NASA/CR-2000-210553
ICASE Report No. 2000-41

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Operated by Universities Space Research Association

National Aeronautics and
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Prepared for Langley Research Center
under Contract NAS1-97046

November 2000

POSSIBLE STATISTICS OF TWO COUPLED RANDOM FIELDS: APPLICATION TO PASSIVE SCALAR

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Abstract. We use the relativity postulate of scale invariance to derive the similarity transformations between two coupled scale-invariant random fields at different scales. We find the equations leading to the scaling exponents. This formulation is applied to the case of passive scalars advected i) by a random Gaussian velocity field; and ii) by a turbulent velocity field. In the Gaussian case, we show that the passive scalar increments follow a log-Levy distribution generalizing Kraichnan's solution and, in an appropriate limit, a log-normal distribution. In the turbulent case, we show that when the velocity increments follow a log-Poisson statistics, the passive scalar increments follow a statistics close to log-Poisson. This result explains the experimental observations of Ruiz *et al.* about the temperature increments.

Key words. scale invariance, scale transformation, scaling, turbulent scalar field

Subject classification. Fluid Mechanics

1. Introduction. Scale invariance [1] usually refers to physical fields keeping the same properties at different scales. Therefore, scale invariance is often associated with the existence of power scaling laws. In this case, the central characteristic quantities of the system are the scaling exponents. When a random field is statistically scale invariant, each of its structure functions follows a power law, and there is a whole family of exponents associated with its statistics.

To understand the underlying properties associated with the scaling exponents one can follow two paths: first, try to measure or compute the scaling exponents with as high a precision as possible. This is the path usually followed in critical phenomena, where scaling exponents are few, and appear to be universal, due to some underlying conformal symmetry [2]. In turbulence, this path seems more difficult to follow, because the number of scaling exponents to measure or compute, in order to know the statistics of the field, is infinite. Therefore, parallel to the measurement or actual computation path, there seem to be the necessity of a more formal path, devoted to understand *link between the various scaling exponents*. That way, the measure or computation of only a few scaling exponents would hopefully enable the unraveling of the statistical nature of the system. A natural way to establish link between scaling exponents is to use the symmetry responsible for the existence of the scaling exponents, namely the scale invariance. This was the basis of conformal theory [2] or multifractal theory [3]. More recently, Dubrulle *et al.* [1, 4, 5, 6] found that new constraints on scaling exponents stemming from scale invariance could be found via an analogy between scale invariance and relativistic dynamics inspired by a theory of Nottale [7] and of Pocheau [8]. This can be used to classify the possible forms of scaling exponents according to the fixed points of the similarity transformation. However, these results were restricted to one random scale invariant field (1D case). In many physical situations, the dynamics involves at least two coupled random fields, such as transports of passive scalars (coupling of velocity and scalar), magnetohydrodynamics (coupling of velocity and magnetic

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field) and turbulence (coupling of transverse and longitudinal velocity increments). The goal of the present paper is therefore to extend some of these results to the case of two coupled scale invariant random fields (2D case) and use the 2D results to explore scaling of passive scalars.

The transport of passive scalars, for example the temperature θ, is governed by the diffusion equation:

$$\partial_t \theta + (\mathbf{u} \cdot \nabla)\theta = \kappa \Delta \theta, \tag{1.1}$$

where the solenoidal velocity field $\mathbf{u}$ is either taken to be random or governed by the Navier-Stokes equation, κ is molecular diffusivity. We are interested in scaling of passive scalar structure functions for given scaling of velocity structure functions:

$$\begin{aligned} S_n^\theta(\ell) &= \langle (\theta(x+\ell) - \theta(x))^n \rangle \propto \ell^{\xi(n)}, \\ S_n^{\mathbf{u}}(\ell) &= \langle (\mathbf{u}(x+\ell) - \mathbf{u}(x))^n \rangle \propto \ell^{\zeta(n)}, \end{aligned} \tag{1.2}$$

where $\zeta(n)$ and $\xi(n)$ are the scaling exponents of velocity structure functions and scalar structure functions respectively. We will use 2D similarity transformations to derive these possible scaling exponents.

This paper is organized as follows. After useful preliminaries regarding scale invariance (Section 2), the 2D similarity transformation of scale invariant fields is presented in Section 3. The results are applied to the case of the transport of a passive scalar by a Gaussian velocity field in section 4 and by a turbulent velocity field in section 5. We conclude the paper in section 6.

2. Preliminaries on scale invariance. Scale invariance is a commonly used concept but sometimes, rigorous definitions are lacking. In this section, we give a brief review on the rigorous definition of scale dilation (or contraction), global and local scale invariance. Then, we discuss the existence of similarity transformation between measurements of physical fields at different scales due to scale invariance.

Let $(\varphi_\ell(\mathbf{x}), \psi_\ell(\mathbf{x}))$ be a two-dimensional positive homogeneous random field [9] and dependent on scale ℓ. We define a family of scale dilation $\mathcal{S}_{h,h'}(\lambda)$ via:

$$\mathcal{S}_{h,h'}(\lambda) : \ell \to \lambda\ell, \varphi_\ell, \psi_\ell \to \lambda^h \varphi_\ell, \lambda^{h'} \psi_\ell. \tag{2.1}$$

A two dimensional homogeneous random field is said to be globally scale invariant or, simply, scale invariant, if $(\varphi_\ell(\mathbf{x}), \psi_\ell(\mathbf{x}))$ and $\mathcal{S}_{h,h'}(\lambda)(\varphi_\ell(\mathbf{x}), \psi_\ell(\mathbf{x}))$ have the same statistical properties.

