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A Note on Kinetic Energy, Dissipation and Enstrophy

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A NOTE ON KINETIC ENERGY, DISSIPATION AND ENSTROPHY[*]

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Abstract. The dissipation rate of a Newtonian fluid with constant shear viscosity can be shown to include three constituents: dilatation, vorticity, and surface strain. The last one is found to make no contributions to the change of kinetic energy. These dissipation constituents are used to identify typical compact turbulent flow structures at high Reynolds numbers. The incompressible version of the simplified kinetic-energy equation is then cast to a novel form, which is free from the work rate done by surface stresses but in which the full dissipation reenters.

Key words. kinetic energy, dissipation, enstrophy, surface strain

Subject classification. Fluid Mechanics

1. Introduction. We make a general theoretical examination on the relation between kinetic energy transport, dissipation, and enstrophy. First, we derive an exact but simplified transport equation for the kinetic energy of viscous compressible flow, in which the full dissipation function is replaced by the squares of vorticity and dilatation. This forms a local counterpart of the classic Bobyleff-Forsyth formula [1] on the integral equivalence of incompressible dissipation and enstrophy under special boundary condition, and provides a physical clarification on dynamic processes really involved in the evolution of kinetic energy. The dissipation constituents are used to identify typical compact flow structures. Second, we cast the incompressible version of the simplified kinetic-energy equation to a novel form, of which the local spatical average does not explicitly depend on boundary conditions, but the full dissipation reappears along with enstrophy. This result is of relevance to some current studies of intermittency and scaling laws in inhomogeneous and anisotropic turbulence.

2. Dissipation constituents. Let $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ and $\vartheta = \nabla \cdot \boldsymbol{u}$ be the vorticity and dilatation, $\mathbf{S}$ and $\mathbf{T}$ be the strain-rate and stress tensor, respectively, so that for Newtonian fluid $\mathbf{T} = (-p + \lambda \vartheta)\mathbf{I} + 2\mu\mathbf{S}$ (e.g. [1,2]), where $\mathbf{I}$ is the unit tensor, and μ and λ are the shear and second viscosity[1]. Throughout this note we assume μ is constant. The conventional transport equation for kinetic energy $E = |\boldsymbol{u}|^2/2$ reads

$$(1) \qquad \rho \frac{DE}{Dt} = \rho \boldsymbol{f} \cdot \boldsymbol{u} + \vartheta p + \nabla \cdot (\mathbf{T} \cdot \boldsymbol{u}) - \Phi,$$

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[1]One may also replace $\mathbf{S}$ by its derivatoric part $\mathbf{S}' = \mathbf{S} - \frac{1}{3}\vartheta\mathbf{I}$ so that $\mathbf{T} = (-p + \zeta\vartheta)\mathbf{I} + 2\mu\mathbf{S}'$, where $\zeta = \lambda + \frac{2}{3}\mu$ is the bulk viscosity coefficient. This does not affect our argument but is less convenient.

where $\boldsymbol{f}$ is a body force and $\Phi = \lambda\vartheta^2 + 2\mu\mathbf{S} : \mathbf{S}$ is the dissipation rate. It is straightforward to derive an invariant decomposition $\Phi = \widehat{\Phi} + \Phi_B$, where

$$\widehat{\Phi} = (\lambda + 2\mu)\vartheta^2 + \mu|\boldsymbol{\omega}|^2,$$

(2) $$\Phi_B = -\nabla \cdot (2\mu\mathbf{B} \cdot \boldsymbol{u}),$$

are, respectively, a positive-definite part caused by ϑ^2 and $|\boldsymbol{\omega}|^2$, and a non positive-definite "dissipation" caused by surface-strain rate tensor[2] [3]

(3) $$\mathbf{B} \equiv \vartheta\mathbf{I} - \nabla\boldsymbol{u}^T \quad \text{with} \quad \nabla \cdot \mathbf{B} = 0.$$

From (2) the compressible extension of the Bobyleff-Forsyth formula follows:

(4) $$\int_V \Phi dV = \int_V [(\lambda + 2\mu)\vartheta^2 + \mu|\boldsymbol{\omega}|^2]dV - 2\mu \int_{\partial V} \boldsymbol{n} \cdot \mathbf{B} \cdot \boldsymbol{u}dS.$$

Thus, if $\boldsymbol{n} \cdot (\mathbf{B} \cdot \boldsymbol{u}) = 0$ on ∂V, we have an *integral* equivalence of Φ and $\widehat{\Phi}$.

