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GAS-KINETIC THEORY BASED FLUX SPLITTING METHOD FOR IDEAL MAGNETOHYDRODYNAMICS

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Abstract. A gas-kinetic solver is developed for the ideal magnetohydrodynamics (MHD) equations. The new scheme is based on the direct splitting of the flux function of the MHD equations with the inclusion of “particle” collisions in the transport process. Consequently, the artificial dissipation in the new scheme is much reduced in comparison with the MHD Flux Vector Splitting Scheme. At the same time, the new scheme is compared with the well-developed Roe-type MHD solver. It is concluded that the kinetic MHD scheme is more robust and efficient than the Roe-type method, and the accuracy is competitive. In this paper the general principle of splitting the macroscopic flux function based on the gas-kinetic theory is presented. The flux construction strategy may shed some light on the possible modification of AUSM- and CUSP- type schemes for the compressible Euler equations, as well as to the development of new schemes for a non-strictly hyperbolic system.

Key words. magnetohydrodynamics, flux splitting, gas-kinetic scheme

Subject classification. Applied Numerical Mathematics

1. Introduction. The development of numerical methods for the MHD equations has attracted much attention in the past years. Godunov-type schemes are considered particularly useful here. On the basis of Roe’s method [18], Brio and Wu developed the first Flux Difference Splitting (FDS) scheme for MHD equations [3]. Aslan also followed the idea of fluctuation approach to construct a second-order upwind MHD solver [1]. Zachary et al. applied an operator splitting technique and devised a high-order Godunov type method [28]. During the same period, the multidimensional extension of MHD solvers was done by Ryu et al. [19] and Tanaka [22]. On the basis of the nonlinear Riemann solver, Dai and Woodward extended the PPM method [5]. Powell et al. constructed an eight-wave family eigensystem for the approximate Riemann solver [15, 16]. Most recently, based on the Lax-Friedrich flux splitting technique, Jiang and Wu applied a high-order WENO interpolation scheme to the MHD equations [9]. In order to increase the robustness and simplify the complicated Roe-type MHD solver, based on the HLL method, Linde developed an adequate Riemann solver for the heliosphere applications [10]. A majority of the methods mentioned above applied characteristic decomposition for the MHD waves, where the entropy, slow, Alfvén and fast waves have to be considered in the evaluation of a single flux function. Because of the wave decomposition procedure, considerable work is required to evaluate and justify the MHD eigensystem, where the non-strictly hyperbolicity causes additional difficulty. Due to the same reason, the issue related to the direct extension of the Flux Vector Splitting (FVS) scheme to the MHD equations was hardly addressed. The search for robust, accurate and efficient MHD flow solvers is still one of the primary directions in MHD research.

For the Euler and Navier-Stokes equations, the development of gas-kinetic schemes has also attracted attention [25]. A particular strength of kinetic schemes lies precisely where Godunov-type FDS schemes

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often fail, such as carbuncle phenomena, positivity, entropy condition [26]. However, like any other FVS method, the Kinetic Flux Vector Splitting (KFVS) scheme is very diffusive and less accurate in comparison with the Roe-type Riemann solver, especially for shear and contact waves. The diffusivity of the FVS schemes, such as Steger-Warming, van Leer and the KFVS [21, 24, 17], is mainly due to the particle or wave-free transport mechanism, which sets the CFL time step equal to particle collision time. Consequently, the artificial viscosity coefficient is always proportional to the time step. Even though numerically the high-order FVS methods can get crisp shock resolution by using a MUSCL-type reconstruction method, physically it is impossible to develop a second-order FVS scheme without correcting the free transport mechanism. In order to reduce the diffusivity, particle collisions have to be modeled and implemented in the gas evolution stage, such as that in the BGK scheme [27].

The construction of the gas-kinetic FVS scheme for the MHD equations started from Croisille et al. [4], where a MHD KFVS solver was obtained by simply extending the KFVS flux function of the Euler equations. The above MHD KFVS scheme is very robust and reliable, but over-diffusive, especially in the contact discontinuity regions [10]. Recently, another interesting gas-kinetic MHD solver has been developed by Huba and Lyon [6]. Different from the earlier approach, Huba and Lyon constructed two equilibrium states and a transport equation to recover the MHD equations. An important aspect of this method is that it provides a framework to incorporate additional terms into the MHD equations, e.g. anisotropic ion stress tensor, and anisotropic temperature distribution. However, the physical basis of the transport equation and the reliability of the equilibrium states need to be further investigated. Since Huba and Lyon's flux function keeps the FVS nature, large numerical dissipation is intrinsically rooted.

In this paper, we are going to construct a new kinetic flux splitting method for MHD equations. Based on the BGK-type scheme, the KFVS MHD solver of Croisille et al. is generalized by including particle collisions. As a result, the new scheme reduces the numerical dissipation significantly and gives a more accurate representation of wave interactions. In Section 3, the new scheme is compared well with the Roe-type MHD solver [3, 16]. The flux construction method presented in this paper splits the macroscopic flux function directly, therefore it is very useful in the design of numerical methods for complicated hyperbolic system.

2. Gas-Kinetic Approach to MHD Equations. In the one-dimensional case, the MHD equation

$$q_t + F(q)_x = 0$$

has the following forms [3]

$$(2.1) \quad \left\{ \begin{array}{l} \rho_t + (\rho U)_x = 0, \\ (\rho U)_t + (\rho U^2 + p_* - B_x^2)_x = 0, \\ (\rho V)_t + (\rho UV - B_x B_y)_x = 0, \\ (\rho W)_t + (\rho UW - B_x B_z)_x = 0, \\ (B_y)_t + (B_y U - B_x V)_x = 0, \\ (B_z)_t + (B_z U - B_x W)_x = 0, \\ (\rho \epsilon)_t + ((\rho \epsilon + p_*)U - B_x(B_x U + B_y V + B_z W)) = 0, \end{array} \right.$$

where p_* is the total pressure

$$p_* = p + \frac{1}{2}(B_x^2 + B_y^2 + B_z^2),$$

and p is the gas pressure. The total energy density is

$$\rho \epsilon = \frac{1}{2}\rho(U^2 + V^2 + W^2) + \rho e + \frac{1}{2}(B_x^2 + B_y^2 + B_z^2).$$

For ideal equilibrium gas, the internal energy is related to pressure through the relation

$$\rho e = p/(\gamma - 1).$$

Due to having a different physical origin, it should be emphasized that in order to properly split the energy flux function, the splitting of internal energy flux $\rho e U$ and the splitting of work done by the pressure pU should be different, although they are only different by a constant $1/(\gamma - 1)$ for the ideal gases.