In log-variables, $T = \ln \ell$, $X = \ln(\varphi_\ell)$, $Y = \ln(\psi_\ell)$, the globally scale invariance amounts to a translational invariance. This invariance is connected with the possibility to multiply any characteristic quantity by an arbitrary constant, i.e. with the possibility to perform arbitrary changes of units in the system [1]. It is a global scale invariance that leads to the existence of power laws of the moments of random homogeneous fields:

$$\langle \varphi_l^n \rangle \propto \ell^{\xi(n)}, \quad \langle \psi_l^n \rangle \propto \ell^{\zeta(n)}. \tag{2.2}$$

Scale invariance can be also considered as local [8]. Local scale invariance is connected with the invariance of the system with respect to resolution changes. Indeed, consider a discrete slicing of the scale space $\ell \le L$ in the form:

$$\begin{aligned} \ell_i &= \Gamma^i \ell_0, \quad L = \Gamma^N \ell_0, \\ \varphi_i &= \Lambda^i \varphi_0, \quad \psi_i = \Lambda^i \psi_0, \end{aligned} \tag{2.3}$$

where, Γ (or equivalently N) and Λ characterize the resolution of the observations. φ_0 and ψ_0 are the units of the observations. By local scale invariance, we require that the physical properties of the system be invariant under changes of resolution, i.e. under changes of N. In fact, changes of N can be achieved by dilation of Γ: $\Gamma \to \Gamma^\alpha$, which transform i into $i\alpha$. It is also a transformation which transforms ℓ_i/ℓ_j into $(\ell_i/\ell_j)^\alpha$, and ϕ_i/ϕ_j into $(\phi_i/\phi_j)^\alpha$. Therefore:

A local scale dilation can be seen as a transformation for arbitrary α:

$$\ell/\ell_0 \to (\ell/\ell_0)^\alpha; \varphi/\varphi_0, \psi/\psi_0 \to (\varphi/\varphi_0)^\alpha, (\psi/\psi_0)^\alpha, \tag{2.4}$$

A two dimensional homogeneous random field is said to be locally scale invariant if the physical fields $(\varphi_\ell(\mathbf{x}), \psi_\ell(\mathbf{x}))$ are statistically invariant under local scale dilation.

Local scale invariance can be also seen as a change of rationalized unit system because it amounts to multiply ℓ_0, φ_0 and ψ_0 by $\ell_0(\ell/\ell_0)^{1-\alpha}$, $\varphi_0(\varphi/\varphi_0)^{1-\alpha}$ and $\psi_0(\psi/\psi_0)^{1-\alpha}$. Therefore, global scale invariance implies local scale invariance.

When the field $(\varphi_\ell(\mathbf{x}), \psi_\ell(\mathbf{x}))$ is globally scale invariant, all these changes of units can be characterized by only one number: the exponent of their power law. This number labels the system of units and varies with it. It is not an absolute quantity. All these remarks have been so far considered as rather obvious because they do not lead to any specially useful information when applied to deterministic fields. For random fields, however, $< \phi^\alpha >$ and $< \phi^\alpha >$ can be very different, and the application of our remarks can lead to very interesting consequences about the exponents. In fact, they are not necessary additive anymore and can obey more complicated composition law, which can be determined only from the requirement of local scale invariance, as was shown in [6]. We now define the formalism which can be used to derive these composition laws in the case of two coupled fields. This is done by a natural generalization of [1, 6] which we refer to for more details.

3. Formalism and derivation of the composition laws.

3.1. Formalism. As noted previously, scale invariance amounts to a translational invariance when logarithm of scale and fields are considered. We also would like to introduce a field variable which is deterministic but keeps track of the possible different realizations of the process. A good way to reconcile these two requirements is to introduce the log-variables [6]:

$$\begin{aligned} T &= \frac{\ln(\ell/\ell_0)}{\ln(K)} = \log_K(\ell/\ell_0), \\ X_n &= \frac{\langle \ln(\phi_\ell/\phi_0)\ (\phi_\ell/\phi_0)^n \rangle}{\langle(\phi_\ell/\phi_0)^n\rangle}, \\ Y_p &= \frac{\langle \ln(\psi_\ell/\psi_0)\ (\psi_\ell/\psi_0)^p \rangle}{\langle(\psi_\ell/\psi_0)^p\rangle}, \end{aligned} \tag{3.1}$$

where K is the basis of the logarithm, ℓ_0 is an arbitrary length unit, ϕ_0, ψ_0 two arbitrary field units (possibly random and/or scale dependent [1]), and n and p are real numbers. We note that $\ln(K)$, p and n labels the system of subunits used to measure the length and fields. Then, globally scale invariance implies homogeneity in these log-coordinates. Local scale invariance implies the existence of a group transformation connecting different measures of the field at different scales. Formally, this means that we can find a similarity transformation between $(\log_K(\ell/\ell_0), X_n, Y_p)$ and $(\log_{K'}(\ell/\ell_0), X_{n'}, Y_{p'})$. Since ℓ is deterministic, it is equivalent to change K or ℓ to change $T = \ln_K(\ell)$, so we can as well fix K e.g. $K = e$ and let ℓ vary. The transformation is now between $(T = \ln(\ell/\ell_0), X_n, Y_p)$ and $(T' = \ln(\ell'/\ell'_0), X_{n'}, Y_{p'})$. We call it the similarity transformation.

The derivatives $U(n) = dX_n/dT, V(p) = dY_p/dT$ are analogous to the relative velocities. In our formalism, they are just the local multifractal exponents [6].

3.2. Two-dimensional similarity transformation. The possible similarity transformation can be derived provided the system is both globally and locally scale invariant. Global scale invariance implies that the transformation are linear [1]. Since nothing forbids a coupling between fields and scales, their most general representation is a 3×3 matrix coupling X, Y and T. By local scale invariance, the set of such 3×3 matrices must form a (at least semi) group, depending on two parameters U and V [1]. Without loss of generality, they may be written as:

$$\begin{bmatrix} A & B & \Gamma U \\ D & E & \Gamma V \\ GU & HV & \Gamma \end{bmatrix}, \tag{3.2}$$

where A, B, D, E, G, H, U, V and Γ are some unknown functions of U and V to be determined. According to the transformation (3.2), a field characterized by $X = Y = 0$ for any T in one reference field $R = (n, p)$ is transformed into $X' = UT'$, $Y' = VT'$ in another reference field $R' = (n', p')$. This shows that (U, V) are characteristic numbers labeling the change from (n, p) to (n', p'). It is then natural to adopt U and V as the group parameters.

To completely determine $A, ..., \Gamma$ as a function of (U, V), an additional assumption is needed. From physical considerations, we choose to postulate that the group of similarity transformations is commutative. This means that it is equivalent to perform a transformation at scale ℓ then ℓ' or ℓ' then ℓ. This property is generic in the group of scale dilation $\mathcal{S}_\lambda(h, h')$, at any given h and h', which motivates our choice. Moreover, it is a natural extension of the commutativity property in 1D [1]. Note that in such case, the commutativity is guaranteed by the 1D Lie group structure and need not be postulated. Note also that in special relativity, the third additional postulate is isotropy. This leads to a representation in term of rotations, which is NOT commutative in $D > 1$. There is therefore no hope to find the Lorentz group as a special case of our similarity transformations, contrary to the 1D case.

