NASA/CR-97-206250
ICASE Report No. 97-63

The Dissipation Rate Transport Equation and Subgrid-Scale Models in Rotating Turbulence

Robert Rubinstein and Ye Zhou
ICASE

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA

Operated by Universities Space Research Association

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

Prepared for Langley Research Center
under Contract NAS1-19480

November 1997

THE DISSIPATION RATE TRANSPORT EQUATION AND SUBGRID-SCALE MODELS IN ROTATING TURBULENCE*

ROBERT RUBINSTEIN† AND YE ZHOU‡

Abstract. The dissipation rate transport equation remains the most uncertain part of turbulence modeling. The difficulties are increased when external agencies like rotation prevent straightforward dimensional analysis from determining the correct form of the modelled equation. In this work, the dissipation rate transport equation and subgrid scale models for rotating turbulence are derived from an analytical statistical theory of rotating turbulence. In the strong rotation limit, the theory predicts a turbulent steady state in which the inertial range energy spectrum scales as k^{-2} and the turbulent time scale is the inverse rotation rate. This scaling has been derived previously by heuristic arguments.

Key words. Dissipation rate transport equation, subgrid scale models, rotating turbulence

Subject classification. Fluid Mechanics

1. Introduction. Kraichnan's (1959) Direct Interaction Approximation (DIA) and related Lagrangian closures (Kraichnan, 1964; Kaneda, 1968) remain the only fully deductive turbulence theories. Although a study of a simple inhomogeneous flow, like channel flow, using these closures would be of the greatest theoretical and practical interest, the complexity of the calculations required has precluded any but preliminary results (Dannevik, 1992).

Practical application of DIA therefore requires some compromise of rigor in the interest of utility. The most comprehensive attempt to extract turbulence models from DIA remains the two-scale theory (TSDIA) of Yoshizawa (1984, 1996) in which inhomogeneity, anisotropy, and time-dependent nonequilibrium effects are introduced by perturbing about a state of homogeneous, isotropic, stationary turbulence.

Yoshizawa's procedure leads to formulas for quantities familiar in single point phenomenological turbulence closures like the two-equation model. A typical result (Yoshizawa, 1984) is the expression for eddy viscosity

$$\nu = \frac{4}{15}\int d\mathbf{k}\int_0^\infty d\tau\ G(k,\tau)Q(k,\tau) \tag{1}$$

in terms of the DIA descriptors (Kraichnan, 1959) of isotropic turbulence: the response function $G(k,\tau)$ and correlation function $Q(k,\tau)$. The isotropy of the lowest order field implies that these descriptors are scalars, homogeneity permits introduction of the wave-vector argument $\mathbf{k}$, and stationarity in time makes them functions of time difference τ only. Yoshizawa (1984) shows how the familiar eddy viscosity formula of single-point turbulence modeling is deduced from this formula, by substituting Kolmogorov scaling forms for G and Q. As is well-known, Kolmogorov scaling describes the simplest turbulent steady state with a constant flux of kinetic energy from the large to the small scales.

2. DIA for Rotating Turbulence. Eq. (1) suggests that an eddy viscosity for turbulence subject to any external agency can be derived, provided that appropriate formulas for G and Q are known. Rotation

*This research was supported by NASA under Contract No. NAS1-19480 while the authors were in residence at ICASE.

† Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23681

‡ Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA 23681 and IBM Research Division, T.J. Watson Research Center, P.O. Box, 218, Yorktown Heights, NY 10598 (email: *zhou@icase.edu* and/or *yzhou@watson.ibm.com*).

is a particularly simple external effect, since the energy remains an inviscid invariant under rotation, and a steady state with constant energy flux remains possible. But Kolmogorov scaling may no longer apply to this steady state and the occurence of a distinguished time scale precludes the deduction of the applicable scaling law by dimensional analysis. To deduce the scaling, we will appeal to closure in the form of the direct interaction approximation.

For rotating turbulence, the DIA equations of motion take the form

$$\dot{G}_{ij}(\mathbf{k},t,s) + 2P_{ip}(\mathbf{k})\Omega_{pq}G_{qj}(\mathbf{k},t,s) + \int_s^t dr\ \eta_{ip}(\mathbf{k},t,r)G_{pj}(\mathbf{k},r,s) = 0 \tag{2}$$

$$\dot{Q}_{ij}(\mathbf{k},t,s) + 2P_{ip}(\mathbf{k})\Omega_{pq}Q_{qj}(\mathbf{k},t,s) + \int_s^t dr\ \eta_{ip}(\mathbf{k},t,r)Q_{pj}(\mathbf{k},r,s) = \int_0^t dr\ G_{ip}(\mathbf{k},t,r)F_{pj}(\mathbf{k},s,r) \tag{3}$$

where the eddy damping η and forcing F are defined by

$$\eta_{ir}(\mathbf{k},t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}P_{imn}(\mathbf{k})P_{\mu rs}(\mathbf{p}) \times G_{m\mu}(\mathbf{p},t,s)Q_{ns}(\mathbf{q},t,s) \tag{4}$$

$$F_{ij}(\mathbf{k},t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}P_{imn}(\mathbf{k})P_{jrs}(\mathbf{k}) \times Q_{ns}(\mathbf{p},t,s)Q_{mr}(\mathbf{q},t,s) \tag{5}$$

