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Modeling of Aircraft Unsteady Aerodynamic Characteristics

Part 1 - Postulated Models

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SUMMARY

A short theoretical study of aircraft aerodynamic model equations with unsteady effects is presented. The aerodynamic forces and moments are expressed in terms of indicial functions or internal state variables. The first representation leads to aircraft integro-differential equations of motion; the second preserves the state-space form of the model equations. The formulation of unsteady aerodynamics is applied in two examples. The first example deals with a one-degree-of-freedom harmonic motion about one of the aircraft body axes. In the second example, the equations for longitudinal short-period motion are developed. In these examples, only linear aerodynamic terms are considered. The indicial functions are postulated as simple exponentials and the internal state variables are governed by linear, time-invariant, first-order differential equations. It is shown that both approaches to the modeling of unsteady aerodynamics lead to identical models. In the case of aircraft longitudinal short-period motion, potential identifiability problems, if an estimation of aerodynamic parameters from flight data were to be attempted, are briefly mentioned .

SYMBOLS

A_j	coefficient in Fourier series, $j = 0, 1, 2, \ldots$
a, a_1, a_2	parameters in indicial function
B	parameter defined in table I
B_j	coefficient in Fourier series, $j = 0, 1, 2, \ldots$
b_1, b_2	parameters in indicial function, 1/sec
C	parameter defined in table I
C_a	general aerodynamic force and moment coefficient
$\boldsymbol{C}_{a_\xi}(t)$	vector of indicial functions
$\boldsymbol{C}_{a_\xi}(\infty)$	vector of aerodynamic derivatives
C_L	lift coefficient
C_l, C_m, C_n	rolling-, pitching-, and yawing-moment coefficient
c	parameter in indicial function
$\bar{c}$	mean aerodynamic chord, m
$\boldsymbol{F}_{a_\xi}(t)$	vector of deficiency functions
I	integral defined by eq. (46)
I_Y	moment of inertia about lateral axis, kg-m^2
K_0, K_1, K_2	transfer function coefficients
k	reduced frequency, $k = \dfrac{\omega \ell}{V}$
k_α	parameter defined by eq. (20a)
ℓ	characteristic length, m
m	mass, kg
p, q, r	roll rate, pitch rate, and yaw rate, rad/sec or deg/sec
S	wing area, m^2
s	parameter in Laplace transform
T_1, T_α	time lag, sec
t	time, sec
$\boldsymbol{u}$	vector of input variables
V	airspeed, m/sec
$\boldsymbol{x}$	vector of state variables
x_α	state variable in eq. (43)
α	angle of attack, rad or deg
β	sideslip angle, rad

δ	control surface deflection, rad or deg
ξ	vector of state and input variables
η	internal state variable
λ	variable in characteristic polynomial
ρ	air density, kg/m^3
τ	time delay, sec
τ_1	nondimensional time constant, $\dfrac{V}{b_1\ell}$
ϕ, ψ	roll and yaw angle, rad
ω	angular frequency, 1/sec

Subscript:

| A | amplitude |
| 0 | initial value |

Matrix exponent:

| T | transpose matrix |

Derivatives of aerodynamic coefficients C_a where the index $a = L,\ l,\ m,$ or n

$$C_{a_p} = \frac{\partial C_a}{\partial \frac{p\ell}{V}} \qquad C_{a_q} = \frac{\partial C_a}{\partial \frac{q\ell}{V}} \qquad C_{a_{\dot{q}}} = \frac{\partial C_a}{\partial \frac{\dot{q}\ell^2}{V^2}} \qquad C_{a_r} = \frac{\partial C_a}{\partial \frac{r\ell}{V}}$$

$$C_{a_\alpha} = \frac{\partial C_a}{\partial \alpha} \qquad C_{a_{\dot{\alpha}}} = \frac{\partial C_a}{\partial \frac{\dot{\alpha}\ell}{V}} \qquad C_{a_\beta} = \frac{\partial C_a}{\partial \beta} \qquad C_{a_\delta} = \frac{\partial C_a}{\partial \delta} \qquad C_{a_\eta} = \frac{\partial C_a}{\partial \eta}$$

Derivatives $M_{\alpha,\dot{\alpha},q,\delta}$ and $Z_{\alpha,q,\delta}$ defined in table I.