With this additional postulate, the form of the matrix (3.2) can be completely specified. The proof is given in Appendix A. It is:

$$\begin{array}{ll} A = (1 - aU - jV)\Gamma & B = -(jU + dV)\Gamma \\ D = -c(jU + dV)\Gamma & E = (1 - cdU - bV)\Gamma \\ G == -h\Gamma & H = -k\Gamma \end{array} \tag{3.3}$$

where Γ is a function of U and V obeying the transformation rule:

$$\Gamma(U'', V'') = \Gamma(U', V')\Gamma(U, V)(1 - hUU' - kVV'), \tag{3.4}$$

with $U'', U', \ldots V$ following (3.7). We were not able to find the exact expression of Γ in the general case, but in the simple cases we study in the next two sections (where $h = k = 0$), $\Gamma = 1$ is always a solution. In general, we postulate that Γ is a quadratic form in U and V, but the explicit value of the coefficient is not needed in the derivation of the composition law.

The coefficients a,b,c,d,h,k and j are parameters depending on the physical model. They satisfy

$$h = ck, \quad -k + ad + jb = j^2 + d^2 c \tag{3.5}$$

They are real if the scale symmetry is continuous [1].

Composing two transformations from reference fields $R = (n,p)$ to $R' = (n',p')$ and from $R' = (n',p')$ to $R'' = (n'',p'')$, we obtain:

$$(U_{R|R''}, V_{R|R''}) = (U_{R|R'}, V_{R|R'}) \otimes (U_{R'|R''}, V_{R'|R''}) \tag{3.6}$$

where $\otimes$ is the composition law:

$$\left\{ \begin{array}{rcl} U'' & = & \dfrac{U + U' - aUU' - dVV' - j(U'V + UV')}{1 - hUU' - kVV'} \\ V'' & = & \dfrac{V + V' - cjUU' - bVV' - cd(U'V + UV')}{1 - hUU' - kVV'} \end{array} \right. \tag{3.7}$$

Like in 1D, the composition law can be characterized by its fixed points, which depend on the parameters a,b,c,d,h,k,j. It can also be used to determine the scaling exponents, by an appropriate choice of R, R' and R''. In both cases, the general computation is rather cumbersome and complicated by the large number of degenerate situations (collapsing or vanishing fixed points). We treat in the following sections two special cases where the computations are tractable. We emphasize that the case $c = 0$ implies no influence of X over Y and is relevant to the case of the passive scalar dynamics when X is taken as the passive scalar log-coordinate and Y as the velocity field log-coordinate.

3.3. The choice of the reference field. To obtain the composition law for the scaling exponents, we must make precise the link between X,Y and U,V. This depends on the choice of ϕ_0 and ψ_0. As emphasized in [1], a convenient choice is to take $\phi_0 =< \ln(\phi_\ell) >$ and $\psi_0 =< \ln(\psi_\ell) >$. Physically, these quantities are the most probable values of the fields ϕ and ψ. The log coordinate X and Y then characterize deviations with respect to these most probable values, and are a measure of the intermittency of the fields. In such simple case, it is easy to check [6] that the scaling exponents for ϕ and ψ are simply connected to X and Y, via:

$$\begin{aligned} \partial_n \xi(n) - \partial_n \xi(n)|_{n=0} &= \frac{dX_n}{dT}, \\ \partial_p \zeta(p) - \partial_p \zeta(p)|_{p=0} &= \frac{dY_p}{dT}. \end{aligned} \tag{3.8}$$

Now, the connection between U,V and $\xi(n),\zeta(p)$ can be made more clearly by exploiting the remark made in Section III.A. Indeed, considering the transformation from $X = 0, Y = 0$ to $X_n = (\partial_n \xi(n) - \partial_n \xi(n)|_{n=0})T, Y_p = (\partial_p \zeta(p) - \partial_p \zeta(p)|_{p=0})T$, we get:

$$\begin{aligned} U(n) = \partial_n \xi(n) - \partial_n \xi(n)|_{n=0} &= \frac{dX_n}{dT}, \\ V(p) = \partial_p \zeta(p) - \partial_p \zeta(p)|_{p=0} &= \frac{dY_p}{dT}. \end{aligned} \tag{3.9}$$

This shows the existence of a bijection between (U,V) and (n,p). It unambiguously defines an internal composition law $\tilde{\otimes}$ on (n,p), through a transport of the group structure $\otimes$:

$$\begin{aligned} (U,V)((n,p)\tilde{\otimes}(n',p') &= (U,V)(n,p)\otimes(U,V)(n',p'), \\ (n,p)\tilde{\otimes}(n',p') &= (U^{-1},V^{-1})((U,V)(n,p) \\ &\quad \otimes(U,V)(n',p')). \end{aligned} \tag{3.10}$$

From this property, we can deduce that the shape of $\tilde{\otimes}$ is the same as (3.7), with different parameters, characterizing the fixed points in the (n,p) space[1]. We have $(n'',p'')=(n,p)\tilde{\otimes}(n',p')$ with:

$$\left\{\begin{array}{lll} n'' & = & \dfrac{n+n'-\alpha nn'-\delta pp'-\gamma(n'p+np')}{1-\eta nn'-\chi pp'}, \\ p'' & = & \dfrac{p+p'-\tau\gamma nn'-\rho pp'-\tau\delta(n'p+np')}{1-\eta nn'-\chi pp'}. \end{array}\right. \tag{3.11}$$

where the parameters $\alpha,\rho,\gamma,\delta,\eta,\tau,\chi$ characterize the fixed points of $\tilde{\otimes}$ and satisfy:

$$\eta=\tau\chi, \quad -\chi+\alpha\delta+\gamma\rho=\gamma^2+\delta^2\tau \tag{3.12}$$