In Eqs. (4) and (5),

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$
$$P_{im}(\mathbf{k}) = \delta_{im} - k^{-2}k_i k_m$$

and Ω_{pq} is the antisymmetric rotation matrix. The solution of these equations in complete generality is not known. A useful simplification, EDQNM, effectively replaces the response equation Eq. (2) by a phenomenological hypothesis, and solves a simplified, Markovianized version of the correlation equation Eq. (3) (Cambon and Jacquin, 1986).

A perturbative solution of these equations is suggested by Leslie's (1972) treatment of shear turbulence: treat the rotation terms as small, and perturb about an isotropic turbulent state. This approach is adopted by Shimomura and Yoshizawa (1986) who derive a TSDIA theory in which both inhomogeneity and rotation are described by small parameters.

A complementary limit is also of interest. Namely, in the response equation, balance the time derivative by the rotation term, and treat the eddy damping as small. This linear theory of the response equation treats strongly rotating turbulence as a case of *weak turbulence* (Zakharov *et al*, 1992) in which nonlinear decorrelation of Fourier modes is dominated by linear dispersive decorrelation (Waleffe, 1993). The result is conveniently expressed in terms of the *Craya-Herring* basis

$$\mathbf{e}^{(1)}(\mathbf{k}) = \mathbf{k}\times\mathbf{\Omega}/\mid \mathbf{k}\times\mathbf{\Omega} \mid$$
$$\mathbf{e}^{(2)}(\mathbf{k}) = \mathbf{k}\times(\mathbf{k}\times\mathbf{\Omega})/\mid \mathbf{k}\times(\mathbf{k}\times\mathbf{\Omega}) \mid$$

or the equivalent basis of Cambon and Jacquin (1989), and the corresponding tensors

$$\begin{aligned}\xi^0_{ij} &= e^{(1)}_i e^{(2)}_j - e^{(1)}_j e^{(2)}_i \\ \xi^1_{ij} &= e^{(1)}_i e^{(2)}_j + e^{(1)}_j e^{(2)}_i \\ \xi^2_{ij} &= e^{(1)}_i e^{(1)}_j - e^{(2)}_i e^{(2)}_j \\ \xi^3_{ij} &= e^{(1)}_i e^{(1)}_j + e^{(2)}_i e^{(2)}_j\end{aligned} \tag{6}$$

Note that $\xi^3_{ij} = P_{ij}(\mathbf{k})$.

The leading order solution of Eq. (2), obtained by dropping the eddy damping term, is

$$\begin{aligned}G_{ij}(\mathbf{k}, t, s) = \{&\cos(\mathbf{k} \cdot \mathbf{\Omega}(t-s)/k) P_{ij}(\mathbf{k}) \\ &+ \sin(\mathbf{k} \cdot \mathbf{\Omega}(t-s)/k) \xi^0_{ij}(\mathbf{k})\} H(t-s)\end{aligned} \tag{7}$$

where H is the unit step function. Let us adopt the *fluctuation-dissipation* hypothesis relating the two-time correlation function to the response function and single-time correlation function

$$Q_{ij}(\mathbf{k}, t, s) = G_{im}(\mathbf{k}, t, s) Q_{mj}(\mathbf{k}) + G_{jm}(\mathbf{k}, s, t) Q_{mi}(\mathbf{k}) \tag{8}$$

Conditions under which this approximation is reasonable are discussed by Woodruff (1992). Substituting Eqs. (7) and (8) in Eq. (3) shows that the single-time correlation function must take the general form containing all of the ξ tensors of Eq. (6)

$$Q_{ij}(\mathbf{k}) = \sum Q^p(\mathbf{k}) \xi^p_{ij}(\mathbf{k}) \tag{9}$$

which is equivalent to the form of the correlation function noted by Cambon and Jacquin (1989).

