlinear term in the energy equation. It is apparent that our calculation is in qualitative agreement with that from the closure theory (Kraichnan, 1976; Chollet and Lesieur, 1981). and direct numerical measurements (Damaradski et al., 1987; Lesieur and Rogallo, 1991; Zhou and Vahala, 1993). In particular, it predicts the correct asymptotic behaviors of the eddy viscosity as  $k \rightarrow 0$  and  $k \rightarrow k_c$  (Kraichnan, 1976).

## 5.8 Numerically evaluated eddy viscosity

These conclusions can be tested directly using numerical simulation databases. Indeed, energy transfer and spectral eddy viscosity can be analyzed using results from direct numerical simulations by introducing an artificial cut at a wavenumber  $k_c$  that is smaller than the maximum resolved wavenumber  $k_m$  of the simulation. With this fictitious separation between the subgrid and resolvable scales, it is possible to evaluate the effect of the subgrid  $k_c < k < k_m$  on the resolved scales  $k < k_c$ . We form an energy equation from the momentum equation and introduce the following notation:  $T^{><}(k)$  and  $T^{>>}(k)$  represent the spectrum of energy transfer to mode  $\mathbf{k}$  resulting from interactions with *one* and *both* modes above the cutoff  $k_c$  respectively. Measurements of numerical simulation databases indicate the following (Zhou and Vahala, 1993a):

- $T^{>>}(k)$  removes energy throughout the resolvable scales in a manner consistent with the notion of eddy viscosity.
- $T^{><}(k)$  removes energy from the last resolved octave that was transferred there by the resolved scale transfer; that is, it allows the local flow of energy through  $k_c$ .



It is the most important subgrid effect near  $k_c$  and accounts for most of the energy flow from the resolved scales.

The subgrid spectral eddy viscosity  $\nu^{>>}(k)$  and  $\nu^{><}(k)$  can be determined from  $T^{>>}(k)$  and  $T^{><}(k)$  for a given energy spectrum,  $E(k)$ . Specifically,  $\nu^{>>}(k) = -T^{>>}(k)/2k^2E(k)$  and  $\nu^{><}(k) = -T^{><}(k)/2k^2E(k)$ . Two important features of the quadratic contribution  $\nu^{>>}(k)$  should be stressed. First, its positive constant asymptote at small  $k$  indicates that the concept of modeling this contribution as an eddy viscosity in analogy to the molecular viscosity is plausible, and second, its value decreases monotonically as  $k$  increases toward  $k_c$ . This indicates that if we include only the contribution of quadratic velocity products, there is no eddy viscosity cusp at the cutoff  $k_c$ . The most important feature of  $\nu^{><}(k)$  is the sharp increase at  $k \rightarrow k_c$ .

## 6 Reconsideration of the YO theory

Robert Rubinstein  
 Institute for Computer Applications in Science and Engineering  
 NASA Langley Research Center  
 Hampton, VA 23681

The first comprehensive application of RG methods to turbulence was the Yaghot-Orszag (1986) theory (YO). This theory built upon earlier work by Forster *et al* (1977) and Fournier and Frisch (1983), but completed the development suggested by its predecessors with impressive successes including simple analytical calculations of the Kolmogorov constant <sup>3</sup> and several constants of interest in turbulence modeling (Speziale, 1991). Nevertheless, this theory continues to generate controversy. Three points will be addressed in this section about which particularly strong objections have been raised:

1. YO's treatment of mode elimination
2. the "correspondence principle"
3. the  $\epsilon$ -expansion and distant interaction limit

This section draws heavily on work of Woodruff (1992, 1993) which emphasizes the connections between the YO theory and Kraichnan's (1959) direct interaction approximation (DIA). Although this viewpoint tends to de-emphasize the importance of mode elimination characteristic of renormalization group theories, it is consistent with the remark of Eyink (1994) that YO is not a true RG theory in any case.

In this Section, the following notation will be used:

$$\hat{k} = (\mathbf{k}, \Omega) \quad \hat{k} = (\mathbf{p}, \omega) \quad \hat{q} = (\mathbf{q}, \Omega - \omega)$$

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<sup>3</sup>The Kolmogorov constant had been computed earlier by Kraichnan (1964) by numerical integration of the LHDIA closure.

where the triangle condition

$$\mathbf{k} = \mathbf{p} + \mathbf{q}$$

holds throughout.

## 6.1 YO's treatment of mode elimination

It was noted at the end of Sect. 3.2 that iterated mode elimination leads to two fundamental problems: it generates nonlinearities of higher order than appear in the Navier-Stokes equations, and it requires some simplifying assumptions to permit averaging over the high wavenumber components of the motion. These difficulties arise because, by itself, mode elimination, like the field theoretic functional formulations of the turbulence problem, does nothing more than reformulate the equations of motion. In order to make progress, statistical hypotheses must be introduced. This circumstance is not surprising: for example, equilibrium statistical mechanics requires Gibbs' hypothesis, a very strong assumption which certainly does not follow from Newtonian mechanics.

Nevertheless, subsequent work has shown that the original treatment of mode elimination by YO was not entirely satisfactory. For example, their proposal that higher order nonlinearities are negligible in a certain perturbative sense, was contradicted by Eyink (1994).<sup>4</sup> But by placing the YO theory in the setting of DIA-like closures, these problems are understood by invoking the statistical hypotheses of these closures. Thus, higher order nonlinearities can indeed be considered, but in the context of higher order versions of DIA (Martin *et al*, 1978). Accordingly, the absence of these nonlinearities in the YO theory merely reflects the order of approximation chosen, and requires no further justification.

The treatment of averaging over the small scale motion, which appears to proceed as if motions of different scale were independent, has been discussed in Kraichnan's (1959) original presentation of DIA: the same requirement arises in the derivation of the DIA response equation. Heuristically, the DIA closure assumes that the velocity field is only weakly non-Gaussian. Therefore, the motions of different scales, which are uncorrelated because of the kinematic hypothesis of homogeneity, are independent to leading order. This statistical hypothesis also justifies the breakup of fourth order moments into products of second order moments. A theory in which the motions of different scale are not independent to leading order, but in which fourth order moments are treated by the quasi-Gaussian hypothesis, must be carefully formulated to avoid inconsistency.

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<sup>4</sup>YO had asserted that higher order nonlinearities are 'irrelevant' as this is understood in Wilson's (1974) theory when  $\epsilon = 0$ , although they are only marginal when  $\epsilon = 4$ . Eyink (1994) demonstrated that, on the contrary, higher nonlinearities are marginal regardless of  $\epsilon$ .