3.4. Computation of the scaling exponents. The correspondence established through eq. (3.10) enables to classify the statistics of scale invariant random process, i.e., of the possible shapes for $\xi(n),\zeta(p)$, on universal grounds. Consider eq. (3.10) with $(n,p)=(n',p')=(1,1)$. This provides a recursive equation which allows the computation of (U,V) for any integer m, and then, by continuation, on any real number, namely:

$$\begin{aligned} E(m) &= (1,1)\tilde{\otimes}(1,1)\tilde{\otimes}...\tilde{\otimes}(1,1)=(1,1)^{\tilde{[m]}}, \\ (U,V)(E(m)) &= (U(1),V(1))\otimes...\otimes(U(1),V(1)) \\ &= (U(1),V(1))^{[m]}. \end{aligned} \tag{3.13}$$

Here the notation $[m]$ (resp. $\tilde{[m]}$), stands for the m^{th} iterate via $\otimes$ (resp. $\tilde{\otimes}$): we have introduced the notation $E(m)=(E_1(m),E_2(m))$ to generalize the exponentiation m to real, defined by using infinitesimal n and p. It is now technically possible to calculate the values of the $\zeta(n)$ and $\xi(p)$. The set of equations (3.7, 3.10, 3.11, 3.13) is a mathematically well-posed problem. Indeed, inverting (3.8), we get:

$$\begin{aligned} \frac{d\xi(n)}{dn} &= \frac{d\xi(n)}{dn}|_{n=0}+U(1)^{[E_1^{-1}(n)]}, \\ \frac{d\zeta(n)}{dn} &= \frac{d\zeta(n)}{dn}|_{n=0}+V(1)^{[E_2^{-1}(n)]}. \end{aligned} \tag{3.14}$$

This provides a symbolic representation of the possible shape for the scaling exponents, depending on the fixed points of the composition law. As an example, we treat below some tractable examples, where the fixed points are simple. Since we are interested in the case where no coupling exist from the field ϕ onto ψ (passive scalar limit), we see that this imposes $c=\tau=0$. These values are adopted from now on.

4. Passive scalar advected by a Gaussian field. We consider here the case where $\mathbf{u}$ is a Gaussian velocity field, with $2n$-th order structure functions scaling like:

$$\left\langle(\delta u_\ell)^{2n}\right\rangle \equiv \left\langle(u(x+\ell)-u(x))^2\right\rangle \propto \ell^{\zeta(2n)}. \tag{4.1}$$

Because $\mathbf{u}$ is Gaussian, it follows the simple "normal" scaling property:

$$\zeta(2n)=n\zeta(2). \tag{4.2}$$

We are interested in the scaling properties of a passive scalar, advected by such a velocity field. Specifically, we are looking for the scaling exponents $\xi(2n)$ of the $2n^{th}$ order structure function of temperature increments defined as:

$$<(\theta(x+\ell)-\theta(x))^{2n}>\sim \ell^{\xi(2n)}. \tag{4.3}$$

[1]Physically, these fixed points correspond to the limit beyond which the moments of the distribution in ϕ or ψ are divergent, see [6].

In the Kraichnan model [10] of the eq.(1.1) where $\mathbf{u}$ is delta correlated in time, a number of exact results have been recently established. Kraichnan [10] and Gawedzki and Kupianen [11] have shown the second order scaling exponent $\xi(2)$ is related to the exponent $\zeta(2)$ of the second order structure function of the velocity field by:

$$\xi(2) = 2 - \zeta(2). \tag{4.4}$$

For scaling exponents of higher order structure functions, both perturbative and non-perturbative results are available. In the limit where $\zeta(2) \to 0$, it was shown by Bernard et al. [12] that the scaling exponents follow the "anomalous" law:

$$\xi(2n) = n\xi(2) - \frac{2n(n-1)}{D+2}\zeta(2), \quad \zeta(2) \to 0, \tag{4.5}$$

where D is the space dimension. In the limit where $\zeta(2) \to 2$, it was shown by [13, 14] that the anomalous correction vanish, so that:

$$\xi(2n) = n\xi(2), \quad \zeta(2) \to 2. \tag{4.6}$$

Chertkov *et al.* [15] used an expansion in the inverse $1/D$ of the space dimensionality to obtain:

$$\xi(2n) = n\xi(2) - \frac{2n(n-1)}{D}\zeta(2), \quad D^{-1} \to 0, \tag{4.7}$$

which is consistent with the result (4.5) obtained by Bernard *et al.* [12].

On the other hand, Kraichnan [10] used the linear ansatz for the molecular-diffusion term and found a non-perturbative result:

$$\xi(2n) = \frac{1}{2}\sqrt{4nD\xi(2) + (D - \xi(2))^2} - \frac{1}{2}(D - \xi(2)). \tag{4.8}$$

Note that this result is inconsistent with the $\zeta(2)$ expansion of Bernard *et al* [12]: using (4.4), one finds that in the limit $\zeta(2) \to 0$, the exponents $\xi(2n) - n\xi(2)$ given by (4.8) tend to a finite limit which contradicts (4.5).

We finally mention the exact non-perturbative calculation of the anomalous correction to fourth scaling exponent $\rho_4 = 2\xi(2) - \xi(4)$ obtained by Benzi *et al.* [16] in the case of a random shell-model for passive scalar. This discrete toy model mimics most of the properties of the equation (1.1) [17]. In such model, the properties of the scaling exponent are independent of ultraviolet and infrared boundary conditions, except in the case where $\xi(2) \to 0$ if the diffusive scale is kept fixed. In this last case, diffusive effects destroy the inertial range scaling, and $\rho_4 \to 0$, like in the models of [12] or [15], in the limit $\zeta(2) \to 2$. Chertkov et al also shows $\rho_4 = 4(2 - \zeta(2))/D$ [18].

4.1. The scaling exponents. Let us now examine those predictions within the framework of the scale invariant theory obtained in Section 3. In the stationary situation and in the non-diffusive limit, the equation (1.1) is invariant under the transformation $\mathcal{S}_{h,h'}(\lambda)$. This shows that, for the range of scale in which the non-diffusive limit is meaningful (the so-called inertial range), the structure functions obey power laws (4.3), where the scaling exponents $\xi(n)$ may be a priori computed by using scale invariant theory.