The DIA inertial range energy balance (Kraichnan, 1971), which states that a steady state with constant energy flux exists, is

$$\begin{aligned}\varepsilon = [I^+ - I^-] P_{imn}(\mathbf{k}) \int_0^\infty d\tau \; &2P_{\mu rs}(\mathbf{p}) G_{m\mu}(\mathbf{p}, \tau) Q_{ns}(\mathbf{q}, \tau) Q_{ir}(\mathbf{k}, \tau) \\ &- P_{jrs}(\mathbf{k}) G_{ij}(\mathbf{k}, \tau) Q_{ns}(\mathbf{p}, \tau) Q_{mr}(\mathbf{q}, \tau)\end{aligned} \tag{10}$$

where the integration operators in Eq. (10) are defined by

$$\begin{aligned}I^+(k_0) &= \int_{k \geq k_0} d\mathbf{k} \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}; p,q \leq k_0} d\mathbf{p} d\mathbf{q} \\ I^-(k_0) &= \int_{k \leq k_0} d\mathbf{k} \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}; p,q \geq k_0} d\mathbf{p} d\mathbf{q}\end{aligned}$$

and $\tau = t - s$ denotes time difference. The time integrals in Eq. (10) will take the form

$$\Theta(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \Omega^{-1} \delta(\pm \mathbf{p} \cdot \mathbf{\Omega}/p\Omega \pm \mathbf{q} \cdot \mathbf{\Omega}/q\Omega \pm \mathbf{k} \cdot \mathbf{\Omega}/k\Omega) \tag{11}$$

where

$$\Omega = (\Omega_{pq} \Omega_{pq})^{1/2}$$

Thus, wave-vector integrations take place over resonant triads only (Waleffe, 1993) and these integrals scale as Ω^{-1} (Zhou, 1995).

Introduce the *ansatz*

$$Q^p(\mathbf{k}) = k^{-\alpha-2} f^p(k, \mathbf{\Omega} \cdot \mathbf{k}/k\Omega) \tag{12}$$

for the functions in Eq. (9). In Eq. (12), f^p is homogeneous of degree zero in k. In view of Eq. (11), the energy balance Eq. (10) requires $\alpha = 2$, as in the heuristic argument of Zhou (1995). The goal of this work is a two-equation model of the standard form. Accordingly, we only attempt to quantify the overall effect of rotation on turbulent energy transfer, ignoring the *polarization* of turbulence (Cambon and Jacquin, 1989) by rotation and the distribution of energy in $\mathbf{k}$ space. For this purpose, it suffices to replace the anisotropic energy spectrum tensor by the energy transfer-equivalent isotropic spectrum given by Zhou (1995),

$$E(k) \sim \sqrt{\varepsilon\Omega} k^{-2} \tag{13}$$

It is of interest to exhibit the eddy damping correction to the leading order results represented by Eqs. (7) and (13). Namely, substituting these results in Eq. (4) for the eddy damping factor,

$$\eta \sim k^2 \int_k^\infty dp \; \frac{1}{\Omega} \sqrt{\varepsilon\Omega} p^{-2} \sim k\sqrt{\varepsilon/\Omega} \tag{14}$$

The corrections to the time scale and energy spectrum have the form

$$\begin{aligned} \Theta &\sim \frac{1}{\Omega}\{1 + O(\Omega^{-3/2})\}^{-1} \\ E &\sim \sqrt{\varepsilon\Omega} k^{-2}\{1 + O((k^2\varepsilon/\Omega^3)^{1/2}\} \end{aligned} \tag{15}$$

The low rotation rate expansion of Shimomura and Yoshizawa (1986) gives the complementary expansions in positive powers of Ω,

$$\begin{aligned} \Theta &\sim \varepsilon^{-1/3} k^{-2/3}\{1 + O(\Omega)\} \\ E &\sim \varepsilon^{2/3} k^{-5/3}\{1 + O(\Omega/\varepsilon^{1/3} k^{2/3}\} \end{aligned} \tag{16}$$

The expansions in Eqs. (15) and (16) could be consolidated into Padé approximations for the energy spectrum and decorrelation time applicable for intermediate rotation rates.

2.1. Locality of energy transfer. The scaling law of Eq. (13) is purely formal: to prove that a steady state scales this way, the convergence of the integral in Eq. (10) must be demonstrated. Divergence would imply strong dependence on the cutoff at large or small scales, and would therefore alter the scaling law (Kraichnan, 1959). Even if the dependence of $E(k)$ on $\Omega^{1/2}$ is known, Eq. (13) cannot be asserted on dimensional grounds: dimensional analysis assumes that k^{-1} is the only relevant length scale; it therefore postulates the locality which convergence of the flux integral proves.

However, it is not difficult to prove that the flux integral does in fact converge when evaluated for an infinite k^{-2} inertial range. The proof only requires the transverse character of the tensors ξ^p and does not differ conceptually from the proof originally given by Kraichnan (1959). Accordingly, we can assert the locality of energy transfer in a rotating inertial range and can apply the scaling laws of Eq. (15) to develop turbulence models.