Theoretically, it is very difficult to construct an equilibrium state and a transport equation to exactly recover the above ideal MHD equations. However, instead of constructing the equilibrium distribution for the flow and magnetic field, we can split the MHD flux function directly on the macroscopic level with the consideration of gas-kinetic theory.

2.1. Gas-Kinetic Flux Splitting Method. In the gas-kinetic theory, the flux is associated with the particle motion across cell interface. For 1D flow, such as the x -direction, the particle motion in this dimension is most important in the determination of the flux function. Other quantities, such as y -direction velocity, thermal energy, and magnetic field, can be considered as passive scalars, which are transported with x -direction particle velocity. Normally, particles are randomly distributed around the average velocity. From statistical mechanics, the moving particles in x -direction can be most favorably described by the Maxwellian-Boltzmann distribution function,

$$(2.2) \quad g = \rho \left(\frac{\lambda}{\pi}\right)^{1/2} e^{-\lambda(u-U)^2},$$

where U is the average velocity and λ is the normalization factor of the distribution of random velocity, which is related to the temperature of the gas flow, i.e. $\lambda = -m/2kT$, where m is molecule mass, k is the Boltzmann constant, and T is the temperature.

The transport of any flow quantities is basically due to the movement of particles. With the above equilibrium state g , we can split the particles into two groups. One group is moving to the right with $u > 0$, and another group moving to the left with $u < 0$. Before splitting the fluxes, let's first define the useful moments of the particle distribution function,

$$\langle u^n \rangle_{a-b} = \int_a^b u^n \left(\frac{\lambda}{\pi}\right)^{1/2} e^{-\lambda(u-U)^2} du,$$

where integration limit (a, b) of the particle velocity can be $(-\infty, +\infty)$, $(-\infty, 0)$ or $(0, +\infty)$. There is a recursive relation for the moments $\langle u^n \rangle$, which is

$$\langle u^{n+2} \rangle_{a-b} = U \langle u^{n+1} \rangle_{a-b} + \frac{n+1}{2\lambda} \langle u^n \rangle_{a-b}.$$

For example, we have

$$\langle u^0 \rangle_{u>0} = \frac{1}{2} \operatorname{erfc}(-\sqrt{\lambda}U) \quad ; \quad \langle u^0 \rangle_{u<0} = \frac{1}{2} \operatorname{erfc}(\sqrt{\lambda}U),$$

where erfc is the error function, and

$$\langle u^1 \rangle_{u>0} = U \langle u^0 \rangle_{u>0} + \frac{1}{2} \frac{e^{-\lambda U^2}}{\sqrt{\pi\lambda}} \quad ; \quad \langle u^1 \rangle_{u<0} = U \langle u^0 \rangle_{u<0} - \frac{1}{2} \frac{e^{-\lambda U^2}}{\sqrt{\pi\lambda}}.$$

Obviously, if the integration limit is $(-\infty, \infty)$, the following relations hold

$$\langle u^0 \rangle = 1 \quad , \quad \langle u^1 \rangle = U.$$

Note that we have dropped the subscript if $(a, b) = (-\infty, \infty)$.

Depending on the particle moving directions, the total density ρ can be split into

$$\begin{aligned}\rho^+ &= \int_0^\infty g du \\ &= \rho \langle u^0 \rangle_{u>0}\end{aligned}$$

and

$$\begin{aligned}\rho^- &= \int_{-\infty}^0 g du \\ &= \rho \langle u^0 \rangle_{u<0}.\end{aligned}$$

Any macroscopic quantities Z without containing explicitly the x -component velocity U , such as density ρ , y - and z -direction momentum ρV and ρW , and magnetic field $B_x B_y$, can be split similarly

$$Z^+ = Z \langle u^0 \rangle_{u>0}$$

and

$$Z^- = Z \langle u^0 \rangle_{u<0}.$$

The above relations mean that the quantity Z is simply advected with the x -direction particle velocity.

The x -direction momentum ρU can be split into

$$\begin{aligned}(\rho U)^+ &= \int_0^\infty u g du \\ &= \rho \langle u^1 \rangle_{u>0}\end{aligned}$$

and

$$\begin{aligned}(\rho U)^- &= \int_{-\infty}^0 u g du \\ &= \rho \langle u^1 \rangle_{u<0}.\end{aligned}$$

Similarly, any quantities containing U term, such as $B_x U$, $B_y U$, ρU , $\rho V U$, $\rho W U$, can be split as

$$(ZU)^+ = Z \langle u^1 \rangle_{u>0}$$

and

$$(ZU)^- = Z \langle u^1 \rangle_{u<0}.$$

For magnetic field, the above splitting implies that the field is frozen into the particle motion and transported with the fluid. Note ZU does not include pU , and the splitting of pU will be derived later.