## 6.2 The correspondence principle

The starting point of YO's analysis is the Navier-Stokes equations driven by a random force:

$$-i\Omega u_i(\hat{k}) - M_{imn}(\mathbf{k}) \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}d\hat{q} u_m(\hat{p})u_n(\hat{q}) = f_i(\hat{k}) \quad (78)$$

where the Gaussian random force  $f_i$  is characterized by its correlation function

$$\langle f_i(\hat{k})f_j(\hat{k}') \rangle = 2D(2\pi)^{d+1}k^{-y}D_{ij}(\mathbf{k})\delta(\hat{k} + \hat{k}') \quad (79)$$

This force is white noise in time. The exponent  $y$  is treated as a variable for purposes of the subsequent  $\epsilon$ -expansion, in which  $\epsilon = 4 + y - d$  and  $d = 3$  is the dimension of space. The analysis leads to a Kolmogorov spectrum when  $y = 3$  or  $\epsilon = 4$ ; this is therefore the case of physical interest. YO's conclusion is that the nonlinear term the Navier-Stokes equation is replaced, in the limit of infinite Reynolds number, by the combination of random forcing by  $f_i$  and a scale dependent viscosity  $\nu(k)$ , so that

$$-i\Omega u_i(\hat{k}) + \nu(k)k^2 u_i(\hat{k}) = f_i(\hat{k}) \quad (80)$$

Mode elimination is used to obtain the recurrence relation

$$\frac{d\nu(k)}{dk} = A \frac{D}{\nu^2 k^5} \quad (81)$$

where the constant  $A$  is computed from the theory.

Whereas it is generally agreed that Eq. (78) provides a plausible model of isotropic turbulence provided the random force  $f$  is concentrated at large scales and therefore provides an energy source, the introduction of a force acting on all inertial range scales appears to lack fundamental justification. YO's model Eqs. (79) and (80) can be compared to the DIA Langevin model (Kraichnan, 1976)

$$-i\Omega u_i(\hat{k}) + \eta(\hat{k})u_i(\hat{k}) = f_i(\hat{k}) \quad (82)$$

where the damping function and force correlation are expressed in terms of the DIA response and correlation functions  $G$  and  $Q$  by

$$\eta(\hat{k}) = 2iM_{rmn}(\mathbf{k}) \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}d\hat{q} D_{mr}(\mathbf{p})D_{ns}(\mathbf{q})G(\hat{p})Q(\hat{q}) \quad (83)$$

$$F(\hat{k}) = \langle f_i(\hat{k})f_i(\hat{k}') \rangle / \delta(\mathbf{k}+\mathbf{k}') = -4M_{imn}(\mathbf{k})M_{jrs}(\mathbf{k}) \int_{\hat{k}=\hat{p}+\hat{q}} d\hat{p}d\hat{q} D_{mr}(\mathbf{p})D_{ns}(\mathbf{q})Q(\hat{p})Q(\hat{q}) \quad (84)$$

Eq. (82) is a generic model in statistical mechanics which replaces the effects of an infinity of nonlinear interactions on any one mode by a random force acting against a generalized damping; DIA applies this description to a problem which is far from thermal equilibrium. The "fixed point" RG model Eqs. (79) and (80) formally resem-

bles the DIA Langevin equation model. But the damping  $\nu(k)k^2$  in in the RG model Eq. (79) is Markovian, so that

$$\eta(\hat{k}) = \eta(k) \quad (85)$$

only and the forcing in Eq. (80) is white noise in time, so that

$$F(\hat{k}) = F(k) \quad (86)$$

only. Neither of these conditions holds for the DIA Langevin model.

To investigate the connection between these models, write following Kraichnan

$$Q(\hat{p}) = Q(p)R(\hat{p}) \quad (87)$$

where  $R$  is the time correlation function. Perform the frequency integration in Eq. (84) and evaluate the result in the long time limit in which  $\Omega = 0$ . This limit corresponds to observing the system over times long compared to any characteristic correlation time of the true DIA random force. The result is

$$F(\hat{k}) = -4M_{imn}(\mathbf{k})M_{jrs}(\mathbf{k}) \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q} D_{mr}(\mathbf{p})D_{ns}(\mathbf{q})Q(p)Q(q)\Theta(k, p, q) \quad (88)$$

where

$$\Theta(k, p, q) = \int_{-\infty}^{\infty} d\omega R(\hat{p})R(\hat{q}) |_{\Omega=0} \quad (89)$$

In this limit, the random force is white in time. Further, in Kolmogorov scaling,

$$Q(\lambda p) = \lambda^{-11/3}Q(p) \quad (90)$$

$$\Theta(\lambda k, \lambda p, \lambda q) = \lambda^{-2/3}\Theta(k, p, q) \quad (91)$$

consequently, the scaling dimension of the random force is found to be -3:

$$F(\lambda k) = \lambda^{-3}k \quad (92)$$

Formally, in the long time limit, the random force in the DIA Langevin model has the same space-time correlation as the force postulated at the outset by YO.

It should be noted that the power counting which leads to Eq. (92) is purely formal, since the actual force correlation integral in Eq. (84), like the integral of Eq. (83), is infrared divergent when evaluated for an infinite Kolmogorov inertial range. We recall that these divergences actually cancel in the DIA energy equation; however, the assumption of a -3 force in DIA requires *a priori* infrared regularization.

That the -3 force is natural in the context of any steady state far from equilibrium with a constant flux of some inviscid invariant is also suggested by the derivation (Rubinstein, 1994a) of Bolgiano scaling inertial range for buoyant turbulence by applying the YO formalism with forcing of the temperature equation only. From this point of view, the introduction by Lam (1992) of a distinguished infrared scale in the RG force is debatable: it corresponds to a loss of the *locality* of the inertial range postulated by Kolmogorov.

### 6.3 The $\epsilon$ -expansion and the distant interaction approximation

The  $\epsilon$ -expansion is the subject of an especially large number of re-evaluations and reconsiderations, among which are Ronis (1987), Lam (1992) and Wang and Wu (1993). In YO's original presentation, the  $\epsilon$ -expansion is an expansion about a logarithmically divergent theory. An interesting alternative was suggested by Carati (1990a), who suggested expanding about a theory with vanishing energy transfer (Fournier and Frisch, 1978). Here, this expansion will be considered, following Woodruff (1992), as an approximation in DIA.

To complete the transition from the DIA Langevin model to the YO theory, further approximations are required. They are

- (a) evaluate the DIA integrals in the distant interaction limit in which  $k/p, k/q \rightarrow 0$
- (b) Markovianize the damping
- (c) introduce an infrared cutoff so that the integrals in Eqs. (83),(84) are restricted to  $p \geq k$  and  $q \geq k$  only
- (d) evaluate the amplitudes using the  $\epsilon$ -expansion

It has been emphasized by Woodruff that these approximations are closely related. First, as noted by Kraichnan (1987), the  $\epsilon$ -expansion is an expansion about a theory in which distant interactions are dominant; accepting this point provisionally, we outline how the distant interaction limit brings about the Markovianization of the damping and forcing.