3. Subgrid-Scale Models. Since the lowest order field in TSDIA is arbitrary, the transport properties of weakly strained strongly rotating turbulence can be derived by perturbing about the steady rotating turbulent state just described instead of perturbing about a Kolmogorov steady state. In carrying out this program, it should be stressed that formulas like Eq. (1) for the eddy viscosity are the result of evaluating an integral over interacting triads which satisfy a resonance condition. This condition provides an additional source of anisotropy in the exact theory of rotating turbulence, since only two-dimensional triads for which

$$\mathbf{k} \cdot \mathbf{\Omega} = \mathbf{p} \cdot \mathbf{\Omega} = \mathbf{q} \cdot \mathbf{\Omega} = 0$$

satisfy the resonance condition automatically. However, since our goal is to evaluate the over-all effect of rotation on turbulent energetics, we will evaluate the eddy viscosity integral using the isotropic expressions of Eq. (15).

The simplest way to derive a sub-grid scale viscosity from Eq. (1) is to integrate only over scales satisfying $k \geq 2\pi/\Delta$, where Δ denotes the filter width. It follows from substitution of Eq. (15) in Eq. (1) that in strongly rotating turbulence, to leading order

$$\nu = C_\nu^\Omega \Delta \sqrt{\varepsilon/\Omega} \tag{17}$$

where Δ is the filter size. Corrections for finite rotation rates are suggested by Eq. (15). Setting Δ equal to the integral scale of turbulence leads to the single point result

$$\nu \sim \frac{K}{\Omega} \tag{18}$$

A subgrid scale model is derived by eliminating the dissipation rate from Eq. (17). We follow the derivation of the Smagorinsky model by equating the dissipation rate to the resolved production:

$$\varepsilon = \frac{1}{2}\nu S_{ij} S_{ij} \tag{19}$$

where $S^2 = S_{ij}S_{ij}/2$ and S_{ij} is the resolved strain rate. Therefore,

$$\varepsilon = C_\nu^\Omega \sqrt{\varepsilon/\Omega} \Delta S^2 \tag{20}$$

Solving Eq. (20) for ε and substituting in Eq. (17),

$$\nu = (C_\nu^\Omega)^2 \frac{S^2 \Delta^2}{\Omega} \tag{21}$$

which is the Smagorinsky model for strongly rotating turbulence.

This calculation supplements the evaluation by Shimomura and Yoshizawa (1986) of the sub-grid scale viscosity in weakly rotating turbulence. But the effect of rotation becomes weaker with decreasing scale size; accordingly, the weak rotation correction of Shimomura and Yoshizawa may be appropriate even in some rapidly rotating flows.

Additional production and dissipation mechanisms are often modeled by Richardson number modifications of the subgrid-scale viscosity. Thus, if the energy balance at the grid scale is written as

$$\nu S^2 (1 - Ri_t) = \varepsilon \tag{22}$$

where Ri_t denotes an appropriate turbulent Richardson number, and Kolmogorov scaling

$$\nu = C \varepsilon^{1/3} \Delta^{4/3} \tag{23}$$

can be assumed, then Eqs. (22) and (23) give

$$\nu = C^{3/2} \Delta^2 S \sqrt{1 - Ri_t} \tag{24}$$

If strong rotation is also present, as in a rotating buoyant flow, Eqs. (17), (18), and (22) imply

$$\nu = (C_\nu^\Omega)^2 \frac{S^2\Delta^2}{\Omega}(1 - Ri_t) \tag{25}$$

Note the difference between the Richardson number dependence in Eq. (24), corresponding to no rotation, and in Eq. (25), corresponding to strong rotation. Eq. (25) predicts that stabilization by Richardson number effects is enhanced by rotation, but that destabilizing Richardson number effects are also enhanced by rotation. The second prediction should be tested in numerical simulations.

Centrifugal instability is another external agency which has often been treated (Bradshaw, 1969) by Richardson number modifications of turbulence models. A representative proposal (Launder *et al*, 1977), applicable to flows in which the mean velocity field has circular streamlines

$$u_r = u_z = 0, u_\theta = U(r)$$

is

$$Ri_t = C_R(\frac{K}{\varepsilon})^2 \frac{U}{r^2}\frac{\partial}{\partial r}(rU) \tag{26}$$

where r denotes distance to the rotation axis and $\Omega + U(r)/r$ is the total mean angular velocity. In Eq. (26), the velocity U should be understood as the velocity relative to the rotation axis: this identification maintains the Galilean invariance of the theory. For a subgrid model, it is appropriate to take the turbulent frequency scale proportional to the resolved strain rate. With this modification, the turbulent Richardson number of Eq. (26) becomes

$$Ri_t = C'_R S^{-2}\frac{U}{r^2}\frac{\partial}{\partial r}(rU) \tag{27}$$

Any part of the mean velocity which corresponds to rigid rotation must be included in Ω; equivalently, we assume that $\partial^2 U/\partial r^2$ is nonzero.

