NASA/CR-1998-208958
ICASE Report No. 98-54

3D Characteristics

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Operated by Universities Space Research Association

National Aeronautics and
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Prepared for Langley Research Center
under Contract NAS1-97046

December 1998

3D CHARACTERISTICS

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Abstract. Contributions to the Method of Characteristics in Three Dimensions, which previously received incomplete recognition, are reviewed. They mostly follow from a fundamental paper by Rusanov which led to several developments in Russia, described by Chushkin.

Key words. 3D characteristics, Rusanov schemes

Subject classification. Applied & Numerical Mathematics

1. Introduction. In Holt (1984) Chapter 4 is devoted to a review of computational methods for solving Gas Dynamic problems in three dimensions by the Method of Characteristics. This adequately covered contributions from the U.S. and Western Europe but, owing partly to oversight and partly to limited knowledge of relevant references, overlooked important work in Russia (then part of the U.S.S.R). This originated in the doctoral thesis of V.V. Rusanov, which was completed in 1951 but not known outside Russia until an openly published version appeared in 1963. The thesis was of general and fundamental character and its development for practical application to Aerodynamic problems extended over a decade or so in a series of papers by other Russian authors. These, together with an assessment of Rusanov's original contribution, are described in an excellent comprehensive article by Chushkin (1968). The present paper highlights the Russian components of the article and will be used to revise and expand Chapter 4 in the third edition of Holt (1984).

In his doctoral dissertation, Rusanov discussed the characteristic properties of a general system of quasi-linear partial differential equations in several independent variables and specialized for applications to three dimensional unsteady and steady gas dynamics. In the steady flow case he outlined a finite difference scheme based on a tetrahedral network formed by characteristic surfaces.

Two versions of Rusanov's tetrahedral scheme were developed for Gas Dynamic applications. The first, due to Podladchikov (1965), is of direct type, in which tetrahedra are projected downstream from points on a surface where the solution is known towards an adjacent surface where the solution is to be found. The second scheme, by Minostsev (1967), is inverse, in which tetrahedra project upstream towards a known surface from points on a surface unknown.

Other schemes developed in Russia for steady supersonic flow, following Rusanov's general analysis, used alternative combinations of characteristic properties. Magomedov (1966) developed a numerical technique employing derivatives along bicharacteristic directions only, while Katskova and Chushkin (1965) proposed a quasi-characteristics scheme in which difference equations are solved in successive coordinate planes by the two-dimensional version of the method of characteristics.

In general, methods of characteristics for three-dimensional problems can be classified under five headings.

1. Methods based on characteristic surfaces.
2. Methods based on bicharacteristics only.

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3. Methods based on two bicharacteristic directions and one non-characteristic direction.
4. Optimal methods of type (3).
5. Methods based on characteristic lines not in the bicharacteristic direction.

The schemes of Podladchikov and Minostsev are of type 1), that of Magomedov is of type 2) while that of Katskova and Chushkin is of type 5). Methods of types 3) and 4) have not been developed in Russia and, in general, have had limited application.

2. Direct Tetrahedral Scheme. Podladchikov (1965) developed this scheme for calculating steady supersonic flow past a body of revolution at angle of attack. The governing Euler equations are written in cylindrical polar coordinates (x, r, Ψ) with x replaced by ξ

$$\xi = (r - r_b(x, \Psi))/\epsilon, \quad \epsilon = r_s(x, \Psi) - r_b(x, \Psi)$$

r_s and r_b are the shock and body radii respectively. With dependent variables V, velocity, p, pressure, ρ, density, S, entropy, Rusanov derived the following compatibility relations satisfied along characteristic surfaces

$$\rho a \mathbf{A}_2 \cdot d_{s1}\mathbf{V} - B_2 d_{s1} p = \rho a \mathbf{A}_1 \cdot d_{s2}\mathbf{V} - B_1 d_{s2} p \tag{2.1}$$

where

$$\mathbf{A}_1 = \mathbf{V} \wedge \mathbf{s}_j, \quad B_j = \mathbf{n} \cdot \mathbf{A}_j \ \ (j = 1, 2) \tag{2.2}$$

Here $\mathbf{s_1}$, $\mathbf{s_2}$ are two independent unit vectors in the characteristic plane with external normal vector $\mathbf{n}$. d_{s1} d_{s2} denote differentials along $\mathbf{s}_1, \mathbf{s}_2$ respectively. Eq. (2.1) is supplemented by the equation of state (perfect gas) and the condition $S =$ constant on stream surfaces. a represents the speed of sound.

Podladchikov's tetrahedral scheme connects known flow conditions on a non-characteristic initial surface with a point downstream at which the flow variables are to be determined. The basic computational cell for internal points in the flow field is shown in Fig. 2.1.