Let the real function  $H(\xi), 0 \leq \xi < \infty$  satisfy

$$H(0) = 1, \quad H(\xi) < 1 \text{ for } \xi > 0, \quad \int_0^\infty H(\xi) d\xi < \infty \quad (93)$$

Then standard properties of delta functions imply

$$\lambda H(\lambda(t-s)) \sim \delta(t-s) \text{ for } \lambda \rightarrow \infty \quad (94)$$

Rewrite Eq. (83) in the time domain, and evaluate the wavevector integrals in the distant interaction approximation in which  $k \rightarrow 0, p, q \rightarrow \infty$ . Then

$$\begin{aligned} \eta(\mathbf{k}, t, s) &= \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q} B(\mathbf{k}, \mathbf{p}, \mathbf{q})G(p, t, s)Q(q, t, s) \\ &\sim \int d\mathbf{p} \left\{ k_m \frac{\partial B}{\partial q_m}(\mathbf{k}, \mathbf{p}, \mathbf{p})G(p, t, s)Q(p, t, s) \right\} \end{aligned}$$

$$-B(\mathbf{k}, \mathbf{p}, \mathbf{p})G(p, t, s)k_m p_m p^{-1} \frac{dQ}{dp}(p, t, s)\}$$

where  $B(\mathbf{k}, \mathbf{p}, \mathbf{q})$  denotes the product of projection operators in Eq. (83). Assuming time stationary similarity forms

$$\begin{aligned} G(p, t, s) &= G(p^r(t-s)) \\ Q(p, t, s) &= R(p^r(t-s))Q(p) \end{aligned}$$

the properties Eqs. (93) of  $H$  may reasonably be postulated of the product  $GR$ . Therefore, Eq. (94) implies that in this limit the damping is Markovian

$$\eta(\mathbf{k}, t, s) = \delta(t-s)\eta(\mathbf{k})$$

and the DIA response equation implies that the Green's function is exponential,

$$G(k, t, s) = \exp [(s-t)\eta(\mathbf{k})] \quad \text{for } t \geq s \quad (95)$$

Likewise evaluating the force correlation Eq. (84) in the distant interaction limit implies that the forcing is white noise in time:

$$\langle f_i(\mathbf{k}, t) f_j(\mathbf{k}', s) \rangle = \delta(t-s)\delta(\mathbf{k} + \mathbf{k}') F_{ij}(\mathbf{k}) \quad (96)$$

Computing the two-time correlation function from the relation

$$\begin{aligned} &Q_{ij}(\mathbf{k}, t, s)\delta(\mathbf{k} + \mathbf{k}') \\ &= \int_0^t dr_1 G(k, t, r_1) \int_0^{r_1} dr_2 G(k', r_1, r_2) \times \langle f_i(\mathbf{k}, r_1) f_j(\mathbf{k}', r_2) \rangle \end{aligned}$$

using Eqs. (95), (96) shows that the fluctuation dissipation relation

$$Q(k, t, s) = Q(k)[G(k, t, s) + G(k, s, t)] \quad (97)$$

expressing the time dependence of the correlation functions in terms of the response function is also valid in this limit.

These simplifications of DIA permit analytical evaluation of the inertial range constants. Although values of these constants could be inferred from numerical solutions of DIA, say for decaying turbulence, it is natural to attempt analytical evaluation as well.

Introducing Eqs. (95) and (96) with the Kolmogorov scaling forms

$$E(p) = C_K \varepsilon^{2/3} k^{-5/3} \quad (98)$$

$$\eta(p) = C_D \varepsilon^{1/3} p^{2/3}$$

into the DIA response equation integrated over all time separations,

$$\frac{C_D^2}{C_K} = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q} 2iM_{rmm}(\mathbf{k})D_{mr}(\mathbf{p})D_{ns}(\mathbf{q}) \frac{p^{-11/3}}{(p^{2/3} + q^{2/3})k^{2/3}} \quad (99)$$

Integrating the single time equation for the correlation function with respect to wavenumber  $k$  leads to a second equation, which for Kolmogorov scaling gives

$$\frac{C_D}{C_K^2} = .1904 \quad (100)$$

This method of evaluating the inertial range constants  $C_D$  and  $C_K$  fails because of the well-known divergence of Eq. (99) at low wavenumbers. The simplest infrared regularization which is consistent with Kolmogorov scaling is to restrict the region of integration to triads satisfying  $p \geq \alpha k$ . Values of  $C_D(\alpha)$  and  $C_K(\alpha)$  have been tabulated by Leslie (1972).

The  $\epsilon$ -expansion can be considered as a method of infrared regularization by analytic continuation. Namely, continue to assume Eq. (97), but replace Eq. (98) by the general form

$$E(p) = C_K D^{2/3} k^{1-2\epsilon/3} \quad (101)$$

The scale independence of the integrated response equation demands

$$\eta(p) = C_D D^{1/3} k^{2-\epsilon/3} \quad (102)$$

The units of  $D$ , consistent with Eq. (79), make these equations dimensionally correct. Substituting these scalings in the integrated response equation gives the  $\epsilon$ -dependent form of Eq. (99),

$$\frac{C_D^2}{C_K} = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q} \, 2iM_{rmm}(\mathbf{k})D_{mr}(\mathbf{p})D_{ns}(\mathbf{q}) \frac{p^{-1-2\epsilon/3}}{(p^{2-\epsilon/3} + q^{2-\epsilon/3})k^{2-\epsilon/3}} \quad (103)$$

The integral in Eq. (103) is ultraviolet divergent when  $\epsilon < 0$  and is logarithmic when  $\epsilon = 0$ . Woodruff observes that it is reasonable to evaluate Eq. (103) for  $\epsilon > 0$  by asymptotic expansion about  $\epsilon = 0$ . This expansion greatly simplifies the integration since the ultraviolet divergence for  $\epsilon < 0$  implies that the integral is dominated by distant interactions, namely by wavevector triangles such that  $p, q \rightarrow \infty$ . In this limit, a simple analytical evaluation of the integrals is possible. The calculation gives

$$\frac{C_D^2}{C_K} = \frac{1}{\epsilon} A(\epsilon) = \frac{A_{-1}}{\epsilon} + A_0 + A_1\epsilon + \dots \quad (104)$$

where

$$A_{-1} = \frac{3}{5}$$

The constant  $A_{-1}$  is distinguished since it is the only one in the series Eq. (104) which has been evaluated exactly in two senses. First, increasing the number of ‘‘loops,’’ that is, considering terms in the perturbative solution of the Navier-Stokes equations with a larger number of force correlations, will correct  $A_p$  only for  $p \geq 0$ . It can also be shown (Rubinstein, 1994b) that even at the one loop level, correcting the distant

interaction approximation by power series expansions in  $k/p$  also only corrects  $A_p$  for  $p \geq 0$ . Accordingly, it is reasonable to evaluate Eq. (104) by taking the leading term only. Setting  $\epsilon = 4$ ,

$$\frac{C_D^2}{C_K} = \frac{3}{20}$$

which is easily shown to be equivalent to YO's calculation.