Using the formulas for mean vorticity

$$\bar{W} = \frac{\partial U}{\partial r} + \frac{U}{r} = \frac{1}{r}\frac{\partial}{\partial r}(rU)$$

and mean strain rate

$$\bar{S} = \frac{\partial U}{\partial r} - \frac{U}{r} = r\frac{\partial}{\partial r}(\frac{U}{r})$$

we can replace Eq. (27) by the explicitly Galilean-invariant expression

$$Ri_t = \frac{1}{2}C'_R S^{-2}\bar{W}(\bar{W} - \bar{S}) \tag{28}$$

The definition of Ri_t in Eq. (28) can be substituted in Eq. (25) to provide a subgrid model suitable for rotating channel flow or for the problem of rotating Kolmogorov flow recently simulated by Shimomura (1995).

We emphasize that in this treatment of external agencies, Richardson number effects are always *production*-related; they never modify the dissipation rate. Moreoever, stabilization by rotation always corresponds to a *decrease* in the energy transfer by turbulence, hence to a *decrease* in dissipation rate. A different viewpoint is sometimes advanced in engineering modeling: see for example, the survey of models of this type by Rodi and Scheuerer (1983).

4. The Dissipation Rate Transport Equation. Our earlier work (Rubinstein and Zhou, 1996) attempts to implement Leslie's (1972) suggestion that the dissipation rate transport equation be derived from the expression Eq. (10) for energy transfer into the inertial range. A complete treatment would require TSDIA in order to evaluate the essentially inhomogeneous diffusion effects. The present account will be limited to the production and destruction terms, which are amenable to a homogeneous theory.

The starting point is then the DIA equations Eqs. (2)-(3) in which the strain rate term

$$\mathcal{S}_{im}(t) = \frac{\partial U_i}{\partial x_m} - 2k^{-2}k_i k_p \frac{\partial U_p}{\partial x_m} + \delta_{im} k_s \frac{\partial U_s}{\partial x_r}\frac{\partial}{\partial k_r}$$

is added to the rotation term. Differentiating the time dependent form of Eq. (10) with respect to time, and assuming stationarity of the lowest order TSDIA field,

$$\begin{aligned}
&\dot{\varepsilon} = [I^+ - I^-]P_{imn}(\mathbf{k})\{P_{mrs}(\mathbf{p})Q_{ns}(\mathbf{q})Q_{ir}(\mathbf{k})\\
&- P_{irs}(\mathbf{k})Q_{ns}(\mathbf{p})Q_{mr}(\mathbf{q}) + \int_0^t ds[\\
&\quad P_{\mu rs}(\mathbf{p})\dot{G}_{m\mu}(\mathbf{p},t,s)Q_{ns}(\mathbf{q},t,s)Q_{ir}(\mathbf{k},t,s)\\
&- P_{jrs}(\mathbf{k})\dot{G}_{ij}(\mathbf{k},t,s)Q_{ns}(\mathbf{p},t,s)Q_{mr}(\mathbf{q},t,s)\\
&+ P_{\mu rs}(\mathbf{p})G_{m\mu}(\mathbf{p},t,s)\dot{Q}_{ns}(\mathbf{q},t,s)Q_{ir}(\mathbf{k},t,s)\\
&- P_{jrs}(\mathbf{k})G_{ij}(\mathbf{k},t,s)\dot{Q}_{ns}(\mathbf{p},t,s)Q_{mr}(\mathbf{q},t,s)\\
&+ P_{\mu rs}(\mathbf{p})G_{m\mu}(\mathbf{p},t,s)Q_{ns}(\mathbf{q},t,s)\dot{Q}_{ir}(\mathbf{k},t,s)\\
&- P_{jrs}(\mathbf{k})G_{ij}(\mathbf{k},t,s)Q_{ns}(\mathbf{p},t,s)\dot{Q}_{mr}(\mathbf{q},t,s)]\}
\end{aligned} \tag{29}$$

The time-dependent form of this equation might be of interest in the context of non-equilibrium turbulence modeling. Note that the first two terms, distinguished by the absence of any time integration, arise from a quasi-normal hypothesis. The remaining terms are corrections due to DIA.

We follow the program outlined before: to find the dissipation rate transport equation in weakly rotating turbulence, we will substitute Kolmogorov scaling forms for the descriptors G and Q; to find this transport equation in strongly rotating turbulence, we will substitute descriptors appropriate to strong rotation.