INTRODUCTION

One of the basic problems of flight dynamics is the formulation of aerodynamic forces and moments acting on an aircraft in arbitrary motion. For many years the aerodynamic functions were approximated by linear expressions leading to a concept of stability and control derivatives. The addition of nonlinear terms, expressing, for example, changes in stability derivatives with the angle of attack, extended the range of flight conditions to high-angle-of-attack regions and/or high-amplitude maneuvers. In both approaches, using either linear or nonlinear aerodynamics, it is assumed that the parameters appearing in polynomial or spline approximations are time invariant. However, this assumption was many times questioned based on studies of unsteady aerodynamics which go back to the twenties.

A fundamental study of unsteady lift on an airfoil due to abrupt changes in the angle of attack was made by Wagner in reference 1. This work was extended by Theodorsen to computing forces and moments on an oscillating airfoil, whereas Küssner and Sears studied the lift on an airfoil as it penetrates a sharp-edge or harmonically-varying gust, respectively (see reference 2). One of the first investigations of unsteady aerodynamic effects on aircraft motion was made by R. T. Jones in reference 3. He studied the effect of the wing wake on the lift of the horizontal tail. A more general formulation of linear unsteady aerodynamics in the aircraft longitudinal equations in terms of indicial functions was introduced by Tobak in reference 4. Later, in reference 5, Tobak and Schiff expressed the aerodynamic forces and moments as functionals of the state variables. This very general approach includes linear unsteady aerodynamics as a special case. A different approach to unsteady aerodynamics in aircraft equations of motion was introduced by Goman and his colleagues in reference 6. They used additional state variables, which they called internal state variables, in the functional relationships for the aerodynamic forces and moments.

Despite the advancements of theoretical works, only a limited number of attempts were made to estimate aerodynamic parameters from experimental data and to demonstrate the importance of unsteady terms in aircraft equations of motion. In reference 7, a procedure for the estimation

of aerodynamic forces and moments from flight data was proposed. It starts with the estimation of stability and control derivatives. Then, the resulting residuals of the response variables are used for the estimation of unsteady terms. Reference 8 addresses identifiability problems for parameters in integro-differential equations. Examples of estimated indicial functions from simulated and flight data are given. Fourier functional analysis for unsteady aerodynamic modeling was applied to wind tunnel data of a triangular wing and a fighter aircraft in references 9 and 10, respectively. It was shown that this modeling method was successful in computing the aerodynamic responses to large-amplitude harmonic and ramp-type motions. Finally, a concept of internal state variables for expressing unsteady aerodynamics was applied to wind tunnel oscillatory data and flight data in references 6 and 11.

The purpose of this report is to summarize the approaches of references 5 and 6 to the formulation of aerodynamic model equations suitable for parameter estimation from experimental data. The report starts with expressing aerodynamic forces and moments in terms of indicial functions and internal state variables. Then, two examples of aerodynamic models for aircraft in small-amplitude motion are given. A discussion of these examples is completed by concluding remarks.

AERODYNAMIC CHARACTERISTICS IN TERMS OF INDICIAL FUNCTIONS

Using the results of reference 5, aircraft aerodynamic characteristics can be formulated as

$$C_a(t) = C_a(0) + \int_0^t \boldsymbol{C}_{a_\xi}\left(t - \tau; \boldsymbol{\xi}(\tau)\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}\tau} \boldsymbol{\xi}(\tau)\mathrm{d}\tau \qquad (1)$$

where
    $C_a(t)$ is a coefficient of aerodynamic force or moment,
    ξ is a vector of aircraft state and input variables upon which the
        coefficient C_a depends,
    $\boldsymbol{C}_{a_\xi}(t)$ is a vector of indicial functions whose elements are the responses
        in C_a to unit steps in ξ, and
    $C_a(0)$ is the value of the coefficient at initial steady-state conditions.