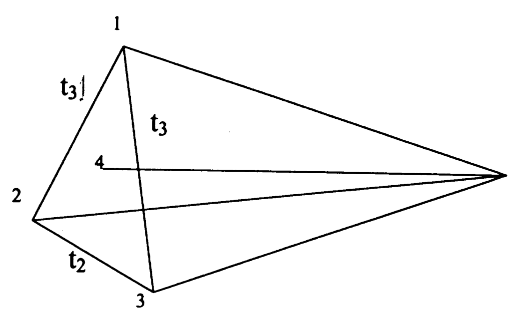

FIG. 2.1. *Network for direct tetrahedral scheme*

Points 1, 2, and 3 are on the initial surface where flow conditions are known. We use the compatibility conditions 2.1 to determine conditions at point 0, the intersection of the three characteristic planes through the non-characteristic lines 1-2, 2-3, and 3-1. The equations of these planes and the location of 0, their point of intersection, are determined uniquely from known data at points 1, 2, and 3.

Flow conditions at point 0 are now determined from difference forms of the compatibility conditions 2.1 applied along the characteristic surfaces 012, 023, and 031. In Eqs. 2.1, on each face take $\mathbf{s}_2 = \mathbf{t}_i$ while $\mathbf{s}_1$ is a unit vector joining the mid-point of $\mathbf{t}_i$ with 0. These difference equations, together with the constant entopy condition and Bernoulli's equation applied on 04 determine all flow conditions at 0.

This procedure is modified at surface boundary and shock points.

Podladchikov used this scheme to calculate flow past a body of revolution with spherical nose and conical afterbody at angle of attack 5°, free stream Mach number 4.

3. Inverse Tetrahedral Scheme. Minostsev (1967) considered the same problem as Podladchikov but proposed an inverse version of Rusanov's tetrahedral scheme. He used the same independent variables x, ξ, Ψ. Minostsev now connects flow conditions to be determined at point 0, in the interior flow field, with those on a non-characteristic surface upstream where conditions are known.

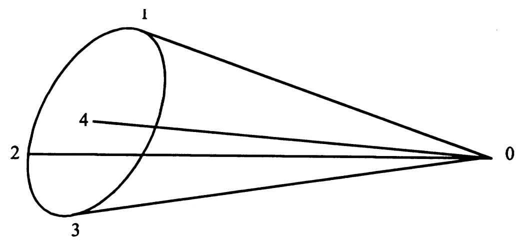

FIG. 3.1. *Network for inverse tetrahedral scheme*

Fig. 3.1 shows three generators (bicharacteristics) 01, 02 and 03 of the backward Mach cone through 0, intersecting the initial surface upstream at points 1, 2, and 3. The equation of the Mach cone has coefficients dependent on the velocity components and the speed of sound at point 0. These are first approximated by values at point $0'$ on the initial surface, in the same meridian plane and with the same values of Ψ and ξ as at 0. The curve of intersection of this Mach cone with the initial surface is then determined and three points on it, 1,2, and 3 are selected. We can now construct the tetrahedral cell bounded by the characteristic surfaces 012, 023, 031 and the initial triangle 123, and use compatibility relations 2.1, together with conditions along the streamline direction 04 to determine flow variables at 0. The latter values should be used to recalculate the equation of the backward Mach cone and repeat the finite difference calculation until converged.

Minostsev used the scheme to calculate supersonic flow past ellipsoids and cones with spherical noses at various angles of attack, and at Mach numbers up to 20.

4. Bicharacteristics Only Scheme. The scheme proposed by Magomedov (1966) uses only relations along bicharacteristics.

In Fig. 4.1 0 is the point where flow variables are to be determined, 01, 02, 03, and 04 are generators (bicharacteristics) of the backward Mach cone through 0 intersecting the known initial surface in points 1, 2, 3, 4, and 05 is the backward streamline.

The compatibility relations 2.1 satisfied along characteristic surfaces contain, as well as a derivative in the characteristic direction, a derivative in a non-characteristic direction. If we apply these on surfaces containing four bicharacteristics 01, 02, 03, 04, the non-characteristic components can be eliminated and relations 2.1 reduce to difference relations in the flow variables along the four bicharacteristic directions only. Together with conditions along streamlines 05 these suffice to determine the flow variables at 0.

Magomedov used his scheme to calculate supersonic flow past cones with spherical noses and past a delta wing with blunt leading edges.