A similar decomposition of $\mathbf{T}$ has been introduced by Wu and Wu [4-5]:

(5) $$\mathbf{T} = \widehat{\mathbf{T}} + \mathbf{T}_B, \quad \widehat{\mathbf{T}} \equiv -\Pi\mathbf{I} + 2\mu\boldsymbol{\Omega}, \quad \mathbf{T}_B \equiv -2\mu\mathbf{B},$$

where $\Pi \equiv p - (\lambda + 2\mu)\vartheta$ is the isotropic part of $\mathbf{T}$ and $\boldsymbol{\Omega}$ is the vorticity tensor. Substituting (5) into the Cauchy motion equation reveals that $\mathbf{T}_B$ plays no role in momentum balance. Now, substituting (5) into $\nabla \cdot (\mathbf{T} \cdot \boldsymbol{u})$ in (1) leads to a splitting of the work rate W per unit volume done by surface stress:

$$W = \nabla \cdot (\mathbf{T} \cdot \boldsymbol{u}) = W_p + W_\vartheta + W_\omega + W_B,$$

where

$$W_p = -\nabla \cdot (p\boldsymbol{u}), \quad W_\vartheta = \nabla \cdot [(\lambda + 2\mu)\vartheta\boldsymbol{u}],$$

$$W_\omega = -\nabla \cdot (\mu\boldsymbol{\omega} \times \boldsymbol{u}), \quad W_B = -\nabla \cdot (2\mu\mathbf{B} \cdot \boldsymbol{u}).$$

Thus, W_B and Φ_B are always self-balanced, making no contribution to the change of E. Hence, Eq. (1) can be simplified to

(6) $$\rho\frac{DE}{Dt} = \rho\boldsymbol{f} \cdot \boldsymbol{u} + \vartheta p + \nabla \cdot (\widehat{\mathbf{T}} \cdot \boldsymbol{u}) - \widehat{\Phi},$$

which proves a *local* "equivalence" of Φ and $\widehat{\Phi}$ under the condition that $\mathbf{T}$ be replaced by $\widehat{\mathbf{T}}$ simultaneously. Whenever $\mathbf{T}_B$ does a work (positive or negative), it is directly and locally dissipated to Φ_B which, as is easily proved, does cause a change of the internal energy. Therefore, so far as its *mechanical aspects* are concerned, a Newtonian fluid can well be simplified to a hypothetic medium of which the motion consists of only an isotropic expansion/compression and a spin. A significant saving of this simplification in numerical computation was cited in [5]. The surface deformation, which is generically most complicated, plays no role except on free surface [4].

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[2]In this report and Dishington's paper there is a sign difference in the definition of $\mathbf{B}$.

In the above analysis a key issue is the distinction between $\widehat{\Phi}$ and Φ_B or the relation between Φ and $\widehat{\Phi}$. Studying their distribution characters allows us to identify different coherent and compact structures in high Reynolds-number flows. Typically, these include two shearing structures: thin vortex layers and tubes, and a compressing structure, shock waves. While across a shock there is evidently an extremely strong peak of ϑ^2, the distribution character of ω^2 and Φ_B for shearing structures can be easily illustrated by idealized models. For a unidirectional shear flow with $\boldsymbol{u} = (U(y,t),0,0)$ in Cartesian coordinates (x,y,z), there is

$$\widehat{\Phi} = \Phi = \mu\omega^2 = \left(\frac{\partial U}{\partial y}\right)^2, \quad \Phi_B = 0;$$

while for a pure vortex with $\boldsymbol{u} = (0, v(r,t), 0)$ and $\boldsymbol{\omega} = (0,0,\omega(r,t))$ in cylindrical coordinates (r,θ,z), there is

$$\widehat{\Phi} = \mu\omega^2, \quad \Phi_B = W_B = -4\mu\frac{v}{r}\frac{\partial v}{\partial r}, \quad \Phi = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2.$$