The indicial responses, $\boldsymbol{C}_{a_\xi}$, are functions of elapsed time $(t - \tau)$ and are continuous single-valued functions of $\xi(t)$. The indicial functions approach steady-state values with increasing values of the argument $(t - \tau)$. To indicate this property, each indicial function can be expressed as

$$C_{a_{\xi,j}}\left(t - \tau; \xi(\tau)\right) = C_{a_{\xi,j}}\left(\infty; \xi(\tau)\right) - F_{a_{\xi,j}}\left(t - \tau; \xi(\tau)\right) \tag{2}$$

where

$C_{a_{\xi,j}}\left(\infty; \xi(\tau)\right)$ is the rate of change of the coefficient C_a with ξ_j, in steady flow, evaluated at the instantaneous value of ξ_j with the remaining variables ξ fixed at the instantaneous values $\xi(\tau)$ and

the function $F_{a_{\xi,j}}$ is called the deficiency function. This function approaches zero for $(t - \tau) \to \infty$.

When equations (2) are substituted into equation (1), the terms involving the steady-state parameters can be integrated and equation (1) becomes

$$C_a(t) = C_a\left(\infty; \xi(t)\right) - \int_0^t \boldsymbol{F}_{a_\xi}\left(t - \tau; \xi(t)\right)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi(\tau) \mathrm{d}\tau \tag{3}$$

where

$C_a\left(\infty; \xi(t)\right)$ is the total aerodynamic coefficient that would correspond to steady flow with ξ fixed at the instantaneous values $\xi(t)$, and

$\boldsymbol{F}_{a_\xi}$ is a vector of deficiency functions

If the indicial response $\boldsymbol{C}_{a_\xi}$ is only a function of elapsed time, equations (1) and (3) are simplified as

$$C_a(t) = C_a(0) + \int_0^t \boldsymbol{C}_{a_\xi}(t - \tau)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi(\tau) \mathrm{d}\tau$$

$$\tag{4}$$

$$= C_a(0) + \boldsymbol{C}_{a_\xi}(\infty)^{\mathrm{T}} \xi(\tau) - \int_0^t \boldsymbol{F}_{a_\xi}(t - \tau)^{\mathrm{T}} \frac{\mathrm{d}}{\mathrm{d}\tau} \xi(\tau) \mathrm{d}\tau$$

When analytical forms of deficiency functions are specified, the aerodynamic model based on equations (3) or (4) can be used in the aircraft equations of motion for stability and control studies involving either linear or nonlinear aerodynamics. The resulting equations of motion will be represented by a set of integro-differential equations.

FORMULATION OF AERODYNAMIC FUNCTIONS USING INTERNAL STATE VARIABLES

When the indicial functions are used in aircraft aerodynamic model equations, it is not clear, either from theory or experiment, what analytical form these functions should have. After postulating models for indicial or deficiency functions, questions about the physical meaning of terms in these models may still be asked. In order to avoid, at least partially, these questions, a concept of internal state variables for modeling of unsteady aerodynamics was proposed in reference 6. This approach retains the state-space formulation of aircraft dynamics, that is

$$\dot{x} = f\big(x(t), u(t)\big); \quad x(0) = x_0 \tag{5}$$

by augmenting the aircraft states with the additional state variable $\eta(t)$. Then, the aerodynamic coefficients are formulated as