It is sometimes claimed that YO evaluate the inertial range constants by setting  $\epsilon$  to four and zero at different places in the same equation. Therefore, it must be emphasized that *in this calculation,  $\epsilon$  is never set to any value but 4*. The analytical procedure which leads to Eq. (104) is entirely routine: it is the evaluation of the leading term in an asymptotic expansion, not a novel procedure unique to YO.

The  $\epsilon$ -expansion was described earlier as an infrared regularization necessary to evaluate the right side of Eq. (99), which diverges when  $p \rightarrow 0$ . Triads with  $p \sim 0, q \sim k$  correspond to sweeping of modes with wavevector  $|\mathbf{k}| = k$  by modes of much larger scale. The dynamic significance of this divergence has been elucidated by Kraichnan (1982). This divergence is removed in YO, and indeed in all renormalization group approaches by focusing exclusively on interactions for which  $p, q \geq k$ .

In fact, the  $\epsilon$  expansion is constructed so that when  $\epsilon = 0$ , the dominant interactions actually are the distant interactions for which  $p, q \rightarrow \infty$ : in this case, the integral in Eq. (99) is logarithmically divergent in this limit. However, as Woodruff (1993) notes, the integral becomes infrared divergent when  $\epsilon = 3$ , and the analytic continuation from  $\epsilon = 0$  to  $\epsilon = 4$  in the YO theory becomes problematic. Thus, although it is satisfying to be able to compute the inertial range constants, and even to obtain satisfactory values by a straightforward computation, the fact remains that the analytic continuation which underlies the calculation requires justification. Moreover, Woodruff also suggests that one might attempt an  $\epsilon$  expansion about this infrared divergence. Not unexpectedly, the results are quantitatively unsatisfactory, but this possibility suggests that the expansion about  $\epsilon = 0$  is not the only one possible.

Another objection to this procedure can be raised in connection with Eq. (100): the constant has been obtained by exact evaluation of the triangle integrals making neither the  $\epsilon$ -expansion nor the distant interaction approximation. However, the integral can be shown to be ultraviolet divergent for  $\epsilon < 4$  and logarithmic exactly when  $\epsilon = 4$ . Thus, there is no possibility of an  $\epsilon$  expansion for this integral, which must be evaluated exactly.



## 7 Appendix: Brief Description of RG Publications Relevant to YO

### 7.1 The work of Kraichnan

Kraichnan (1987) provides an extensive discussion on YO. First, he pointed out the logical distinction between two procedures: the multiple scale elimination and the  $\epsilon$ -expansion. First, the evaluation of eddy viscosity by perturbative elimination of successive small spherical shells of high-wavenumber modes can be carried out by specifying an actual energy spectrum (Rose, 1977) rather than introducing a forcing spectrum. Kraichnan discussed next the method of “ $\epsilon$ -expansion”. In this procedure, the properties of a  $E(k) \sim k^{1-2\epsilon/3}$  spectrum range are examined through an expansion, in powers of  $\epsilon$ , about the properties of a spectrum  $E(k) \sim k$ .

Kraichnan’s analysis is based on estimation of the qualitative nature of eddy damping in a spectrum of the form

$$E(k) \propto k^s \quad (k_c < k < k_d). \quad (105)$$

The corresponding eddy damping is given by RG as

$$\nu(k|p) \propto (p/k)^{-\epsilon/3} \quad (0 < \epsilon < 4), \quad (106)$$

where  $s = 1 - \epsilon$ . For the Kolmogorov inertial range spectrum,  $\epsilon = 4$ , the eddy damping is local for any positive  $\epsilon$  since  $\nu(k|p) \rightarrow 0$  as  $p \rightarrow \infty$ . In YO, the explicit calculation of eddy damping is first made in the near neighborhood of this reference spectrum. Now distant interactions really are weakly dominant for  $E(k) \sim k$ . At  $s = 1$  or  $\epsilon = 0$ , Eq. (106) is replaced by

$$\nu(k|p) \propto \ln(k_c/p). \quad (107)$$

The results are then mapped to the Kolmogorov spectrum by taking  $\epsilon \rightarrow 4$ . In the meantime, YO retained terms through first order in  $\epsilon$ , whereas 4 is not a small number. Hence, the convergence properties of the expansion are unclear.

Kraichnan (1987) also stressed that this type of estimation can be done completely by dimensional analysis (Fourier and Frisch, 1978; 1983; Kraichnan, 1982). The only element of the Navier-Stokes equation involved in these discussions are the overall energy conservation by nonlinear terms and the coefficients of interaction of distant wave vector triads.

### 7.2 The work of Teodorovich and Wang & Wu

Teodorovich (1987, 93, 94) used the field-theoretical method to re-evaluate the results of YO. We will discuss two issues discussed in his most recent publication in 1994.

The first issue investigated by Teodorovich is a reassessment of the eddy viscosity calculation of YO. This is exactly the same subject examined in Wang & Wu (1993). Recall that in YO the parameter  $\epsilon$  is used as a small perturbation expansion and is an important parameter in YO's eddy viscosity expression. YO then took the limit  $\epsilon \rightarrow 0$  when they compute the coefficient but set  $\epsilon = 4$  in the powerlaw exponent. However, both Teodorovich and Wang & Wu (among many others workers) noted that there is no mathematical justification for doing this, and the choice of  $\epsilon = 4$  leads to unacceptable results. Furthermore, they found an algebraic error in YO. Specifically, in the integration over wavevector  $\mathbf{q}$ , YO introduced a substitution  $\mathbf{q} \rightarrow \mathbf{k} + \mathbf{q}/2$ . However, YO did not take this substitution into account in the corresponding transformation in the associated domain of integration. Correcting this algebraic error, they found an eddy viscosity which is independent of the parameter  $\epsilon$ . Hence, the eddy viscosity in Teodorovich and Wang & Wu is independent of  $\epsilon$ , but has the numerical value of the eddy viscosity as YO who took the  $\epsilon \rightarrow 0$  limit. Without  $\epsilon$  dependence in the eddy viscosity, Teodorovich and Wang & Wu propose that a good agreement with experimental data can be obtained.

It is important to note the limit  $\epsilon \rightarrow 0$  in YO was not only used to obtain an acceptable value for the eddy viscosity, but the limit  $\epsilon \rightarrow 0$  also plays an important role in eliminating higher order nonlinearities in the RG procedure. However, Teodorovich and Wang & Wu do not address the closure problem if  $\epsilon = 4$  is maintained throughout the analysis.