4.1. The destruction term. The destruction terms D_ε are those which are independent of strain. Consider first the destruction term in non-rotating turbulence. The quasi-normal terms can be shown (Rubinstein and Zhou, 1996) to contribute

$$D_\varepsilon = -C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{30}$$

when expressed in terms of single-point quantities. The most important conclusion is that the integral which leads to Eq. (30) is convergent in the large k limit. This implies that there is no Reynolds number dependence in the destruction term. The remaining terms in Eq. (29) lead to the same result. The third term, for example, has the form

$$[I^+ - I^-]\eta(p)\Theta(k,p,q)P(\mathbf{k})P(\mathbf{p})Q(q)Q(k)$$

Since the combination $\eta\Theta$ is homogeneous of degree zero, the integral remains convergent, and the form Eq. (30) again follows.

To evaluate the destruction term in strongly rotating turbulence, we observe first that the terms of lowest order in Ω are the quasi-normal terms which we have already analyzed (Rubinstein and Zhou, 1996).

Substituting the energy spectrum with the scaling of Eq. (15) in these terms, there results

$$D_\varepsilon = -C^{\Omega}_{\varepsilon 2}\varepsilon \mid \Omega \mid \tag{31}$$

The constant $C^{\Omega}_{\varepsilon 2}$ is expressed as a convergent integral. The result of Eq. (31) agrees in the strong rotation limit with the rotation correction proposed by Bardina *et al* (1985). Dimensional analysis obviously cannot predict this limit, and other limits for D_ε have been proposed. The result Eq. (31) is consistent with Aupoix's (1987) finding that the stability of rotating decaying turbulence requires

$$D_\varepsilon \sim - \mid \Omega \mid^{\alpha} \quad \text{with} \quad \alpha < 2$$

4.2. The production term. The production terms P_ε depend on the mean velocity gradient. Beginning again with turbulence without rotation, we find at once that there can be no production of dissipation rate without weak breaking of the isotropy of small scales (Xu and Speziale, 1996). This weak anisotropy is introduced using Leslie's (1972) perturbative DIA theory of shear turbulence. Expanding the single time correlation function in a power series in the strain rate,

$$Q_{ij}(\mathbf{k},t) = Q^{(0)}_{ij}(\mathbf{k},t) + Q^{(1)}_{ij}(\mathbf{k},t) + \cdots$$

where $Q^{(0)}$ is the correlation function of isotropic turbulence and setting $G_{ij} = G^{(0)}_{ij}$ the response function of isotropic turbulence, Leslie found

$$\begin{aligned} Q^{(1)}_{ij}(\mathbf{k},t) = \int_0^t ds\{\ & G^{(0)}(k,t,s)(-\frac{\partial U_i}{\partial x_r} + 2k_i k_p k^{-2}\frac{\partial U_p}{\partial x_r})Q^{(0)}_{rj}(k,t,s) \\ & + G^{(0)}(k,t,s)(-\frac{\partial U_j}{\partial x_r} + 2k_j k_p k^{-2}\frac{\partial U_p}{\partial x_r})Q^{(0)}_{ri}(k,t,s) \\ & + G^{(0)}(k,t,s)k_r\ \frac{\partial U_r}{\partial x_n}\ \frac{\partial}{\partial k_n}\ Q^{(0)}_{ij}(k,t,s) \\ & - Q^{(0)}_{ij}(k,t,s)k_r\ \frac{\partial U_r}{\partial x_n}\ \frac{\partial}{\partial k_n} G^{(0)}(k,t,s)\} \end{aligned} \tag{32}$$

Nonzero contributions to production are possible from terms which contain the combination $\mathcal{S}Q^{(1)}$. These terms will contibute production of dissipation terms proportional to quadratic invariants of the mean velocity gradient $S_{ij}S_{ij}$ and $W_{ij}W_{ij}$ where W_{ij} is the antisymmetric part of the mean velocity gradient. For example, the third term in Eq. (15) contributes

$$\begin{aligned} P_\varepsilon &= [I^+ - I^-]\mathcal{S}\Theta P(\mathbf{k})P(\mathbf{p})Q^{(1)}(\mathbf{q})Q^{(0)}(\mathbf{k}) \\ &= [I^+ - I^-]\mathcal{S}\Theta^2 P(\mathbf{k})P(\mathbf{p})\mathcal{S}Q^{(0)}(\mathbf{q})Q^{(0)}(\mathbf{k}) \end{aligned} \tag{33}$$

where indices have been dropped. The second power of Θ in Eq. (33) arises from the time integration in Eq. (32) required to express $Q^{(1)}$ in terms of $Q^{(0)}$. The contribution to P_ε of Eq. (33) differs from a contribution from the quasi-normal terms only in the factors of Θ. Since these factors scale like $k^{-2/3}$, they do not introduce any high wavenumber divergence, and we find that in terms of single-point quantities, they contribute

$$P_\varepsilon \sim KS^2 \sim \frac{\varepsilon}{K}\nu \nabla U^2 \tag{34}$$

where ∇U^2 denotes terms quadratic in the mean velocity gradient. These terms must be proportional to the invariants $S_{ij}S_{ij}$ or to $W_{ij}W_{ij}$