Note that inside the vortex core $(r < r_0)$ with $\partial v/\partial r > 0$ there is a negative peak of Φ_B which cancels the peak of $\widehat{\Phi}$ ($\widehat{\Phi} = \widehat{\Phi}_{max}$ at $r = 0$), so that Φ remains small as it should ($\Phi = 0$ in a solid-like core). But outside the core Φ_B reaches a hollow-tube like (macaroni-like) positive peak Φ_{Bmax} at an $r = r_m > r_0$, which becomes the main contributor to Φ. For $r > r_m$ the contribution of enstrophy to Φ is neglegible. The peak behaviors of $\widehat{\Phi}$ and Φ_B for layers and tubes can be easily checked by considering, say, $U(y) = \tanh(y/\delta)$ with $\delta \ll 1$ and an Oseen vortex with $\mu \ll 1$, respectively.

When $Re \gg 1$, these idealized models can serve as local building blocks of more generic layer and tube structures, provided that the strain components along the layers or tube axes are weaker than the vorticity therein. In particular, in a compressible turbulence the strong peaks of ϑ^2, ω^2, and the simultaneous peaks of ω^2 and Φ are good indicators of random shocklets, filaments, and shearing sheets, respectively. Indeed, in a numerical simulation of two-dimensional supersonic turbulence, Porter et al. [6] have observed complex sheet-like shocklet structures by visualizing dilatation field. From their direct numerical simulations of a uniformly sheared incompressible turbulence, Tanaka and Kida [7] used the ratio $\widehat{\Phi}/\Phi$ to identify many thin vortex filaments with $\widehat{\Phi}/\Phi > 2$ (as shown above, these high-enstrophy structures must be surrounded by weaker hollow-tube peaks of strain), while at the high enstrophy region with comparable strain $1/2 < \widehat{\Phi}/\Phi < 4/3$, the vortex sheet structures were observed. These results are confirmed by Boratav and Pelz [8], who conducted direct numerical simulations of an unforced flow and investigated the strain-vorticity correlation in a plane spanned by Φ/μ and enstrophy $\widehat{\Phi}/\mu$. Now, our result provides a theoretical support for these identifications. Interestingly, for the Oseen vortex at $t = 1$ we found $\Phi_{Bmax}/\widehat{\Phi}_{max} = 0.09$, completely independent of the Reynolds number. The ratio r_m/r_0, which is about 1.2, has only a very weak dependence on Re. Thus, as $Re \to \infty$ and $r_0 \to 0$, the distribution of $\widehat{\Phi}$ and Φ_B (or the full Φ) for a filament becomes a singular "spaghetti" enclosed by a singular "macaroni", having the *same* fractal dimension (approaching 1).

3. Kinetic-energy equation. We now further focus on incompressible flow with $\rho = 1$, which is a typical situation where the thermodynamics is not involved. In this case (6) reduces to

$$(7) \qquad \frac{DE}{Dt} = \boldsymbol{f} \cdot \boldsymbol{u} - \nabla \cdot (p\boldsymbol{u}) - \nu\nabla \cdot (\boldsymbol{\omega} \times \boldsymbol{u}) - \widehat{\Phi}.$$

In the studies of turbulent-energy budget one removes the divergence terms in (7) by taking average over a region with periodic boundary conditions. Then for statistically steady turbulence and as $\nu \to 0$, one obtains a balance solely between the work of stiring force $\boldsymbol{f}$ and energy cascade in integral range, and a balance solely

between the energy cascade and dissipation (enstrophy) in inertial range (e.g. Frisch [9]). This procedure is necessary to obtain the Kolmogorov scaling law and its modern revisions. However, for the local scaling of structure functions in an inhomogeneous and anisotropic turbulence, the boundary condition on a small sphere (in which the average is to be taken) must be kept generic. Therefore, it will be highly interesting if a general incompressible energy equation can be cast to a form free from work-rate terms, so that the averaging process does not explicitly rely on specific boundary conditions. We now consider this issue.