The discussion now leads naturally to the question on how the local and nonlocal interactions are being treated by YO – a second major issue in the work of Teodorovich. For a forcing correlation which is a function of  $\epsilon$ , an energy spectrum can be deduced with an  $\epsilon$  dependence. As discussed already, Kraichnan (1987) found that the interactions are local when  $\epsilon \rightarrow 0$  but nonlocal when  $\epsilon = 4$ . Hence the results computed at one limit may not be used at another limit. Teodorovich claims that  $\epsilon \rightarrow 0$  corresponds to the dominance of local interaction. An analytic continuation in  $\epsilon$  from a pole-type singularity at  $\epsilon = 0$  to the point  $\epsilon = 4$  means neglecting the nonlocal interactions with the large-scale modes (Teodorovich, 1994). His analysis is not transparent.

### 7.3 The work by Ronis

Ronis (1987) analyzed a model for randomly stirred fluids using RG on a path-integral representation of the Navier-Stokes equation. Unlike YO, he found that a choice of the random force correlation exponent ( $y = -1.5851$  in three dimensions) is needed to give the Kolmogorov 5/3 law at high wavenumber.

Of particular interest, Yuan and Ronis (1992) discussed the issue on the generation of random force. They noted that YO and Ronis (1987) ignored the actual generation of the turbulence, e.g., at the boundaries of the system, and the precise nature of the stirring force is not clear. In particular, they found that there was no *a priori* theory

for the exponents used to characterize the random-force autocorrelation function. First of all, for incompressible turbulence,

$$\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + \mathbf{u}(\mathbf{r}, t) \cdot \nabla \mathbf{u}(\mathbf{r}, t) - \nabla [p/\rho] + \nu_0 \nabla^2 \mathbf{u}(\mathbf{r}, t) = \mathbf{F}(r, t) \quad (108)$$

where the force  $\mathbf{F}$  results from the interactions with the boundary and as such is not stochastic in nature. They took the view that a statistical force arises from the random force which represents the effective force felt at smaller length scales that results from turbulent, but deterministic, motion at larger scales as ‘transmitted’ by the nonlinear terms in NSE. They introduced a projection operator  $\mathcal{P}$  such that

$$\mathbf{u}'(\mathbf{r}, t) \equiv \mathcal{P} \mathbf{u}(\mathbf{r}, t) \quad (109)$$

contains only high-wavenumber information. By applying  $\mathcal{P}$  to Navier-Stokes equation it follows

$$\frac{\partial \mathbf{u}'(\mathbf{r}, t)}{\partial t} + \mathbf{u}'(\mathbf{r}, t) \cdot \nabla \mathbf{u}'(\mathbf{r}, t) - \nabla [p'/\rho] + \nu_0 \nabla^2 \mathbf{u}'(\mathbf{r}, t) = \mathbf{f}(r, t) \quad (110)$$

where

$$\mathbf{f}(r, t) \equiv \mathcal{P} \mathbf{F}(r, t) - \mathcal{P}[\mathbf{u}(\mathbf{r}, t) \cdot \nabla \mathbf{u}(\mathbf{r}, t)] + \mathbf{u}'(\mathbf{r}, t) \cdot \nabla \mathbf{u}'(\mathbf{r}, t). \quad (111)$$

Yuan and Ronis (1992) noted that this ‘new’ force contains information about boundaries as well as the mode-coupling effects associated with velocity components on the injection scales. Since  $\mathcal{P} \mathbf{F}(r, t) = 0$  away from boundaries, they defined the random stirring force used in RG studies to result from the mode coupling between the energy containing and the smaller scales. Since the motion on all scales is expected to be ‘chaotic’ including the energy containing scale, Yuan and Ronis (1992) expected that  $\mathbf{f}(r, t)$  will have complicated, chaotic time and space dependences. Therefore, they identified this as the quantity which is actually modeled by a stochastic force in random stirring models of turbulence.

The autocorrelation of the transverse parts of  $\mathbf{f}(r, t)$  is assumed with a non-zero correlation time (colored noise). The major problem with such an assumption is that the resulting theory would not be invariant under Galilean transformation. Yuan and Ronis (1992) argued that there is no *a priori* reason why global Galilean invariance must hold. Their reason is that the random forces represents the effects of boundaries and these are not included in a Galilean transformation. Note that Eq. (110) is the subgrid NSE after the full fluctuating NSE is divided into the super- and sub- grid scales. However, the filtering operation is not used in the later development, and in particular, the path-integral RG works only with the full fluctuation Navier-Stokes equation.

## 7.4 The work of Carati

The objective of Carati (1990a,b) is to modify the  $\epsilon$ -expansion of YO.

Carati (1990a) first noticed the difference between the  $\epsilon$ -RG applications in Ma and Mazenko (1975) and YO. In YO, the dimensionality  $d$  is considered as a fixed. A stochastic forcing is introduced to replace the initial and boundary conditions in Navier-Stokes equation. A correlation function of this forcing is assumed to following a powerlaw with exponent  $y$ . YO then defined  $y = \epsilon + d - 4$  where  $\epsilon$ -expansion is performed for  $y$ . Carati (1990a) attempted to modify the YO procedure by introducing a parametric dimension  $d = d_0 - 0(\epsilon)$ . But, as Carati (1990a) noted, the physical meaning of  $d_0$  is not clear *a priori*.

The next effort of Carati (1990b) involved the extension of the white noise of the forcing correlation function to a colored one (again a colored correlation refers to a non-zero correlation in time). He argued, as did Yuan and Ronis (1992), that the correlation time as well as the correlation length can be important due to the appearance of macroscopic scale structures in turbulence. It is then possible to relate the expansion parameter to the stochastic forcing correlation. The result is that the parameter  $\epsilon$  is not treated as a small parameter and is fixed in its original physical value 4. The frequency correlation provides another free parameter through the powerlaw exponent  $\gamma/3$  where  $-1 < \gamma < 1$ . The original YO's  $\epsilon$ -expansion is replaced by arguing that the frequency integral is nearly divergent because of the assumed value of  $\gamma$ . Alternatively, the  $\epsilon$ -expansion has now been replaced by a  $\gamma$ -expansion where  $\gamma = 1 - \delta$ . This  $\delta$ -expansion was interpreted as a scheme in which a large amount of energy is injected into the system (Carati, 1990b).

Although Carati viewed the colored forcing correlation as a good way to sidestep the  $\epsilon$ -expansion, he did not consider that the introduction of the colored noise violates the Galilean invariance. This important issue was pointed out by Yuan and Ronis (1992) in their analysis (see previous subsection).

## 7.5 The work of Liang and Diamond

Liang and Diamond (1993) conducted a careful examination on the applicability of  $\epsilon$ -RG to two dimensional (2D) flows. They examined both 2D fluid and 2D Magnetohydrodynamic (MHD) turbulence but our discussion will be limited to the fluid turbulence part of the paper only.

The most important difference between 2D and 3D turbulence is the direction of energy transfer. There is a dual-cascade phenomena, in which the energy is transferred from large to small scales, while enstrophy (defined as mean-square vorticity), is transferred from large scale to small scales (Kraichnan and Montgomery, 1980). Liang and Diamond (1993) found that when a proper inertial range energy spectrum is assumed, no fixed point can be found for positive or negative eddy viscosity. There-

fore,  $\epsilon$ -RG can not be used to analyze the physics of 2d turbulence. They found that the reason of this difficulty can be traced to the dual energy cascade.