The occurrence of terms proportional to $W_{ij}W_{ij}$ in Eq. (34) would lend theoretical support to the procedure of 'sensitizing to irrotational strains' introduced by Hanjalić and Launder (1980). Since such a term cannot appear in energy production P_K, we conclude that P_K and P_ε may not be related by the proportionality

$$P_\varepsilon = C_{\varepsilon 1}\frac{\varepsilon}{K}P_K \tag{35}$$

Further evidence against the proportionality of Eq. (35) occurs if the series for P_ε is taken to higher order in ∇U by substituting the higher order terms in Leslie's expansion Eq. (32) in Eq. (29). The result,

$$P_\varepsilon = \nu\frac{\varepsilon}{K}\{O(\nabla U)^2 + \frac{K}{\varepsilon}O(\nabla U)^3 + \cdots\} \tag{36}$$

can be compared term by term with the result of substituting Yoshizawa's (1984) expansion of the Reynolds stress in the definition of energy production,

$$P_K = \nu\{O(\nabla U)^2 + \frac{K}{\varepsilon}O(\nabla U)^3 + \cdots\} \tag{37}$$

There is no reason to anticipate term-by-term equality of the series in braces in Eqs. (36) and (37).

Evaluating Eq. (33) using the descriptors Eq. (15) of strongly rotating turbulence leads to the scaling

$$P_\varepsilon \sim \Omega^{-1}\varepsilon\nabla U^2 \sim \frac{\varepsilon}{K}\nu(\Omega)\nabla U^2 \tag{38}$$

where $\nu(\Omega)$ denotes the rotation-dependent viscosity with the strong rotation limit Eq. (18).

5. Interpolation Formulas. In principle, the series expansions of ν, D_ε and P_ε, in positive powers of Ω following Shimomura and Yoshizawa (1986) for weak rotation, and in negative powers of Ω for strong rotation, can be continued to arbitrary order. The problem arises to interpolate rationally between these limits to obtain a model valid at intermediate rotation rates. Thus, for the turbulent viscosity, we could propose a series of approximations of Padé type for eddy viscosity

$$\nu = C_\nu\frac{K^2}{\varepsilon}\Big\{\frac{1}{1 + C_1(\Omega K/\varepsilon)^2}\Big\}^{1/2} \tag{39}$$

$$\nu = C_\nu\frac{K^2}{\varepsilon}\Big\{\frac{1 + D_1(\Omega K/\varepsilon)^2}{1 + C_1(\Omega K/\varepsilon)^2 + C_2(\Omega K/\varepsilon)^4}\Big\}^{1/2} \tag{40}$$

which reduce to the usual eddy viscosity formula with $O(\Omega)$ corrections for low rotation rates, and to the limiting form given by Eq. (18) for strong rotation. The constants C_i and D_i could be determined in principle by matching to the high and low rotation rate expansions constructed above. The corresponding interpolation formulas for D_ε follow by analogy.

Interpolation formulas can also be proposed for subgrid scale viscosity including additional production or stabilization mechanisms modeled as Richardson number effects. For example, for rotating turbulence with added centrifugal stabilization or destabilization, we could set

$$\nu = C_S\Delta^2 S^2\sqrt{1 - Ri_t}\big\{S^2 + \Omega^2/(1 - Ri_t)\big\}^{-1/2} \tag{41}$$

where, repeating the definition given earlier in Eq. (27),

$$Ri_t = C_R' S^{-2}\frac{U}{r^2}\frac{\partial}{\partial r}(rU) = \frac{1}{2}C_R' S^{-2}\bar{W}(\bar{W} - \bar{S})$$

This formula generalizes the subgrid scale model for strong rotation, Eq. (25), to arbitrary rotation rates.

Younis (1997) has observed that the lowest order interpolation, Eq. (39), when used with analogs for D_ε and P_ε does not prove satisfactory in computations. At moderate rotation rates, the reduction in ε brought about by the modified dissipation rate transport equation causes the leading factor K^2/ε in ν to increase, unless the constant C_1 is made sufficiently large. Only when Ω is asymptotically large does the factor in braces become small enough to reduce the turbulent viscosity.

This defect may be due to the property of Eq. (39) that the strong rotation limit fixes the constant C_1, which then also determines the rotation correction in the weak rotation limit. It appears that these limits are not consistent. The more complex model of Eq. (40) may be more satisfactory, but further investigation is essential in order to at least suggest the size of the constants.

This observation recalls a fact noted earlier: stabilization of turbulence is sometimes treated empirically as an *increase* in the dissipation rate ε, because this increase will reduce the (unmodified) turbulent viscosity K^2/ε. From our viewpoint, the stabilization of turbulence by rotation means that energy transfer is blocked; therefore, the dissipation rate ε must *decrease*. But this decrease does not incorrectly enhance the eddy viscosity, because the eddy viscosity is also modified by rotation according to Eq. (18).