The splitting of energy can be written as

$$\begin{aligned}(\rho \epsilon)^+ &= \int_0^\infty \frac{1}{2} u^2 g du \\ &= \frac{1}{2} \langle u^2 \rangle_{u>0} \\ &= \frac{1}{2} \rho U \langle u^1 \rangle_{u>0} + \frac{\rho}{4\lambda} \langle u^0 \rangle_{u>0} \\ &= \frac{1}{2} \rho U \langle u^1 \rangle_{u>0} + \rho e \langle u^0 \rangle_{u>0},\end{aligned}$$

where ρe is the internal energy of the specific distribution function g in Eq.(2.2) with the value of $\rho/4\lambda$. Similarly, we have

$$\begin{aligned}
(\rho e)^- &= \int_{-\infty}^0 \frac{1}{2} u^2 g du \\
&= \frac{1}{2} \langle u^2 \rangle_{u<0} \\
&= \frac{1}{2} \rho U \langle u^1 \rangle_{u<0} + \frac{\rho}{4\lambda} \langle u^0 \rangle_{u<0} \\
&= \frac{1}{2} \rho U \langle u^1 \rangle_{u<0} + \rho e \langle u^0 \rangle_{u<0}.
\end{aligned}$$

The above equations imply that the kinetic energy $\frac{1}{2}\rho U^2$ can be split as

$$\begin{aligned}
\frac{1}{2}\rho U^2 &= (\frac{1}{2}\rho U^2)^+ + (\frac{1}{2}\rho U^2)^- \\
&= \frac{1}{2}\rho U \langle u^1 \rangle_{u>0} + \frac{1}{2}\rho U \langle u^1 \rangle_{u<0},
\end{aligned}$$

and the internal energy

$$\begin{aligned}
\rho e &= (\rho e)^+ + (\rho e)^- \\
&= \rho e \langle u^0 \rangle_{u>0} + \rho e \langle u^0 \rangle_{u<0}.
\end{aligned}$$

In general, besides the thermal energy, ρe can include other kinds of internal energy, such as magnetic energy in MHD equations. For nonideal gases, the internal energy could have a complicated form as a function of ρ and T . However, the above formulation can still be used to split them in terms of $\langle u^0 \rangle_{u>0}$ and $\langle u^0 \rangle_{u<0}$. Since the pressure p is related to the internal energy, it can be split as

$$p = p \langle u^0 \rangle_{u>0} + p \langle u^0 \rangle_{u<0}.$$

Now let's consider the energy transport $\frac{1}{2} \int u^3 g du$. The energy transport in the positive x -direction is

$$\begin{aligned}
\int_0^\infty \frac{1}{2} u^3 g du &= \frac{1}{2} \langle u^3 \rangle_{u>0} \\
&= (\frac{1}{2}\rho U^2 + \rho e) \langle u^1 \rangle_{u>0} + \frac{1}{2} U p \langle u^0 \rangle_{u>0} + \frac{1}{2} p \langle u^1 \rangle_{u>0} \\
&= \rho e \langle u^1 \rangle_{u>0} + \frac{1}{2} U p \langle u^0 \rangle_{u>0} + \frac{1}{2} p \langle u^1 \rangle_{u>0},
\end{aligned}$$

where $\rho e = \frac{1}{2}\rho U^2 + \rho e$ is the total energy density for the specific distribution g . Similarly, the corresponding flux in the negative x -direction is

$$\begin{aligned}
\int_{-\infty}^0 \frac{1}{2} u^3 g du &= \frac{1}{2} \langle u^3 \rangle_{u<0} \\
&= (\frac{1}{2}\rho U^2 + \rho e) \langle u^1 \rangle_{u<0} + \frac{1}{2} U p \langle u^0 \rangle_{u<0} + \frac{1}{2} p \langle u^1 \rangle_{u<0} \\
&= \rho e \langle u^1 \rangle_{u<0} + \frac{1}{2} U p \langle u^0 \rangle_{u<0} + \frac{1}{2} p \langle u^1 \rangle_{u<0}.
\end{aligned}$$

The total energy flux in x -direction is

$$\begin{aligned}
\int_{-\infty}^\infty \frac{1}{2} u^3 g du &= (\frac{1}{2}\rho U^2 + \rho e)U + pU \\
&= \rho e U + pU.
\end{aligned}$$

From the above three equations, we conclude that the total energy flux $\rho\epsilon U$ can be split as

$$\begin{aligned}\rho\epsilon U &= (\rho\epsilon U)^+ + (\rho\epsilon U)^- \\ &= \rho\epsilon\langle u^1 \rangle_{u>0} + \rho\epsilon\langle u^1 \rangle_{u<0},\end{aligned}$$

which is composed of kinetic energy flux splitting

$$\begin{aligned}\frac{1}{2}\rho U^3 &= \left(\frac{1}{2}\rho U^3\right)^+ + \left(\frac{1}{2}\rho U^3\right)^- \\ &= \frac{1}{2}\rho U^2\langle u^1 \rangle_{u>0} + \frac{1}{2}\rho U^2\langle u^1 \rangle_{u<0}\end{aligned}$$

and the internal energy flux splitting

$$\rho\epsilon U = \rho\epsilon\langle u^1 \rangle_{u>0} + \rho\epsilon\langle u^1 \rangle_{u<0}.$$

At the same time, the splitting of the work done by the pressure pU term is

$$\begin{aligned}pU &= (pU)^+ + (pU)^- \\ &= \frac{1}{2}(Up\langle u^0 \rangle_{u>0} + p\langle u^1 \rangle_{u>0}) + \frac{1}{2}(Up\langle u^0 \rangle_{u<0} + p\langle u^1 \rangle_{u<0}).\end{aligned}$$

Note the above splitting formula can be generalized to a hyperbolic system with complicated total energy density.

As a special application of the above splitting principle, let's split the 1D Euler fluxes. The flux function for 1D Euler equations can be separated into

$$\begin{pmatrix} \rho U \\ \rho U^2 + p \\ \rho\epsilon U + pU \end{pmatrix} = F_f^+ + F_f^-$$

where f means free transport. The positive flux F_f^+ is

$$F_f^+ = \langle u^1 \rangle_{u>0} \begin{pmatrix} \rho \\ \rho U \\ \rho\epsilon \end{pmatrix} + \begin{pmatrix} 0 \\ p\langle u^0 \rangle_{u>0} \\ \frac{1}{2}p\langle u^1 \rangle_{u>0} + \frac{1}{2}pU\langle u^0 \rangle_{u>0} \end{pmatrix},$$

and the negative part F_f^- is

$$F_f^- = \langle u^1 \rangle_{u<0} \begin{pmatrix} \rho \\ \rho U \\ \rho\epsilon \end{pmatrix} + \begin{pmatrix} 0 \\ p\langle u^0 \rangle_{u<0} \\ \frac{1}{2}p\langle u^1 \rangle_{u<0} + \frac{1}{2}pU\langle u^0 \rangle_{u<0} \end{pmatrix}.$$

With the above splitting formula, the numerical flux across a cell interface $j + 1/2$ for the Euler equations can be written as

$$F_{j+1/2}^f = F_{j,f}^+ + F_{j+1,f}^-.$$

This is exactly the Kinetic Flux Vector Splitting Scheme for the Euler equations [17, 13], and the positivity and entropy condition for the above scheme have been analyzed by many authors, such as [14, 23, 12] and references therein.