In the Gaussian case, all the moments are convergent. The convergence of all moments implies that the nonzero fixed points of the composition law for p must be $-\infty$ or ∞. That is possible only if $\tau = \eta = \chi = \rho = 0$. Since the scaling exponent of the Gaussian field is linear, that is $\zeta(2n) = n\zeta(2)$, $V(n)$ has to be identically zero according to (3.8). With $c = h = 0$ as imposed by the passive scalar requirement and $b = k = 0$ required by the linear scaling exponent, the relevant composition law becomes:

$$\left\{ \begin{array}{lcl} U'' & = & U + U' - aUU' \\ V'' & = & V + V' \end{array} \right. \tag{4.9}$$

and

$$\left\{ \begin{array}{lcl} n'' & = & n + n' - \alpha n n' \\ p'' & = & p + p' \end{array} \right. \tag{4.10}$$

Note that it is compatible with the requirement $V(p) = 0$. The solution of eqs. (4.9) and (4.10) can be found by iteration, as explained in Section III.D. We now introduce the parameter β via

$$\beta = 1 - aU(1). \tag{4.11}$$

The solution $U(p)$ of the equation (4.9) takes two different forms corresponding to the two following cases:

(1) If $\beta \neq 1$ and $\alpha \neq 0$, then

$$U(n) = \frac{1}{a}\left(1 - (1 - \alpha n)^{\lambda - 1}\right) \tag{4.12}$$

where $\lambda - 1 = \ln\beta / \ln(1 - \alpha n(1))$.

Using eq.(3.8) and integrating over n, we obtain:

$$\xi(2n) = 2nh_0 - \frac{1}{\alpha\lambda a}\left(1 - (1 - 2\alpha n)^{\lambda}\right), \tag{4.13}$$

where h_0 is a constant depending on $\partial_n\xi(0)$. This expression is a slight generalization of Kraichnan's formula. Indeed, setting $h_0 = 0$, $\lambda = 1/2$, $a = -2(D - \xi(2))/D\xi(2)$ and $\alpha = -2D\xi(2)/(D - \xi(2))^2$, we find exactly (4.8). The "log-normal" formula of Chertkov et al (4.7) and Bernard et al (4.5), with $\xi(2n)$ being the sum of a linear and a parabolic contribution is then obtained from (4.13) by taking the limit either $\lambda \to 2$, or the limit as follows.

(2) $\beta = 1$ and $\alpha = 0$. Then,

$$\xi(2n) - n\xi(2) = 2\frac{U(1)}{n(1)}n(n-1). \tag{4.14}$$

Note that this limit is not achieved when the parameters are "tuned" to Kraichnan's value in the limit $\zeta(2) \to 0$, $\xi(2) \to 2$. This illustrates in another way the discrepancy arises between Kraichnan's result and Bernard *et al.*'s asymptotic result.

Finally, we get two additional interesting possibilities when

(3) $\beta = 1$ and $\alpha \neq 0$

$$\begin{aligned} U(n) &= -\frac{U(1)}{\ln(1 - \alpha n(1))}\ln(1 - \alpha n), \\ \xi(2n) &= 2nh_0 + \\ &\frac{U(1)}{\alpha\ln(1 - \alpha n(1))}[2\alpha n - (1 - 2\alpha n)\ln(1 - 2\alpha n)]. \end{aligned} \tag{4.15}$$

This expression tends towards the log-normal asymptotic form when $\alpha \to 0$.

(4) $\beta \neq 1$ and $\alpha = 0$

$$U(n) = \frac{1}{a}\left(1 - \beta^{n/n(1)}\right),$$
$$\xi(2n) = 2nh_0 + \frac{n(1)}{a \ln \beta}\left(1 - \beta^{2n/n(1)}\right). \tag{4.16}$$

This is the log-Poisson law [19, 20, 21], which again tends to a log-normal in the limit $\beta \to 1$.

4.2. Discussion. Our result obviously generalizes the results of [10, 12, 15], using four arbitrary constants. Because we use symmetry arguments, we cannot obtain any quantitative information about the value of the anomalous correction. Only "matching" with exact computations (starting from the original equations) can provide values for these constants. In the present case, we have four arbitrary parameters: a, $U(1)$, α and h_0. These parameters can be arbitrary function of $\zeta(2)$ which need to be determined. The exact result (4.4) enables to write an equation linking our four parameters to $\zeta(2)$. Besides that, the asymptotic expressions of Bernard *et al.* and Chertkov *et al.* only provide asymptotic expressions for $U(1)$, and the condition that α must tend to zero as $D \to \infty$ or $\zeta(2) \to 0$. This obviously does not constrain our parameters enough, so we shall not bother to try to exhibit one out of the many solutions which fits all the requirement. It would be interesting, as one gets more and more precise numerical result as those obtained by Frisch *et al.* [22], to try to derive the four parameters for any given value of $\zeta(2)$, and see if they are enough to fit the whole collection of exponents at this value of $\zeta(2)$. This would provide a test of the present theory.

5. Passive scalar advected by turbulent flow. In this Section, we consider the case where the velocity field $\mathbf{u}$ follows the Navier-Stokes equations:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla P}{\rho} + \nu \Delta \mathbf{v} + \mathbf{f}. \tag{5.1}$$

Here ρ is fluid density, $\mathbf{f}$ force, P pressure and ν viscosity. Eq. (1.1) indicates that the passive scalar is transported by the turbulence. Eq. (5.1) shows that the dynamics of the velocity field is not influenced by the passive scalars.

In the stationary unforced inviscid limit ($\nu, \kappa \to 0$), the set of equations (1.1,5.1) is invariant under the transformation $\mathcal{S}_{h,h'}(\lambda)$. This shows that for the range of scale in which the inviscid unforced limit is meaningful (the so-called inertial range) the structure functions of velocity and passive scalar obey power laws (1.2). Dimensional considerations lead to the prediction $\xi(n) = \zeta(n) = n/3$. However, the experiments and numerical simulations do not support the predictions. These exponents are observed to deviate from the linear laws and the deviation is higher for the temperature than for the velocity. It is believed that the deviations are due to intermittency of velocity and scalar dissipations.