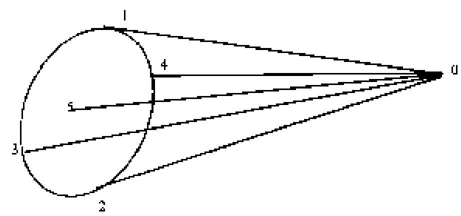

FIG. 4.1. *Element of bicharacteristics only scheme*

5. Quasi-Characteristic Scheme. Katskova and Chushkin (1965) proposed quasi-characteristics scheme to calculate steady supersonic flow past smooth bodies. Starting from equations of motion in the same cylindrical coordinate system already defined they represent variation in the circumferential direction by expansions in sin $k\Psi$ or cos $k\Psi$, and solve a sequence of two-dimensional problems in meridian planes

$$\Psi = \Psi_k = \frac{k\pi}{K}(k = 0, 1, ...K) \qquad 0 \leq \Psi \leq \pi$$

In each meridian plane the coefficients in the trigonometric series satisfy partial differential equations in x and ξ with characteristic properties similar to those for supersonic plane flow, and can be solved by a method of characteristics like that for two-dimensional problems.

At an interior point the network used is shown in Fig. 5.1.

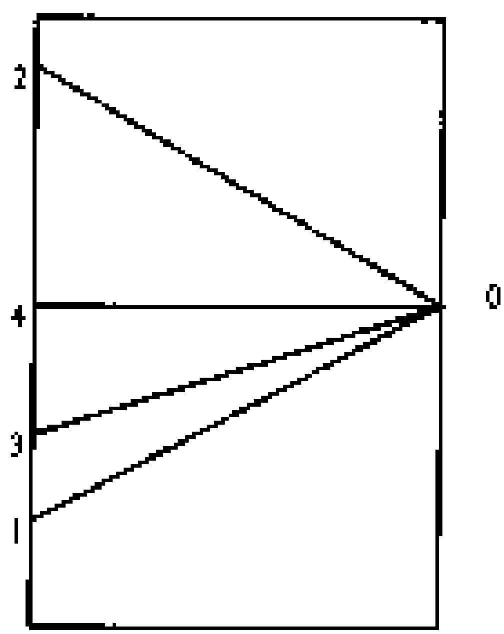

FIG. 5.1. *Element of quasi-characteristics scheme*

0 is the point at $x = x_* + \Delta x$, downstream of the initial line $x_* = .1342$, where the flow variables at 0 an inverse characteristics method is used (similar to that first proposed by Hartree (1953)). Using values at point 4 ($\xi_4 = \xi_0$) characteristics and streamline are drawn upstream of 0 to points 1, 2, and 3 respectively.

Using interpolated values at 1, 2, and 3 found from the initial data the compatibility relations are solved along 20 and 10 and streamline relation along 30, yielding a first estimate of the flow variables at 0. The process is repeated with better estimates of 01, 02, and 03 until converged. Modified schemes are used at body surface and shock surface points.

This computational scheme is applied successively in all planes, $\Psi = \Psi_k$, thus determining a first estimate of the solution on the shock wave. In further iterations the same general scheme is applied, but in each iteration, the latest approximation to the shock shape is made.

Katskova and Chushkin applied their scheme to calculate a series of three-dimensional flows past blunt cones and through inlets.

6. Conclusions. A series of characteristic schemes for solving problems of steady supersonic flow past bodies in three dimensions, which were developed in USSR, mostly in the 1960 decade, are described. They were based on a fundamental paper by Rusanov, originating in his 1951 doctoral thesis, but not published openly until 1963. This material will be included in an updated Chapter 4 of Holt (1984) as part of the third edition of this monograph.

REFERENCES

[1] Chushkin, P.I., *Numerical methods of characteristics for three dimensional supersonic flow*, Progress in Aeronautical Sciences, Pergamon, 1969.

[2] Hartree, D.R., *Some practical methods of using characteristics in the Calculation of non-steady compressible flow*, U.S. Atomic Energy Commission Rep. AECU-2713, 1953.

[3] Holt, M., *Numerical Methods in Fluid Dynamics*, 2nd Edition, Springer-Verlag, 1984.

[4] Katskova, O.N. and Chushkin, P.I., Zh. Vych. Mat. i Mat. Fiz., Vol. 5, 1965, pp. 503-518.

[5] Magomedov, K.M., Zh. Vych. Mat. i Mat. Fiz., Vol. 6, 1966, pp. 313-325.

[6] Minostsev, V.B., V.B. Izv. Akad. Nauk SSSR. Mekh. Zhid. i Gaza, 1967, pp. 126-133.

[7] Podladchikov, Yu N., Izv. Akad Nauk SSSR Mekhan., 1965, pp. 3-12.

[8] Rusanov, V.V., Zh. Vych. Mat. i Mat. Fiz., Vol. 3, 1963, pp. 508-527.