As analyzed in [25], all FVS schemes based on positive (negative) particle velocities suffer from the same weakness. The particle free transport across cell interfaces unavoidably introduces large numerical dissipation, and the viscosity and heat conduction coefficients are proportional to the CFL time step. In order to reduce the over-diffusivity in FVS Schemes, particle collisions have to be added in the transport process.

As a simple particle collisional model, we can imagine that the particles from the left- and right-hand sides of a cell interface collapse totally to form an equilibrium state. In order to define the equilibrium state at the cell interface, we need first to figure out the corresponding macroscopic quantities $\bar{q}_{j+1/2}$ there, which are the combination of the total mass, momentum and energy of the left and right moving beams. For example, for the Euler equations, we have

$$\begin{aligned}\bar{q}_{j+1/2} &= \begin{pmatrix} \bar{\rho} \\ \bar{\rho}\bar{U} \\ \bar{\rho}\bar{e} \end{pmatrix}_{j+1/2} \\ &= \begin{pmatrix} \rho \\ \rho U \\ \rho e \end{pmatrix}_j^+ + \begin{pmatrix} \rho \\ \rho U \\ \rho e \end{pmatrix}_{j+1}^- \\ &= \begin{pmatrix} \rho\langle u^0 \rangle_{u>0} \\ \rho\langle u^1 \rangle_{u>0} \\ (\rho e - \frac{1}{2}\rho U^2)\langle u^0 \rangle_{u>0} + \frac{1}{2}\rho U\langle u^1 \rangle_{u>0} \end{pmatrix}_j \\ &\quad + \begin{pmatrix} \rho\langle u^0 \rangle_{u<0} \\ \rho\langle u^1 \rangle_{u<0} \\ (\rho e - \frac{1}{2}\rho U^2)\langle u^0 \rangle_{u<0} + \frac{1}{2}\rho U\langle u^1 \rangle_{u<0} \end{pmatrix}_{j+1},\end{aligned}$$

where $\rho e - \frac{1}{2}\rho U^2$ is the internal energy density ρe . Then, from the ‘‘averaged’’ macroscopic flow quantities in the above equation, we can construct the equilibrium flux function

$$F_{j+1/2}^e = \begin{pmatrix} \bar{\rho}\bar{U} \\ \bar{\rho}\bar{U}^2 + \bar{p} \\ (\bar{\rho}\bar{e} + \bar{p})\bar{U} \end{pmatrix}_{j+1/2}.$$

The final flux with the inclusion of both free transport (nonequilibrium) and collision (equilibrium) terms is

$$F_{j+1/2} = \eta F_{j+1/2}^f + (1 - \eta) F_{j+1/2}^e,$$

where η is a justifiable parameter, which will be analyzed in the next section. The scheme with fixed $\eta \in [0, 1]$ is called Partial Thermalized Transport method, which is exactly the first order BGK scheme [25]. With the inclusion of equilibrium flux function, the dissipation in the KFVS scheme is reduced substantially. In the next section, we are going to extend the above method to MHD equations. In contrast to the Roe’s approximate Riemann solver for the Euler equations [18], in the above BGK method, we have strived for even less information necessary to form a flux function. So, the above scheme is very efficient. The construction of $\bar{q}_{j+1/2}$ term at the cell interface gives some ideas about how to construct $U_{\frac{1}{2}}$ and $M_{\frac{1}{2}}$ in the AUSM and CUSP-type schemes. It will be interesting to see the results if $U_{\frac{1}{2}}$ and $M_{\frac{1}{2}}$ are replaced by the equivalent values from $\bar{q}_{j+1/2}$ term. The splitting of advection and pressure terms in F_f^+ and F_f^- has the similarity with the AUSM-type methods [11, 8, 20].

2.2. Flux Splitting Method for MHD Equations. For the MHD equations, we can use the same technique in the last section to split the flux directly. Since the splitting of fluxes is closely related to the definition of $\langle u^0 \rangle$ and $\langle u^1 \rangle$ terms, which are functions of x -direction velocity U and the “temperature” λ . For the MHD equations, both gas and magnetic field contribute to the total pressure p_* , and the total internal energy is a combination of gas and magnetic energy. With the definition of normal pressure from the distribution function g

$$\int_{-\infty}^{\infty} (u - U)^2 g du = \frac{\rho}{2\lambda},$$

the total pressure (gas + magnetic) in the MHD equations uniquely determines the value of λ

$$\lambda = \frac{\rho}{2p_*} = \frac{\rho}{2p + (B_x^2 + B_y^2 + B_z^2)},$$

where p is the gas pressure. The velocity U in g can be the same as the macroscopic fluid velocity in the x -direction.