One important point should be stressed. Although Liang and Diamond (1993) followed most of the procedure of YO  $\epsilon$ -RG, they did not apply the  $\epsilon$ -expansion. Instead, they used the proper inertial range scaling as an input.

To identify the reason behind the failure of  $\epsilon$ -RG in 2D, Liang and Diamond (1993) compared it to the EDQNM closure. In  $\epsilon$ -RG procedure, all calculations are one point and only eddy damping term appear. The nonlinear noise (nonlinear coupling) is replaced by adding stochastic forces. The major problem is that the properties of the stochastic forces can not be determined recursively. They are determined instead by the properties of the self-similarity range spectra, which in reality, are the goals of the calculation. On the other hand, a two-point closure calculation determines the form of the spectrum. Therefore, both the eddy damping and the nonlinear model coupling terms will now appear. Furthermore, while the  $\epsilon$ -RG only incorporate the nonlocal effects of the small scales on large scales, EDQNM (and DIA) take both effects into account, in principle.

## 7.6 The work of Lam

Lam's approach (Lam 1992), although belonging to the FNS school of activity, is quite different from the other members of that group. It does not make use of the  $\lambda$ -expansion procedure used by FNS, nor the correspondence principle and the  $\epsilon$ -expansion procedure used in the YO theory. His interpretation of  $\epsilon$ -RG is based on phenomenological approach. Zhou (1995) has shown that Lam's model is essentially the physical space version of the classical closure theory (Leslie and Quarini, 1979) in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

## 7.7 The work of Eyink

Eyink (1994) used the momentum-shell RG method of Kadanoff-Wilson based on the Martin-Siggia-Rose (1978) field-theory formulation of stochastic dynamics.

One of the major points in Eyink (1994) is that, contrary to the claim of YO, the higher-order nonlinear terms generated in their RG analysis are not irrelevant but marginal by power-counting. Because of this, the terms neglected by YO are not necessarily small *even for small*  $\epsilon$ . The problem was traced to issues relevant to Galilean-invariant theories. It is shown not to occur in the FNS original analysis since they were dealing with fluctuation dynamics for equilibrium NS fluids. The issue of Galilean invariance has also been discussed by Zhou and Vahala (1993b).

Eyink (1994) also presented his objections to the YO applications of  $\epsilon$ -RG to turbulence modeling. First, he repeated Kraichnan's (1987) arguments that the power-laws derived by YO can be simply derived from dimensional analysis. Second, he

stated, as Zhou et al. (1988), that by setting  $\epsilon = 4$ , there is no reason to believe that additional nonlinearities are negligible or insignificant to the Physics. Finally, the YO analysis does not include the rescaling procedure, which Eyink (1994) viewed as vital to error estimation. In fact, Eyink (1994) even stated that “YO thory is not an RG analysis at all!”.

## Part IV

# RG-based turbulence modelling

## 8 Modeling of Reynolds stress using RG

Turbulent flows of practical interest involve a broad spectrum of length and time scales and require some type of modeling for Reynolds stresses. The most widely used are the two-equation turbulence models based on the transport equations for parameters that involve the length and the time scales and require the simplest level of Reynolds stress closure that does not depend specifically on the flow geometry. In its standard form the two-equation Reynolds stress turbulence models involve the turbulence kinetic energy and dissipation based on a Boussinesq type approximation involving isotropic eddy viscosity. Such a representation of turbulence is often not effective from both theoretical as well as phenomenological point of view. To overcome this, models that are nonlinear (i.e., quadratic) in the mean strain rate were proposed in the form of a constitutive relation (Speziale, 1987; Yoshizawa, 1984). These models are capable of predicting the anisotropy in the Reynolds stresses in complex flows but require empirical evaluation of the model constants (Speziale, 1991).

Two studies presented here<sup>5</sup> address the need for a more effective approach in the development of two-equation turbulence models, and in this context the renormalization group (*RG*) theory based models are considered (for a summary of the ICASE panel discussion on RG based turbulence modeling, see Zhou and Speziale (1994)). These models fall into two distinct categories: (a)  $\epsilon$ -*RG*, where a small parameter  $\epsilon$  is introduced into the exponent of the forcing correlation function (with the forcing function being introduced into the momentum equation), and the theory is then developed for  $\epsilon \ll 1$  (Yakhot and Orszag, 1986); and (b) recursion-*RG*, which does not rely on an  $\epsilon$ -expansion, and treats explicitly the cubic nonlinearities introduced into the renormalized momentum equation (Zhou, et al. 1988). It should be noted that in the  $\epsilon$ -*RG* while all constants generated are evaluated in the limit  $\epsilon \ll 1$ , at the same time, all exponents that are  $\epsilon$ -dependent are evaluated at  $\epsilon = 4$ . In fact,  $\epsilon = 4$  is required in the  $\epsilon$ -*RG* to recover the Kolmogorov energy spectrum in the inertial range. In addition,  $\epsilon$ -*RG* theory can only take into account non-local interactions (Smith and Reynolds, 1992). On the other hand, recursion-*RG* does not rely on an  $\epsilon$ -expansion, and treats explicitly the cubic nonlinearities introduced into the renormalized momentum equation. Moreover, recursion-*RG* can handle both local and non-local interactions. Effects such as the cusp behavior in the transport coefficients (Zhou

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<sup>5</sup>Note that recently Nagano and Itazu (1997) attempted to derive an eddy viscosity model using the iterative averaging method

and Vahala, 1993a) which are due to local interactions and the cubic nonlinearities are recovered in these theories. As in  $\epsilon - RG$ , the eddy viscosity is readily determined from the solution of a relatively simple differential equation (Zhou and Vahala, 1993a). However, unlike  $\epsilon - RG$ , the transport coefficients are determined over the whole resolvable scales and not just in the wavenumber limit  $k \rightarrow 0$ .

The contributions to the Reynolds stress tensor  $\tau_{ij}$  from the conventional time-averaging of the equations of motion are due to the nonlinear coupling term and its interaction with the velocity field (Zhou et al., 1993). In particular, the Reynolds stress tensor,  $\tau_{ij} = \tau_{ij}^{++} + \tau_{ij}^{--}$ , where the  $\tau_{ij}^{++}$  - part arises from the infrared limit of  $k \rightarrow 0$  and is due to the  $u_i^+ - u_j^+$  distant interaction limit while the  $\tau_{ij}^{+-}$  - part arises from the  $0 < k \leq k_c$  spectrum and is due to the  $u_i^+ - u_j^-$  local interaction limit. It should be noted that in  $\epsilon - RG$  approach, one takes the large-scale infrared limit  $k \rightarrow 0$ . In essence, this forces a spectral gap between the resolvable part of the flow field and the small unresolved scales. If this spectral gap were somehow present initially, it would be quickly populated in just a few eddy turn over times. Thus, retaining only the distant interactions may not be appropriate. In fact, it has been shown (Zhou and Vahala, 1993a) that the energy transfer function that corresponds to local interactions accounts for most of the energy flow out of the resolvable scales. It is, therefore, important to retain both local and nonlocal interactions in the modeling of the Reynolds stress and this can be readily achieved by recursion- $RG$ . But in the  $\epsilon - RG$  model, the Reynolds stress  $\tau_{ij} = \tau_{ij}^{++}$  and is obtained purely from the  $u_i^+ - u_j^+$  interaction in the small unresolved scale momentum equation.