Many models have been proposed to reproduce the scaling exponents of velocity structures. Among them, She and Leveque's model [19] gives an excellent prediction of experimental results without any free parameters. Later, this model is shown to be equivalent to assume that velocity increments follow a log-Poisson statistics [20, 23]. The log-Poisson statistics converges to a log-normal statistics under a suitable limit. In fact, a recent careful analysis [24] of turbulent data using wavelet transform indicated that the turbulence is so close to log-normal that distinguishing between log-Poisson and log-normal statistics is impossible, given the available amount of turbulent data. Therefore, we shall assume in the following discussion that the turbulent velocity is log-Poisson, and then take the suitable limit towards log-normal statistics to examine whether the passive scalar statistics can help in making the distinction.

As an extension of log-Poisson statistics of velocity fields, Cao and Chen [25] assumed a bivariable log-Poisson distribution for joint moments of velocity dissipation ϵ_ℓ and scalar dissipation N_ℓ. This assumption implies that passive scalars also obey log-Poisson statistics, which is in remarkable agreement with the results of numerical simulations. Ruiz, Baudet and Ciliberto's recent experiments [26] also provide strong evidence to support log-Poisson statistics of passive scalars. He *et al.* [21] propose a hierarchical relation for the joint moments to explain why passive scalars must be log-Poisson. In the following, we explore the possibility of log-Poisson statistics of passive scalars within the formalism developed in Section 3.

5.1. The scaling exponents. A remarkable simplification occurs from dynamical considerations: since velocity dynamics is not influenced by the scalar dynamics, that is, the composition law for V must be independent of U, one must choose $c = h = \tau = \eta = 0$. Moreover, to select the log-Poisson statistics for velocity increments [4] requires $k = 0$ and $\beta = \kappa = 0$. Therefore, the relevant composition law in this case is:

$$\left\{ \begin{array}{lcl} U'' & = & U' + U - aU'U - dV'V - j[U'V + UV'] \\ V'' & = & V' + V - bV'V \end{array} \right. \tag{5.2}$$

and

$$\left\{ \begin{array}{lcl} n'' & = & n' + n - \alpha n'n - \delta p'p - \gamma[n'p + np'] \\ p'' & = & p' + p \end{array} \right. \tag{5.3}$$

A rescaling of U and V by an arbitrary factor t, and of a, b, d and j by $1/t$ leaves (5.2) unchanged. At the same time, a rescaling of n and p by $1/t$, and a rescaling of α, δ, γ by a factor t leaves (5.3) unchanged. Since the scaling exponents depends only on the derivative of U and V with respect to n and p, we may assume without loss of generality that, say, $p(1) = 1$.

The equation (5.2) has one stable fixed point

$$U_+ = \frac{b-j}{ab}, \quad V_* = \frac{1}{b} \tag{5.4}$$

and one unstable fixed point

$$U_- = -\frac{j}{ab}, \quad V_* = \frac{1}{b}. \tag{5.5}$$

We solve (5.2) and (5.3) and get an implicit representation of the solution via,

$$\left\{ \begin{array}{lcl} U[n(m)] & = & (U_+ - U_-)(1 - \beta_1^m) + V_*(1 - \beta_2^m) \\ V[p(m)] & = & V_*(1 - \beta_2^m) \end{array} \right. \tag{5.6}$$

where

$$\begin{aligned} \beta_1 &= 1 - \frac{U(1)}{U_+ - U_-} \\ \beta_2 &= 1 - \frac{V(1)}{V_*}. \end{aligned} \tag{5.7}$$

and

$$
\text{(5.8)} \qquad \begin{cases} n(m) &= \left(n(1) - \dfrac{a_2}{1-a_1}\right) \dfrac{1-a_1^m}{1-a_1} + \dfrac{a_2 m}{1-a_1} \\ p(m) &= m \end{cases}
$$

where

$$
\begin{aligned}
a_1 &= 1 - \alpha n(1) - \gamma, \\
a_2 &= -\delta - \gamma n(1).
\end{aligned} \tag{5.9}
$$

Here, $n(1)$ is a free parameters. To avoid any discontinuity while ensuring $n(0) = 0$, we shall only assume that $n(0)$ tends to a_2 when a_1 tends to 0. Note here that the log-normal shape for the velocity increments is obtained by letting $\beta_2 \to 1$, in which case the function U simply becomes:

$$
U(m) = (U_+ - U_-)(1 - \beta_1^m) + m. \tag{5.10}
$$

To get $U(n)$ as a closed form, one needs to invert the function $n(m)$ given by (5.8). This is analytically possible only for special values of the parameters, otherwise one can only use the implicit representation (5.6) and (5.8). The special values where an explicit formulation of $U(n)$ is possible are:

1) when $a_2 = 0$: we get:

$$
\begin{aligned}
U(n) &= (U_+ - U_-)\left(1 - (1 + n/a_3)^{\lambda_1 - 1}\right) \\
&\quad + V_* \left(1 - (1 + n/a_3)^{\lambda_2 - 1}\right), \\
V(p) &= V_* \left(1 - \beta_2^p\right), \\
\xi(n) &= n h_\xi + \frac{a_3 (U_+ - U_-)}{\lambda_1} \left(1 - (1 + n/a_3)^{\lambda_1}\right) \\
&\quad + \frac{V_* a_3}{\lambda_2} \left(1 - (1 + n/a_3)^{\lambda_2 - 1}\right), \\
\zeta(p) &= p h_\zeta - \frac{V_*}{\ln \beta_2} \left(1 - \beta_2^p\right),
\end{aligned} \tag{5.11}
$$

where $\lambda_i - 1 = \ln \beta_i / \ln(a_1)$, for $i = 1, 2$, $a_3 = n(1)/(a_1 - 1)$ and h_ξ and h_ζ are integration constants. The shape for ζ is the usual log-Poisson shape. The shape for ξ is a superposition of log-levy function. In the log-normal limit for the velocity increments, one gets a different shape:

$$
\begin{aligned}
\xi(n) &= n h_\xi + \frac{a_3 (U_+ - U_-)}{\lambda_1} \left(1 - (1 + n/a_3)^{\lambda_1}\right) \\
&\quad + a_3 \left((1 + n/a_3) \ln(1 + n/a_3) - n/a_3\right), \\
\zeta(p) &= p h_\zeta + \frac{p^2}{2}.
\end{aligned} \tag{5.12}
$$

So the log-normal limit is not really simpler than the general solution.