After determining λ and U , we are ready to split the MHD flux function,

$$F = \begin{pmatrix} \rho U \\ \rho U^2 + p_0 \\ \rho UV - B_x B_y \\ \rho UW - B_x B_z \\ B_y U - B_x V \\ B_z U - B_x W \\ \rho \epsilon U + p_0 U - B_x (B_y V + B_z W) \end{pmatrix} = F_f^+ + F_f^-$$

where $p_0 = p_* - B_x^2$. The positive flux F_f^+ is

$$F_f^+ = \langle u^1 \rangle_{u>0} \begin{pmatrix} \rho \\ \rho U \\ \rho V \\ \rho W \\ B_y \\ B_z \\ \rho \epsilon \end{pmatrix} + \begin{pmatrix} 0 \\ p_0 \langle u^0 \rangle_{u>0} \\ -B_x B_y \langle u^0 \rangle_{u>0} \\ -B_x B_z \langle u^0 \rangle_{u>0} \\ -B_x V \langle u^0 \rangle_{u>0} \\ -B_x W \langle u^0 \rangle_{u>0} \\ \frac{1}{2} (p_0 U \langle u^0 \rangle_{u>0} + p_0 \langle u^1 \rangle_{u>0}) - B_x (B_y V + B_z W) \langle u^0 \rangle_{u>0} \end{pmatrix}.$$

Similarly, the negative flux is

$$F_f^- = \langle u^1 \rangle_{u<0} \begin{pmatrix} \rho \\ \rho U \\ \rho V \\ \rho W \\ B_y \\ B_z \\ \rho \epsilon \end{pmatrix} + \begin{pmatrix} 0 \\ p_0 \langle u^0 \rangle_{u<0} \\ -B_x B_y \langle u^0 \rangle_{u<0} \\ -B_x B_z \langle u^0 \rangle_{u<0} \\ -B_x V \langle u^0 \rangle_{u<0} \\ -B_x W \langle u^0 \rangle_{u<0} \\ \frac{1}{2}(p_0 U \langle u^0 \rangle_{u<0} + p_0 \langle u^1 \rangle_{u<0}) - B_x (B_y V + B_z W) \langle u^0 \rangle_{u<0} \end{pmatrix}.$$

Combining the above splitting fluxes, the free transport flux for MHD equations at a cell interface becomes

$$F_{j+1/2}^f = F_{j,f}^+ + F_{j+1,f}^-.$$

This formulation is exactly the one given by Croisille et al. [4]. Numerically, the above flux function is very reliable and robust [10], and the scheme performs well for these problems for which the Roe scheme fails, such as the odd-even decoupling and carbuncle phenomena. However, the accuracy of the above scheme is noticeably worse, especially around contact and tangential discontinuities in the MHD applications.

Now let's construct the corresponding equilibrium flux for the MHD equations. The corresponding macroscopic variables of a equilibrium state at a cell interface are,

$$(2.3) \quad \bar{q}_{j+1/2} = \begin{pmatrix} \bar{\rho} \\ \bar{\rho} \bar{U} \\ \bar{\rho} \bar{V} \\ \bar{\rho} \bar{W} \\ \bar{B}_y \\ \bar{B}_z \\ \bar{\rho} \bar{\epsilon} \end{pmatrix}_{j+1/2} = q_j^+ + q_{j+1}^-,$$

where

$$q_j^+ = \begin{pmatrix} \rho \langle u^0 \rangle_{u>0} \\ \rho \langle u^1 \rangle_{u>0} \\ \rho V \langle u^0 \rangle_{u>0} \\ \rho W \langle u^0 \rangle_{u>0} \\ B_y \langle u^0 \rangle_{u>0} \\ B_z \langle u^0 \rangle_{u>0} \\ (\rho \epsilon - \frac{1}{2} \rho U^2) \langle u^0 \rangle_{u>0} + \frac{1}{2} \rho U \langle u^1 \rangle_{u>0} \end{pmatrix}_j,$$

and

$$\bar{q}_{j+1} = \begin{pmatrix} \rho \langle u^0 \rangle_{u<0} \\ \rho \langle u^1 \rangle_{u<0} \\ \rho V \langle u^0 \rangle_{u<0} \\ \rho W \langle u^0 \rangle_{u<0} \\ B_y \langle u^0 \rangle_{u<0} \\ B_z \langle u^0 \rangle_{u<0} \\ (\rho\epsilon - \frac{1}{2}\rho U^2) \langle u^0 \rangle_{u<0} + \frac{1}{2}\rho U \langle u^1 \rangle_{u<0} \end{pmatrix}_{j+1}.$$

With the above ‘‘averaged’’ macroscopic variables $\bar{q}_{j+1/2}$, the equilibrium flux can be constructed as

$$F_{j+1/2}^e = F(\bar{q}_{j+1/2}) = \begin{pmatrix} \bar{\rho}\bar{U} \\ \bar{\rho}\bar{U}^2 + \bar{p}_* - \bar{B}_x^2 \\ \bar{\rho}\bar{U}\bar{V} - \bar{B}_x\bar{B}_y \\ \bar{\rho}\bar{U}\bar{W} - \bar{B}_x\bar{B}_z \\ \bar{B}_y\bar{U} - \bar{B}_x\bar{V} \\ \bar{B}_z\bar{U} - \bar{B}_x\bar{W} \\ (\bar{\rho}\bar{\epsilon} + \bar{p}_*)\bar{U} - \bar{B}_x(\bar{B}_x\bar{U} + \bar{B}_y\bar{V} + \bar{B}_z\bar{W}) \end{pmatrix}_{j+1/2}.$$

where $\bar{B}_x = B_x$ is a constant in the 1D case and

$$\bar{p}_* = (\gamma - 1) \left(\bar{\rho}\bar{\epsilon} - \frac{1}{2}\bar{\rho}(\bar{U}^2 + \bar{V}^2 + \bar{W}^2) - \frac{1}{2}(\bar{B}_x^2 + \bar{B}_y^2 + \bar{B}_z^2) \right) + \frac{1}{2}(\bar{B}_x^2 + \bar{B}_y^2 + \bar{B}_z^2).$$

The final flux across a cell interface is a combination of nonequilibrium and equilibrium ones,

$$(2.4) \quad F_{j+1/2} = \eta F_{j+1/2}^f + (1 - \eta) F_{j+1/2}^e,$$

where η is an adaptive parameter. The program from the left and right states to the final flux function is given in the Appendix. By removing the contribution from the magnetic field, the above MHD flux function reduces exactly to the BGK flux constructed for the Euler equations in the last section.