$$C_a(t) = C_a\big(\xi(t), \eta(t)\big) \tag{6}$$

where

$$\dot{\eta} = g\big(\eta(t), \xi(t), \dot{\xi}(t)\big) \tag{7}$$

and

$$\xi(t) = \left[x(t)^{\mathrm{T}} \ \ u(t)^{\mathrm{T}} \right]^{\mathrm{T}}$$

An example of equation (7) for a study of aircraft longitudinal dynamics is given in reference 6. Here, the internal state variable represents the vortex burst point location along the chord of a triangular wing. This location is described as

$$T_1\dot{\eta} + \eta = \eta_0\big(\alpha - T_\alpha\dot{\alpha}\big); \quad |\eta| \le 1 \tag{8}$$

where
    η_0 is the vortex burst point location under steady conditions,
    T_1 is the time constant in the vortical flow development, and
    T_α is the time lag in the same process caused by the angle-of-attack rate
      of change.
The experimentally-obtained effect of the angle of attack and pitch rate on vortex point location is taken from reference 12 and is plotted in figure 1. The resulting curves were obtained by flow visualization on a delta wing

undergoing static test and forced pitching oscillations at two reduced frequencies. The effect of pitch rate is seen by comparing the dynamic vortex burst point location with the static point location.

EXAMPLES

The following two examples demonstrate the formulation of aerodynamic equations and equations of motion with unsteady aerodynamics. In these examples, only small-amplitude motion will be considered, thus leading to a system of linear equations. In the first example, aircraft one-degree-of-freedom (one d.o.f.) oscillatory motion about each of the three body axes is considered. The second example deals with short-period longitudinal motion. In both examples, the formulation of unsteady aerodynamics using indicial functions and internal state variables is considered.

Harmonic Oscillatory Motion:

In the development of aerodynamic models of an aircraft performing a one d.o.f. oscillatory motion, an approach using indicial functions and internal state variables will be considered. For the oscillatory motion in pitch, the functional relationships for the lift and pitching moments are

$$C_L(t) = C_L\big(\alpha(t), q(t)\big)$$
$$C_m(t) = C_m\big(\alpha(t), q(t)\big)$$

Applying equation (4), the lift coefficient can be expressed as

$$C_L(t) = C_L(0) + \int_0^t C_{L_\alpha}(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau + \frac{\ell}{V}\int_0^t C_{L_q}(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}q(\tau)\mathrm{d}\tau$$

$$= C_L(0) + C_{L_\alpha}(\infty)\alpha(t) - \int_0^t F_\alpha(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau \qquad (9)$$

$$+ \frac{\ell}{V}C_{L_q}(\infty)q(t) - \frac{\ell}{V}\int_0^t F_q(t-\tau)\frac{d}{d\tau}q(\tau)\,d\tau$$

Similar expressions can be written for $C_m(t)$. Neglecting the effect of $\dot{q}(t)$ on the lift and taking into account only the increments with respect to steady conditions, equation (9) is simplified as

$$C_L(t) = C_{L_\alpha}(\infty)\alpha(t) - \int_0^t F_\alpha(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau + \frac{\ell}{V}C_{L_q}(\infty)q(t) \qquad (10)$$

For obtaining a model with a limited number of parameters, the indicial function is assumed to be in the form of a simple exponential

$$C_{L_\alpha}(t) = a\left(1 - e^{-b_1 t}\right) + c \qquad (11)$$

Because

$$\lim_{t\to\infty} C_{L_\alpha}(t) = a + c = C_{L_\alpha}(\infty),$$

equation (11) can also be written as

$$C_{L_\alpha}(t) = C_{L_\alpha}(\infty) - ae^{-b_1 t} \qquad (11a)$$

After substituting (11a) into (10) and applying the Laplace transform to equation (10), the expression for the lift coefficient is obtained as

$$C_L(s) = \left(C_{L_\alpha} - \frac{as}{s + b_1} + \frac{\ell}{V}C_{L_q}s\right)\alpha(s) \qquad (12)$$

where
    $q(s)$ was replaced by $s\alpha(s)$ and, for simplicity, $C_{L_\alpha} \equiv C_{L_\alpha}(\infty)$ and
      $C_{L_q} \equiv C_{L_q}(\infty)$.