The advection of E and pressure work in (7) can be combined to $\nabla \cdot (\boldsymbol{u} h_0)$, where h_0 is the stagnation enthalpy. To remove this term we split the momentum equation into two parts: a transverse or divergence-free part (including harmonic components), and a longitudinal or curl-free part. Namely, we make a decomposition of $\boldsymbol{f}$ and the Lamb vector $\boldsymbol{l} = \boldsymbol{\omega} \times \boldsymbol{u}$, denoted by $\boldsymbol{f} = \boldsymbol{f}_\perp + \boldsymbol{f}_\parallel$ and $\boldsymbol{l} = \boldsymbol{l}_\perp + \boldsymbol{l}_\parallel$. It then follows that

$$(8) \qquad \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{f}_\perp - \boldsymbol{l}_\perp - \nu \nabla \times \boldsymbol{\omega}, \quad \nabla h_0 = -\boldsymbol{l}_\parallel,$$

where $\boldsymbol{l}_\perp$ is the projection of advection $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ onto the solenoidal space. From (8a) there is

$$(9) \qquad \frac{\partial E}{\partial t} = (\boldsymbol{f}_\perp - \boldsymbol{l}_\perp) \cdot \boldsymbol{u} - \nu \nabla \cdot \boldsymbol{l} - \widehat{\Phi}.$$

Compared to (7), the pressure work is absent, and advection retains a "residue" $\boldsymbol{l}_\perp \cdot \boldsymbol{u}$, which is implicitly boundary-dependent since the projection is a global operator. Then, since

$$\widehat{\Phi} = \nu |\boldsymbol{\omega}|^2, \quad \Phi_B = 2\nu \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = -2\nu \nabla^2 p,$$

there is

$$-\nu \nabla \cdot \boldsymbol{l} = \nu \nabla^2 h_0 = \nu \nabla^2 E - \frac{1}{2}(\Phi - \widehat{\Phi}).$$

Therefore, from (9) a novel diffusion equation for the kinetic energy follows:

$$(10) \qquad \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) E = (\boldsymbol{f}_\perp - \boldsymbol{l}_\perp) \cdot \boldsymbol{u} - \frac{1}{2}(\Phi + \widehat{\Phi}),$$

in which dissipation and enstrophy appear simultaneously and symmetrically, but all boundary-dependent work-rate terms disappear. Note that the reappearance of Φ in (10) does not conflict with (7); it comes from a part of the vorticity work (the other part being the diffusion of E). Moreover, compared to the usual form of the energy cascade [9], we now see that the cascade is neatly represented by the statistic average of $\boldsymbol{l}_\perp \cdot \boldsymbol{u}$, which is dominated by the transverse Lamb vector and has a compact support where $\boldsymbol{l}_\perp \neq 0$.

4. Conclusion. It has been noticed that the relative contributions to the dissipation from strain-dominated and enstrophy-dominated structures may be relevant to the strength of intermittency, as well as to the scalings in longitudinal and transversal structure functions [10]. Now, by placing the enstrophy and dissipation in equal footing and exhibiting their interplay, Eq. (10) clearly emphasizes that, if a more boundary-independent (and hence more universal) theory is to be reached, both strain rate (via dissipation) and vorticity (via enstrophy) are important in the energy transfer process. On the other hand, if (7) is used, one need reconsider the respective roles of full dissipation and enstrophy. In either case a more careful treatment of the effect of boundary condition on the average over a small sphere in inhomogeneous and anisotropic turbulence is necessary.

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