In the current study, we are more interested in the specific numerical flux function for the MHD equations. For the 1st-order scheme, η can be fixed, such as 0.7 or 0.5, in the numerical calculations. Theoretically, the parameter η should depend on the real flow situations: in the equilibrium and smooth flow regions, the use of $\eta \sim 0$ is physically reasonable, and in discontinuity region, η should be close to 1 in order to have enough numerical dissipation to recover the smooth shock transition. A possible choice for η in high-order scheme is to design a pressure-based stencil, such as the switch function in the JST scheme [7]. In the high-order BGK scheme for the Euler and Navier-Stokes equations [25], with the BGK model as the governing equation, the time dependent flux in the gas evolution stage can be obtained by following the BGK solution, and the relation between the collision time τ and viscosity coefficient is well established. For the MHD equations, basically we only split the macroscopic flux function without knowing the explicit microscopic transport equation for the fluid and magnetic field. Nevertheless, we can follow the MUSCL-type approach to extend the current scheme to high order. For example, we can get the left and right states at a cell interface through the nonlinear reconstruction of the initial data, then evaluate the flux according to the formulation given by Eq.(2.4). A high-order Runge-Kutta time-stepping scheme is also recommended. For the high-order scheme,

the interpolated pressure jump p_l and p_r around a cell interface can naturally be used as a switch function for the parameter η , such as

$$\eta = 1 - \exp\left(-\alpha \frac{|p_l - p_r|}{p_l + p_r}\right),$$

where α can be some constants.

3. A Numerical Experiment. For any upwinding schemes, the construction of the flux function, or the 1st-order scheme, is very important in the understanding of the scheme. Since in the high-order extensions, many factors, such as nonlinear limiter, the reconstruction of conservative or primitive variables and time-stepping methods, can all affect the performance of the scheme. In the following, we are going to apply the current method to the Brio-Wu 1D MHD test case [3]. Only the results from first-order method with fixed $\eta = 0.5$ will be presented.

The initial condition of the Brio-Wu case is

$$\rho_l = 1.0, U_l = 0, p_l = 1, B_{x,l} = 0.75, B_{y,l} = 1$$

on the left, and

$$\rho_r = 0.125, U_l = 0, p_r = 0.1, B_{x,r} = 0.75, B_{y,r} = -1$$

on the right. The gas constant γ is equal to 2, which corresponds to an internal degree of freedom $K = -1$ for the simulated molecule [25]. Note the gas-kinetic flux splitting formula presented in the last section can be applied to any reasonable γ .

In order to evaluate the performance of the current method, we are going to compare its numerical results with that from the Roe-type MHD Riemann solver [3, 16]. The Roe-type MHD solver is considered the most accurate MHD solver existing so far [10], although the robustness of the scheme is questionable in some special applications.

There are 400 grid points used from $[-1, 1]$ in the x -direction. The time step is based on $\Delta t/\Delta x = 0.2$, which is equivalent to CFL number 0.8 in this case. The results at 200 time steps are displayed in Fig. 4.1-4.5. The results from the Roe scheme [3, 16], with identical initial condition and time step, are also plotted in these figures. In most regions, the kinetic and Roe-type MHD solvers give almost identical results, except the non-conservative quantities at the fast right-moving rarefaction wave.

Due to the nonconvexity, the MHD equations could present compound waves, which directly connect shock and rarefaction. In Table 1, we list the data at the peak point of the compound wave in the Brio-Wu test case. Both results are compared with the theoretical prediction in [3]. Fig. 4.6 gives a close look at the density distributions around the right moving shock and the middle contact discontinuity wave. Three schemes used here are the current one with $\eta = 0.5$, Croisille et al.'s KFVS MHD solver, and the Roe type MHD solver. The diffusivity of KFVS MHD solver can be clearly observed.

Table 1. The Flow Variables at the Peak Point of Compound Wave

Scheme	ρ	U -Velocity	V -Velocity	B_y	gas pressure p
theory [3]	0.7935	0.4983	-1.290	-0.3073	0.6687
Kinetic	0.8179	0.4679	-1.083	-0.1239	0.7300
Roe	0.8257	0.4623	-0.928	0.0163	0.7400

4. Discussion and Conclusion. In this paper, based on the gas-kinetic theory, we have constructed the kinetic flux splitting formula for the MHD equations. We feel that perhaps there is a wide application of the splitting techniques presented in this paper. Also, the kinetic flux splitting formulation has the similarities with the AUSM and CUSP type schemes [11, 8], where the advection and pressure terms are split differently. The numerical results validate the accuracy of the current approach.

In terms of the current gas-kinetic MHD solver, we have the following remarks:

(1.) Extension of the current method to the multidimensional case is straight-forward using directionally splitting techniques. If there is a jump of magnetic field in the normal direction, such as B_x in the x -direction across a cell interface, the weakly nonconservative form [16],

$$\frac{\partial B_x}{\partial t} + U \frac{\partial B_x}{\partial x} = 0$$

can be simply split by changing U in the above equation to \bar{U} of Eq.(2.3). Also, in order to satisfy $\nabla \cdot \mathbf{B} = 0$ condition, the projection method can be used to clean up the non-zero divergence of the magnetic field [2].

(2.) The current scheme is very efficient in comparison with the Roe-type Riemann MHD solver. For example, for 1-D calculations, the flux evaluation takes about 1/3 of the CPU time of the Roe-type scheme. For 3D calculations, the saving of computational time is enormous. Since we do not use characteristic information of the MHD system, the numerical problems related to nonconvexity, non-strictly hyperbolicity, and linearization are avoided. Also, the Boltzmann-type scheme is very robust, especially for high-speed low density regions [10]. The main reason for this is that the splitting is based on $\langle u^n \rangle_{u>0}$ and $\langle u^n \rangle_{u<0}$, which accounts for all particle velocities, instead of switching the flux function according to the Mach number $M > 1$ or $M < 1$ in many other splitting schemes.

(3.) The extension of the current method to the system with general equation of state $p = p(\rho, e)$ is straightforward. The important point is to distinguish the differences between the splitting of internal energy flux $\rho e U$ and the work done by the pressure pU . No singularity and ambiguity in characteristic decomposition of the MHD equations will be encountered in the gas-kinetic splitting formulation.