2) when $a_1 = 0$, we get:

$$
\begin{aligned}
U(n) &= (U_+ - U_-)\left(1 - \beta_1^{n/n(1)}\right) + V_* \left(1 - \beta_2)^{n/n(1)}\right), \\
V(p) &= V_* \left(1 - \beta_2^p\right), \\
\xi(n) &= n h_\xi - \frac{n(1)(U_+ - U_-)}{\ln \beta_1} \left(1 - \beta_1^{n/n(1)}\right) \\
&\quad - \frac{n(1) V_*}{\ln \beta_2} \left(1 - \beta_2^{n/n(1)}\right), \\
\zeta(p) &= p h_\zeta - \frac{V_*}{\ln \beta_2} \left(1 - \beta_2^p\right),
\end{aligned} \tag{5.13}
$$

where h_ζ and h_ξ are integration constants. In this case, they are scaling exponents connected with the properties of the rarest events (the most intermittent structures), whose connexion with finite size effects is discussed in [5]. The physical interpretation of the parameters appearing in (5.13) is obtained by taking the limit $n \to \infty$ or $p \to \infty$, when β_1 and β_2 are less than one. We find that

$$\xi(n) \sim nh_\xi - n(1)(U_+ - U_-)/\ln\beta_1 - n(1)V_*/\ln\beta_2$$

and

$$\zeta(p) \propto ph_\zeta - V_*/\ln\beta_2.$$

This suggests interpreting $-n(1)(U_+ - U_-)/\ln\beta_1 - n(1)V_*/\ln\beta_2$ and $-V_*/\ln\beta_2$ as the codimensions of the most intermittent structures of the passive scalar and of the velocity. At this stage, we can make an interesting remark: when $n(1)$ is small (this is a case of weak coupling between passive scalar and velocity), the last term in (5.13c) can be neglected. In this case, the scaling exponents of the passive scalar have a log-Poisson shape. We show later by fitting the experimental data that this coupling coefficient is indeed small. This therefore explains the observation in [26]. Also, we note that the scaling exponents (5.13) reassemble a log-Poisson distribution with two "atoms", whose existence was conjectured in [23]. Finally, it is interesting to consider the limit where the velocity increments are log-normal. In that case, we get:

$$\begin{aligned} \xi(n) &= nh_\xi - \frac{n(1)(U_+ - U_-)}{\ln\beta_1}\left(1 - \beta_1^{n/n(1)}\right) + \frac{V(1)n^2}{2n(1)}, \\ \zeta(p) &= ph_\zeta + \frac{V(1)p^2}{2}, \end{aligned} \tag{5.14}$$

In that case, the scaling exponents are the superposition of an exponential log-Poisson shape, and of a parabolic log-normal shape. At large positive n (if $\beta_1 < 1$), the parabolic shape dominates the scaling behavior, and one does not observe the simple linear asymptotics existing in the log-Poisson case. This observation could be used as a tool to discriminate between log-Poisson and log-normal velocity increments, by considering the statistics of passive scalars. This is an indirect check, but in some experimental settings, this might actually prove easier than the direct check.

In all other cases, we may only obtain an implicit representation of the scaling exponent by using the chain rule $d\xi/dm = d\xi/dn \cdot dn/dm$.

5.2. Value of the constants. We obtained different explicit and implicit forms of scaling exponents depending on several constants. In the most general (implicit) case, we have 9 free parameters. In the simplest case (number 2), we have 6 free parameters. We need the same number of constraints to compute these parameters. For h_ζ, β_1 and $V_*/\ln\beta_2$, we can use the values obtained by [19], which provide a good fit of the velocity increments: $h_\zeta = 1/9$, $\beta_2 = 0.88$ and $V_*/\ln\beta_2 = 2$. For the remaining parameters, we need to use values measured for the passive scalar. Given the available precision of the measures, it is not really realistic to consider more than $n = 5$ or $n = 6$ scaling exponents for computing the values of the constants. It is then only possible to determine the constants in the log-Poisson case, where only three remaining parameters need to be determined. This simplification enables a partial determination of the parameters. If the most intermittent structure have respectively vortex filaments and temperature sheets, as observed in numerical simulations [27, 28], their codimensions are approximatively one and two. By considering that the most intense energy dissipation events are only driven by the mean transfer term ($< (\delta v)^3 >$) and using the K62 refined similarity hypothesis [29], She and Léveque [19] obtain $h_\zeta = 1/9$, and

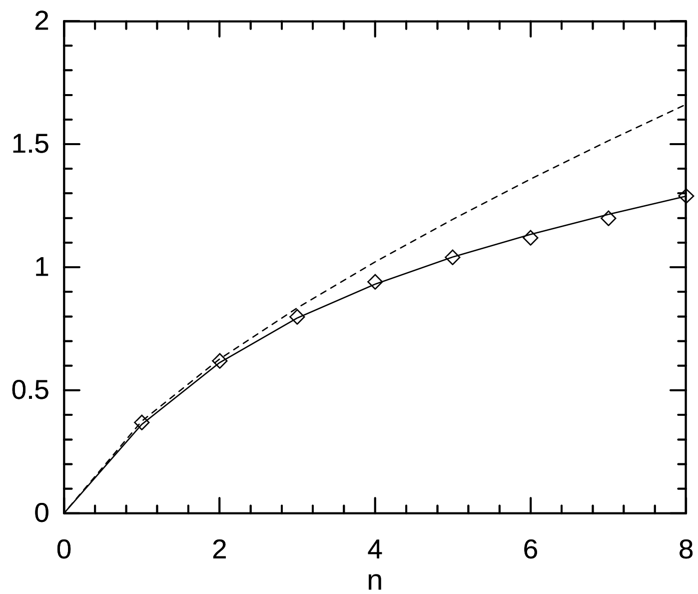

FIG. 5.1. *Comparison between the experimental data of Ruiz et al. (diamonds) and formula (49) with: A set of parameter derived by physical arguments copying the argumentation of She and Leveque [19] for the velocity increments. This gives* $(h_\xi, \beta_1, n(1), U_+ - U_-, h_\zeta, V_*, \beta_2) = (1/9, 0.985, 0.1, 0.015, 1/9, 0.26, 0.88)$ *(dash line); A set of parameters derived by a best fit to the data* $(h_\xi, \beta_1, n(1), U_+ - U_-, h_\zeta, V_*, \beta_2) = (0.06, 0.915, 0.25, 0.12, 1/9, 0.26, 0.88)$ *(solid line).*