Rubinstein and Barton (1990) have derived a Reynolds stress model using the  $\epsilon - RG$  method (which corresponds to the infrared limit of  $k \rightarrow 0$ ) of the following form:

$$\begin{aligned} \tau_{ij}^{++} = & -\frac{2}{3}K\delta_{ij} + v_T(\partial_j U_i + \partial_i U_j) - \frac{K^3}{\epsilon^2}[C_{\tau 1}(\partial_\alpha U_i \partial_\alpha U_j)^* \\ & + C_{\tau 2}(\partial_\alpha U_i \partial_j U_\alpha + \partial_\alpha U_j \partial_i U_\alpha)^* + C_{\tau 3}(\partial_i U_\alpha \partial_j U_\alpha)^*]. \end{aligned} \quad (112)$$

The constants  $C_{\tau 1} = 0.034$ ,  $C_{\tau 2} = 0.104$  and  $C_{\tau 3} = -0.014$ , and  $(\dots)^*$  denotes the deviatoric part of the expression within the parenthesis. The first two terms correspond to the linear model and  $v_T = C_\mu K^2/\epsilon$  is the isotropic eddy-viscosity, where  $\epsilon$  is the turbulence dissipation and  $C_\mu \approx 0.09$  based on empirical data from equilibrium boundary layer flows. The above model is quadratic in mean strain rate, includes the effect of convection and diffusion and is qualitatively similar to other second order models (Speziale, 1991).

Now, if one follows standard recursion- $RG$  procedures (Zhou et al., 1993), the relevant part of the small scale velocity field that contributes to  $\tau_{ij}^{+-}$  is obtained in the wavenumber space. Transforming back to the physical space the following algebraic representation is obtained (Zhou et al., 1993):

$$\tau_{ij}^{+-}(\mathbf{x}) = C_{R1} \frac{K^4}{\epsilon^3} [\partial_\alpha U_i \partial_\beta U_j \partial_\alpha U_\beta + i \leftrightarrow j]$$



$$+C_{R2}\frac{K^7}{\varepsilon^5}[(\partial_\alpha U_i)\partial_j(\partial_\gamma U_\beta\partial_\alpha\partial_\beta U_\gamma) + i \leftrightarrow j] \quad (113)$$

where  $U_i$  is the time-averaged mean velocity (and the second term inside the square brackets is obtained by switching the indices). However, for most flows of interest (inhomogeneous flows), the cut-off wavenumber  $k_c$  varies considerably throughout the flow domain. Thus, if one wishes to use the local approximation, as well as retain  $k_c \sim O(1)$  effects so as to obtain an algebraic form for  $\tau_{ij}^{><}(\mathbf{x})$ , then the coefficients  $C_{R1}$  and  $C_{R2}$  will be functions of the flow quantities. As a first attempt at applying this recursion-*RG* model we make the lowest order approximation that these coefficients are constants. In particular, for the present analysis,  $C_{R1}$  and  $C_{R2}$  are taken to be 0.025 and  $0.342 \times 10^{-3}$ , respectively (Zhou et al., 1994).

Combining (112) - (113) a formal expression for Reynolds stress which includes both the local and nonlocal interactions may be obtained. It is of some interest to note that integrity basis representations are commonly employed to represent the anisotropic part of the Reynolds stress tensor for three dimensional turbulent flows based on a systematic derivation from a hierarchy of second-order closure models (Gatski and Speziale, 1993). It can readily be shown that the tensors that constitute the integrity basis are recovered for most part when the proposed recursion-*RG* model is recast appropriately (Zhou et al., 1994).

The above expression for Reynolds stress (112) - (113) are to be used along with the equations of motion by specifying turbulent kinetic energy and dissipation. In two-equation turbulence models, this closure is achieved through the development of transport equations for the turbulent kinetic energy and dissipation - quantities that are directly related to the length and time scales - of the following general form:

$$\partial_t K + U_j \partial_j K = \mathcal{P} - \varepsilon + \partial_i [(v_0 + v_T/\alpha_K)\partial_i K] \quad (114)$$

$$\partial_t \varepsilon + U_j \partial_j \varepsilon = C_{\varepsilon 1} \mathcal{P} \varepsilon / K - C_{\varepsilon 2} \varepsilon^2 / K + \partial_i [(v_0 + v_T/\alpha_\varepsilon)\partial_i \varepsilon] \quad (115)$$

where,  $\nu = \nu_0 + \nu_T$  is the total viscosity,  $\mathcal{P} = -\tau_{ij}(\partial U_i/\partial x_j)$  is the turbulence production,  $\varepsilon$  is the scalar turbulent dissipation rate. The quantities  $C_{\varepsilon 1}$ ,  $C_{\varepsilon 2}$ ,  $\alpha_K$ ,  $\alpha_\varepsilon$  are dimensionless and taken to be 1.44, 1.92, 1.0 and 1.3, respectively, consistent with the standard form of the two-equation  $K - \varepsilon$  model (based on empirical data obtained from equilibrium boundary layer flows). We note that with various approximations, the  $\varepsilon - RG$ -based formulations computed these constants as 1.42, 1.68, 0.719 and 0.719, respectively (Yakhot and Orszag, 1986).

The *RG* theory is utilized to develop Reynolds stress closure models for the prediction of turbulent separated flows. The combined model includes both the local and nonlocal interaction of all the relevant resolvable scales. The ability of the proposed model to accurately predict separated flows is analyzed from a combined theoretical and computational standpoint by considering turbulent flow past a backward facing step as a test case. The results obtained based on detailed computations demonstrate

that the RG model can yield very good predictions for the turbulent flow of an incompressible viscous fluid over a backward-facing step (Zhou *et al.*, 1994). Thus, in spite of its well known deficiencies, when the anisotropy of the turbulent stresses are properly accounted for, the two-equation turbulence models can be quite effective for the prediction of turbulent separated flows.

## 9 RG based $K - \epsilon$ model

YO derived the  $K - \epsilon$  two equation model using  $\epsilon$ -RG method. Speziale (1990) found that the original YO model performs poorly in homogeneous shear flow. The value  $C_{\epsilon 1}$  derived in YO yields excessively large growth rate for the turbulent kinetic energy in homogeneous shear flow in comparison to both physical and numerical experiments (Speziale, 1991).