There are still many open questions related to the current gas-kinetic approach. First, underlying the macroscopic flux splitting, we do not know the exact microscopic equilibrium state for the whole flow system including gas and magnetic field. Second, different from the BGK scheme for the Euler and Navier-Stokes equations [25], there is no direct way to extend the current method to solve dissipative (including resistivity and dispersive effects) MHD equations due to the lack of microscopic transport equations, although the dissipative terms can be regarded as additional source terms to the current ideal MHD equations. Third, in the plasma calculation, particle method is usually used. How to make the smooth transition from the microscopic particle method to the macroscopic MHD Riemann solver through the gas-kinetic scheme is an important and interesting research direction. Even with many unknowns, the potential advantage of the kinetic approach over Riemann solver in the construction of numerical flux function becomes clear when solving more and more complicated hyperbolic systems.

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Appendix: Evaluation of Kinetic MHD Flux Function.

```

c
c left state = (ADE1,AXM1,AYM1,AZM1,AEN1,ABX1,ABY1,ABZ1)
c right state = (ADE2,AXM2,AYM2,AZM2,AEN2,ABX2,ABY2,ABZ2)
c gas constant GAM  $\gamma$ , PI=3.14  $\pi$  should be given.

c left and right side pressure pl, pr
  APP1=(GAM-1)*(AEN1-0.5*(AXM1**2+AYM1**2+AZM1**2)/ADE1)
  *      +0.5*(2.0-GAM)*(ABX1**2+ABY1**2+ABZ1**2)
  APP2=(GAM-1)*(AEN2-0.5*(AXM2**2+AYM2**2+AZM2**2)/ADE2)
  *      +0.5*(2.0-GAM)*(ABX2**2+ABY2**2+ABZ2**2)
c left and right sides  $\lambda$ , and macroscopic velocities U,V,W$
  AE1=0.5*ADE1/APP1
  AU1=AXM1/ADE1
  AV1=AYM1/ADE1
  AW1=AZM1/ADE1

  AE2=0.5*ADE2/APP2
  AU2=AXM2/ADE2
  AV2=AYM2/ADE2
  AW2=AZM2/ADE2
c left and right side particle velocity moments  $\langle u^0 \rangle$ ,  $\langle u^1 \rangle$ 
  TEU0=0.5*DERFC(-AU1*SQRT(AE1))
  TEU1=AU1*TEU0+0.50*EXP(-AE1*AU1*AU1)/SQRT(AE1*PI)
  TGU0=0.5*(DERFC(AU2*SQRT(AE2)))
  TGU1=AU2*TGU0-0.50*EXP(-AE2*AU2*AU2)/SQRT(AE2*PI)

c intermediate (equilibrium ) state, and corresponding  $\lambda$  and pressure.
  ADE=ADE1*TEU0+ADE2*TGU0
  AU=(ADE1*TEU1+ADE2*TGU1)/ADE
  AV=(ADE1*AV1*TEU0+ADE2*AV2*TGU0)/ADE
  AW=(ADE1*AW1*TEU0+ADE2*AW2*TGU0)/ADE
  ABY=ABY1*TEU0+ABY2*TGU0
  ABZ=ABZ1*TEU0+ABZ2*TGU0
  AE=(AEN1-0.5*ADE1*AU1**2)*TEU0+(AEN2-0.5*ADE2*AU2**2)*TGU0
  *      +0.5*ADE1*AU1*TEU1+0.5*ADE2*AU2*TGU1
  TP=(GAM-1)*(AE-0.5*ADE*(AU**2+AV**2+AW**2))
  *      +0.5*(2.0-GAM)*(ABX1**2+ABY**2+ABZ**2)

c gas-kinetic flux function, ETA  $\eta$  is a justifiable parameter.
  FM=ETA*(TEU1*ADE1+TGU1*ADE2)+(1-ETA)*ADE*AU
  FU=ETA*(TEU1*AXM1+TGU1*AXM2+(APP1-ABX1**2)*TEU0+(APP2-ABX2**2)*TGU0)
  *      +(1-ETA)*(ADE*AU**2+TP-ABX1**2)
  FV=ETA*(TEU1*AYM1+TGU1*AYM2-ABX1*ABY1*TEU0-ABX2*ABY2*TGU0)
  *      +(1-ETA)*(ADE*AU*AV-ABX1*ABY)
  FW=ETA*(TEU1*AZM1+TGU1*AZM2-ABX1*ABZ1*TEU0-ABX2*ABZ2*TGU0)
  *      +(1-ETA)*(ADE*AU*AW-ABX1*ABZ)
  FBY=ETA*(TEU1*ABY1+TGU1*ABY2-ABX1*AV1*TEU0-ABX2*AV2*TGU0)
  *      +(1-ETA)*(ABY*AU-ABX1*AV)
  FBZ=ETA*(TEU1*ABZ1+TGU1*ABZ2-ABX1*AW1*TEU0-ABX2*AW2*TGU0)
  *      +(1-ETA)*(ABZ*AU-ABX1*AW)
  FE=ETA*(TEU1*AEN1+TGU1*AEN2
  *      +0.5*(APP1-ABX1**2)*TEU1+0.5*AU1*(APP1-ABX1**2)*TEU0
  *      -ABX1*(ABY1*AV1+ABZ1*AW1)*TEU0
  *      +0.5*(APP2-ABX2**2)*TGU1+0.5*AU2*(APP2-ABX2**2)*TGU0
  *      -ABX2*(ABY2*AV2+ABZ2*AW2)*TGU0)
  *      +(1-ETA)*((AE+TP)*AU-ABX1*(ABX1*AU+ABY*AV+ABZ*AW))