then, using $\zeta(3) = 1$, $\beta_2^3 = 2/3$. A same kind of argument also leads to $h_\xi = 1/9$. Indeed, if we suppose that the most intense temperature dissipation events are only driven by $< (\delta v)^3 >$ and $< \delta v(\delta\theta)^2 >$, we obtain $N^\infty \sim \Theta^2 < (\delta v)^3 >^{1/3} / < \delta v(\delta\Theta)^2 >$, where Θ is some constant temperature. Since $< (\delta v)^3 > \sim \ell$ and $< \delta v(\delta\theta)^2 > \sim \ell$, we obtain $N^\infty \sim \ell^{-2/3}$. Then, by using the refined Obukhov-Corrsin hypothesis ([30]), we get $h_\xi = 1/9 + h_N/2 + 1/3 = 1/9$. We may then use these values and vary $n(1)$ and β_1 to obtain the best fit with the data. The result is given in Figure 1 for $n(1) = 0.1$ and $\beta_1 = 0.86$. It is not really good, especially at large n where the formula gives a much larger value than the data. This is because the value $h_\xi = 1/9$ is too high. A three parameter fit of the data gives $h_\xi = 0.06$, $n(1) = 0.25$, $\beta_1 = 0.7$, resulting in a codimension of 0.84 for the passive scalar rarest events. The result is displayed in Figure 1.

Certainly, more work is needed to understand the meaning of the parameters. A recent work of Dubrulle [5] [31] suggest that some of them might be linked with finite size effects. Work is in progress in that direction.

6. Conclusion and discussion. In this paper, we develop a formalism leading to the equations followed by the scaling exponents of two-dimensional scale invariant random fields. It is the extension of [1] on one-dimensional random fields. We do not make any assumption on (statistical) relations of the two random fields. For example, they may be independent of each other or anisotropic. Thus, the two random fields possess independent scaling exponents and they may have different limit velocities. This situation is completely different from the relativity theory of motion, where the limit velocity is the same one (light velocity) in each direction. We begin with the structures of transformation group, and then compute the transformation matrix. The composition law, derived from the similarity transformation, indicates the equations relating the different scaling exponents. Based on the scaling exponent equations, the classification of all exponents is possible by the computation of the fixed points of the transformation. This is performed in a simple case, the passive scalar advected by i) a random Gaussian velocity field; and ii) a turbulent velocity field.

In the Gaussian case, we show that the passive scalar increments follow a log-levy distribution generalizing the solution of Kraichnan. This distribution tends in an appropriate asymptotic limit, towards a log-normal distribution. The anomalous correction to the scaling exponent can be matched with the Kraichnan solution [10] or with recent perturbative expansions of Chertkov *et al.* [15] and Bernard *et al.* [12] via an appropriate tuning of the parameters, which cannot be predicted by our simple symmetry arguments.

In the turbulent case, we show that when the velocity increments follow log-Poisson statistics, the passive scalar increments follow various different statistics, the simplest one being close to log-Poisson. It is in fact a log-Poisson distribution with two "atoms", thereby providing an explicit example of a random system with more than one atom whose existence was conjectured in [23]. This result provides an explanation of the experimental findings of [26] about the temperature increments.

We note that scale invariance may play a role in many other physical problems. It would then be interesting to consider other values of the parameters to describe other physical situations such as scaling laws in magneto-hydrodynamics.

Acknowledgments G. W. He expresses his gratitude to Lachieze-Rey for discussion around the relativity of motion. He also would like to thank SAP for its hospitality. B. Dubrulle acknowledges useful discussions with M. Vergassola.

Appendix A: Derivation of Similarity Transformations

We show that the similarity transformation compatible with the following postulates are the solution (3.3) by generalizing the method [1] to the two-dimensional case:

Posulate 1. The log-coordinates are invariant under global translation.

Posulate 2: Among all imaginable reference fields, there exist a continuous class of equivalent reference fields built on scale invariant fields.

Posulate 3: the group of similarity transformations is commutative.

Postulate 2 implies that the composition of two similarity transformations with respective parameters (U, V) and (U', V') should be a similarity transformation, associated to a third parameter noted (U'', V''). Composition of two transformations in (3.2) then gives

$$\begin{aligned}
A'' &= AA' + BD' + \Gamma G'UU' \\
B'' &= AB' + BE' + \Gamma H'UV' \\
\Gamma''U'' &= A\Gamma'U' + B\Gamma'V' + \Gamma\Gamma'U \\
D'' &= DA' + ED' + \Gamma G'VU' \\
E'' &= DB' + EE' + \Gamma H'VV' \\
\Gamma''V'' &= D\Gamma'U' + E\Gamma'V' + \Gamma\Gamma'V \\
G''U'' &= GA'U + HD'V + \Gamma G'U' \\
H''V'' &= GB'U + HE'V + \Gamma H'V' \\
\Gamma'' &= G\Gamma'UU' + H\Gamma'VV' + \Gamma\Gamma'
\end{aligned} \tag{6.1}$$

We now use the Postulate 3 about commutativity. We first apply the transformation (U', V') and then the transformation (U, V). By comparison with (6.1), we obtain several equations. For example, equations (6.1.1) and (6.1.5) give

$$\begin{aligned}
BD' - B'D &= UU'(\Gamma'G - \Gamma G'), \\
DB' - D'B &= VV'(\Gamma'H - \Gamma H').
\end{aligned} \tag{6.2}$$

Factorizing (6.2) leads to

$$D = cB, G = -h\Gamma, H = -k\Gamma, \tag{6.3}$$

where c, h, k are adjustable parameters.

Substituting (6.3) into (6.1.2) and (6.1.4), and then comparing their coefficients, we obtain

$$h = ck.$$

The same technique applied to (6.1.3) and (6.1.6) gives

$$\begin{array}{rcl} A & = & (1 - aU - jV)\Gamma \\ B & = & -(jU + dV)\Gamma \\ E & = & (1 - cdU - bV)\Gamma \end{array} \tag{6.4}$$

where a, b, d, j are adjustable parameters. (6.3) and (6.4) are solutions of equations (6.1) provided the parameters follow

$$-k + ad + jb = j^2 + d^2c.$$

This completes our proof.

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