Smith and Reynolds (1992) found some algebraic error in the original derivation of YO. The coefficients of the dissipation term in  $\epsilon$  equation is not in good agreement with generally accepted values. Furthermore, YO's derivation did not yield a term responsible for the production in the  $\epsilon$  equation.

The original derivation of YO was revised by Yakhot and Smith (YS) (1992) by the following features:

1. The 'infrared cutoff' of the random force,  $\langle f^> f^> \rangle = 0$  when  $0 < k < \Lambda_L$

$$\langle f_\alpha(k, t) f_\beta(k', t') \rangle = D_0 k^{-y} D_{\alpha\beta}(k) \delta(k+k') \delta(t-t'), \quad \Lambda_L < k < \infty = 0, \quad 0 < k < \Lambda_L \quad (116)$$

This property is needed in the derivation of the equation for the mean rate of energy dissipation  $\epsilon$  (YS).

2. The input of energy spectrum for the interval  $0 < k < \Lambda_L$

$$E(k) \sim k^\alpha \quad (117)$$

is required to evaluate the integrals (with  $\alpha = 2$ ).

3. Performing a Reynolds decomposition of  $T_1 = -2\nu_0(\nabla_j u_i)(\nabla_j u_l)(\nabla_l u_i)$  into mean  $\mathbf{U}$  and fluctuating  $\mathbf{u}$  velocities.

The derivation of YO and YS starts from dynamical equations for the homogeneous part of the instantaneous rate of energy dissipation per unit mass  $\epsilon \equiv \nu_0(\nabla_j u_i)^2$

$$\frac{\partial \epsilon}{\partial t} + u_i \nabla_i \epsilon = \overbrace{2\nu_0(\nabla_j u_i)(\nabla_j f_i)}^{\mathcal{P}_\epsilon} - \overbrace{2\nu_0(\nabla_j u_i)(\nabla_j u_l)(\nabla_l u_i)}^{T_1}$$

$$- \overbrace{2\nu_0^2(\nabla_j \nabla_l u_i)^2}^{T_2} - 2\nu_0(\nabla_j u_i)(\nabla_i \nabla_j p) + \nu_0 \nabla_i \nabla_i \varepsilon \quad (118)$$

After some work, at the stirred fluids at the long-time and large-distance limit, the  $\epsilon$ -RG dissipation equation (YS, 1992) is found to be

$$D_t \varepsilon = C_{\epsilon 1}(\varepsilon/K)\tau_{ij}\partial_j u_i - C_{\epsilon 2}\varepsilon^2/K + \partial_i(\alpha\nu_T\partial_i \varepsilon) - \mathcal{R} \quad (119)$$

where  $C_{\epsilon 1} = 1.42$ ,  $C_{\epsilon 2} = 1.68$  and

$$\mathcal{R} = 2\nu_0 S_{ij} \overline{\frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_j}} \quad (120)$$

The  $\epsilon$  expansion procedure and above mentioned assumptions have been employed.

The above  $\epsilon$ -RG dissipation equation is not closed. The neglect of  $\mathcal{R}$  is a formally justified approximation at high Reynolds number if the hypothesis of local isotropy is invoked. Durbin and Speziale (1991) have questioned the validity of local isotropy in strongly strained turbulence flows. Yakhot et al. (1992) have proposed a model where  $\mathcal{R} = \mathcal{R}(\eta)$ , where the standard form of the model is recovered for  $\mathcal{R} \rightarrow 0$  in the limit of weak strains. Note that Durbin (1990) has already developed a model for the production of dissipation along these lines that was quadratic in the ratio of production to the dissipation and, hence, quartic in  $\eta$ . Lam (1994) published a critique on the YS derivation.

Iterating the expression for  $\mathcal{R}$  using the Navier-Stokes equation will generate a power series

$$\mathcal{R} = \nu_T S^3 \sum_{n=0}^{\infty} r_n \left(\frac{SK}{\varepsilon}\right)^n \quad (121)$$

where  $S = (2S_{ij}S_{ij})^{1/2}$ . It is not possible to evaluate the summation since the values of coefficients are unknown.

The  $\mathcal{R}$  is modeled via three steps:

1. The summation is performed for the *geometric series* for *every three terms*. This procedure reduces the numbers of unknown coefficients to *one*,  $\beta$ .

$$\mathcal{R}^0 = \nu_T S^3 \sum_{n=0}^{\infty} (-\beta)^n \left(\frac{SK}{\varepsilon}\right)^{3n} = \frac{\nu_T S^3}{1 + \beta\eta^3}. \quad (122)$$

2. Assuming that the fixed point value  $\eta_0 = 4.38$  of the homogeneous shear flow in the equilibrium states, is invariant to dropping all terms but those in (122) above, Yakhot et al. (1992) postulated that

$$\mathcal{R} = \frac{\nu_T S^3}{1 + \beta\eta^3} (1 - \eta/\eta_0). \quad (123)$$

3. One now further assumes that the isotropic Reynolds stress  $\tau_{ij} = -2C_\nu K \eta_{ij}$  ( $\eta_{ij} = S_{ij}K/\varepsilon$ )

$$\mathcal{R} = \frac{C_\nu \eta^3 (1 - \eta/\eta_0) \varepsilon^3}{1 + \beta \eta^3} \frac{\varepsilon}{K} = \frac{\eta(1 - \eta/\eta_0) \varepsilon}{1 + \beta \eta^3} \frac{\varepsilon}{K} \tau_{ij} S_{ij}. \quad (124)$$

The undertermined constant  $\beta = 0.012$  for the von Kármán constant 0.4. The final  $\varepsilon$ -RG dissipation rate transport equation is given

$$D_t \varepsilon = C_{\varepsilon 1}^* (\varepsilon/K) \tau_{ij} \partial_j u_i - C_{\varepsilon 2} \varepsilon^2 / K + \partial_i (\alpha \nu_T \partial_i \varepsilon) \quad (125)$$

where the coefficient  $C_{\varepsilon 1}^*$  is given by

$$C_{\varepsilon 1}^* = C_{\varepsilon 1} - \frac{\eta(1 - \eta/\eta_0)}{1 + \beta \eta^3} \quad (126)$$

The model of Yakhot et al. (1992) has been tested for homogeneous shear flows and for flow over a backward facing step. Excellent results are obtained in both cases. Recently, Yakhot and Orszag have extended the model and applied it to complex flows using the FLUENT code.

## Part V

# Conclusion

In this review, we explained the concepts of terms *renormalization* and *renormalization group* by referencing to various physical systems, such as the ising model. We then present a comprehensive review on applications of the method of renormalization group to turbulence. These parts should be sufficient for readers who wishes to get a balanced view of RG in turbulence. For a few selected approaches, we have provided further technical details. We conclude with a discussion of the relevance and application of renormalization group to turbulence modelling.

## Part VI

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