```

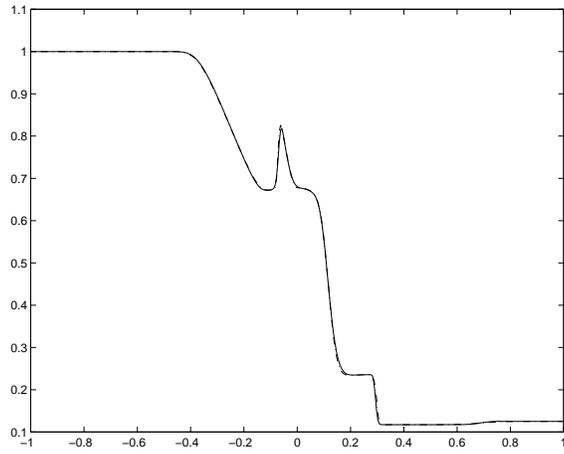


FIG. 4.1. Density distributions with 400 grid points, solid line: first-order BGK-type scheme, dashdot line: first-order Roe-MHD solver

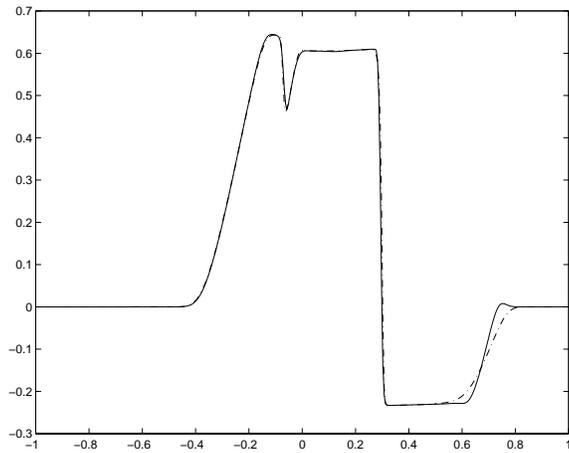


FIG. 4.2. x -component velocity distributions with 400 grid points, solid line: first-order BGK-type scheme, dashdot line: first-order Roe-MHD solver

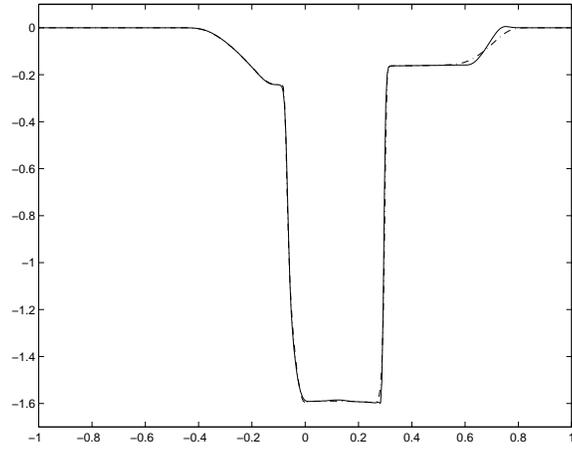


FIG. 4.3. y -component velocity distributions with 400 grid points, solid line: first-order BGK-type scheme, dashdot line: first-order Roe-MHD solver

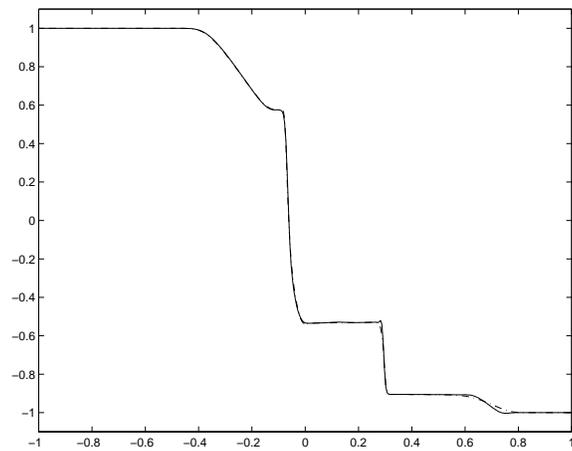


FIG. 4.4. B_y distributions with 400 grid points, solid line: first-order BGK-type scheme, dashdot line: first-order Roe-MHD solver

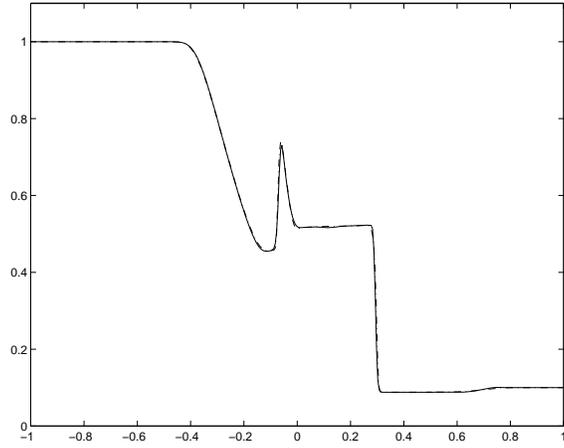


FIG. 4.5. Gas pressure p distributions with 400 grid points, solid line: first-order BGK-type scheme, dashdot line: first-order Roe-MHD solver

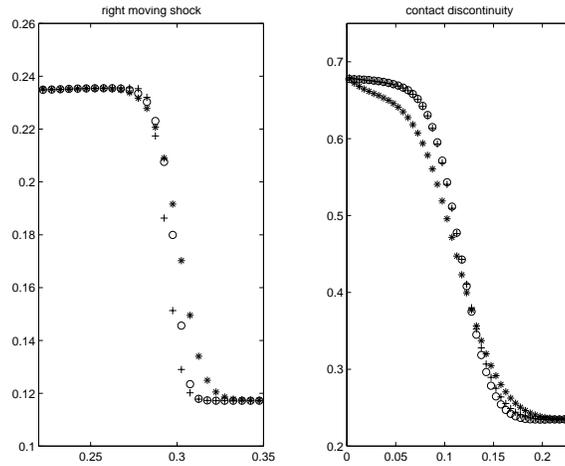


FIG. 4.6. Density profiles around right moving shock and middle contact discontinuity using three first-order schemes, $+$: current kinetic method with $\eta = 0.5$, \circ : Roe-type MHD solver, $*$: Croisille et al.'s KFVS MHD solver (corresponding to $\eta = 1.0$ in the current scheme).