6. Conclusions. The extension of the direct interaction approximation to rotating turbulence following the suggestions of Zhou (1995) makes possible a rational derivation of a dissipation rate transport equation and subgrid scale models for strongly rotating turbulence. Empirical turbulence modeling is unable to produce these results because rotation precludes simple dimensional arguments from determining the correct form of the modelled equations.

REFERENCES

[1] Aupoix, B., 1987, *Applications de modèles dans l'éspace spectral,* Thesis, l'Université Claude Bernard, Lyon.

[2] Bardina, J., Ferziger, J. H., and Rogallo, R. S., 1985, *Effect of rotation of isotropic turbulence: computation and modeling,* J. Fluid Mech., Vol. 154, p. 321.

[3] Bradshaw, P., 1969, *The analogy between streamline curvature and buoyancy in turbulent shear flows,* J. Fluid Mech. Vol. 36, p. 177.

[4] Cambon, C., and Jacquin, L., 1989, *Spectral approach to non-isotropic turbulence subjected to rotation,* J. Fluid Mech. Vol. 202, p. 295.

[5] Cambon, C., Jacquin, L., and Lubrano, J. L., 1992, *Toward a new Reynolds stress model for rotating turbulent flows,* Phys. Fluids A, Vol. 4, p. 812.

[6] Dannevik, W. P., 1986, *Efficient solution of non-Markovian covariance evolution equations in fluid turbulence,* J. Sci. Comput Vol. 1, p. 151.

[7] Hanjalić, K. and Launder, B. E., 1980, *Sensitizing the dissipation rate equation to irrotational strains, J. Fluids Eng.* Vol. 102, p. 34.

[8] Launder, B. E., Priddin, C. H., and Sharma, B. I., 1977, J. Fluids Eng. Vol. 99, p. 237.

[9] Leslie, D. C., 1972, *Modern Developments in the Theory of Turbulence*, Oxford University Press.

[10] Kaneda, Y., 1981, *Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function,* J. Fluid Mech. Vol. 107, p. 131.

[11] Kraichnan, R. H., 1959, *The structure of isotropic turbulence at very high Reynolds number,* J. Fluid Mech. Vol. 5, p. 497.

[12] Kraichnan, R. H., 1965, *Lagrangian-history closure approximation for turbulence,* Phys. Fluids, Vol. 8, p. 575.

[13] Kraichnan, R. H., 1971, *Inertial range transfer in two and three dimensional turbulence,* J. Fluid Mech. Vol. 47, p. 525.

[14] Rodi, W. and Scheuerer, G., 1983, *Calculation of curved shear layers with two-equation turbulence models,* Phys. Fluids, Vol. 26, p.1422.

[15] Rubinstein, R. and Zhou, Y., 1996, *Analytical theory of the destruction terms in dissipation rate transport equations,* Phys. Fluids, Vol. 8, p. 3172.

[16] Shimomura, Y., and Yoshizawa, A., 1986 *Statistical analysis of anisotropic turbulent viscosity in a rotating system,* J. Phys. Soc. Japan Vol. 55, p. 1904.

[17] Shimomura, Y., 1995, *Performance of the subgrid-scale algebraic stress model in large eddy simulation,* International Symposium on Mathematical Modeling of Turbulent Flows, Tokyo, Proceedings, p. 321.

[18] Waleffe, F., 1993, *Inertial transfers in the helical decomposition,* Phys. Fluids Vol. 5, p. 677.

[19] Woodruff, S. L., 1992, *Dyson equation analysis of inertial-range turbulence,* Phys. Fluids A, Vol. 5, p. 1077.

[20] Xu, X. H. and Speziale, C. G., 1996, *Explicit algebraic stress model of turbulence with anisotropic dissipation,* AIAA J. Vol. 34, p. 2186.

[21] Yoshizawa, A., 1984, *Statistical analysis of the deviation of the Reynolds stress from its eddy viscosity representation,* Phys. Fluids Vol. 27, p. 3177.

[22] Yoshizawa, A. and Yokoi, N., 1995, *Statistical analysis of the effects of helicity in inhomogeneous turbulence, Phys. Fluids* Vol. 7, p. 12.

[23] Yoshizawa, A., 1996, *Simplified statistical approach to complex turbulent flows and ensemble-mean compressible turbulence modeling,* Phys. Fluids, submitted.

[24] Younis, B. A., 1996, private communication.

[25] Zakharov, V. E., L'vov, V. S., and Falkovich, G., 1992, *Kolmogorov Spectra of Turbulence* I, Springer.

[26] Zhou, Y., 1995, *A phenomenological treatment of rotating turbulence,* Phys. Fluids Vol. 7, p. 2092.