Using a complex expression for harmonic changes in $\alpha(t)$, that is

$$\alpha(t) = \alpha_A e^{i\omega t} = \alpha_A\left(\cos(\omega t) + i\sin(\omega t)\right),$$

and replacing s by $i\omega$, the steady-state solution to equation (12) is

$$\begin{aligned} C_L(t) = &\left(C_{L_\alpha} - a\frac{\omega^2}{b_1^2 + \omega^2}\right)\alpha_A \sin(\omega t) \\[2mm] &+ \left(\frac{\ell}{V}C_{L_q} - a\frac{b_1}{b_1^2 + \omega^2}\right)\alpha_A \omega \cos(\omega t) \end{aligned} \qquad (13)$$

The introduction of reduced frequency

$$k = \frac{\omega \ell}{V}$$

and nondimensional time constant

$$\tau_1 = \frac{V}{b_1 \ell}$$

yields

$$C_L(t) = \overline{C_{L_\alpha}} \alpha_A \sin(\omega t) + \overline{C_{L_q}} \alpha_A k \cos(\omega t) \qquad (14)$$

where

$$\overline{C_{L_\alpha}} = C_{L_\alpha}(\infty) - a \frac{\tau_1^2 k^2}{1 + \tau_1^2 k^2}$$

$$\overline{C_{L_q}} = C_{L_q}(\infty) - a \frac{\tau_1}{1 + \tau_1^2 k^2} \qquad (15)$$

Similarly, the steady-state solution for the pitching-moment coefficient will be

$$C_m(t) = \overline{C_{m_\alpha}} \alpha_A \sin(\omega t) + \overline{C_{m_q}} \alpha_A k \cos(\omega t) \qquad (16)$$

where

$$\overline{C_{m_\alpha}} = C_{m_\alpha}(\infty) - a \frac{\tau_1^2 k^2}{1 + \tau_1^2 k^2}$$

$$\overline{C_{m_q}} = C_{m_q}(\infty) - a \frac{\tau_1}{1 + \tau_1^2 k^2} \qquad (17)$$

The parameters a and τ_1^2 in equation (17) have, in general, different values from those in equation (15).

When the internal state variable is used in formulating the unsteady aerodynamic effect, the development of a model for the lift coefficient starts with the equations

$$C_L(t) = C_L\big(\alpha(t), q(t), \eta(t)\big) \qquad (18)$$

$$T_1 \dot{\eta} + \eta = \eta_0\big(\alpha - T_\alpha \dot{\alpha}\big) \qquad (8)$$

For small perturbations, both equations can be linearized around a steady-state condition. Then, the linearized equations (18) and (8) will have the form

$$C_L(t) = C_{L_\alpha}\alpha(t) + \frac{\ell}{V}C_{L_q}q(t) + C_{L_\eta}\eta(t) \tag{19}$$

$$T_1\dot{\eta} + \eta = -\left(T_1 + T_\alpha\right)\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}\dot{\alpha} \tag{20}$$

Applying the Laplace transform, these equations will be changed as

$$C_L(s) = C_{L_\alpha}\alpha(s) + \frac{\ell}{V}C_{L_q}q(s) + C_{L_\eta}\eta(s) \tag{21}$$

$$\left(T_1s + 1\right)\eta(s) = -\left(T_1 + T_\alpha\right)\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}s\alpha(s) \tag{22}$$

When equation (22) is substituted into (21) and $q(s)$ is replaced by $s\alpha(s)$,

$$C_L(s) = C_{L_\alpha}\alpha(s) - \frac{T_1 + T_\alpha}{1 + T_1s}\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}C_{L_\eta}s\alpha(s) + \frac{\ell}{V}C_{L_q}s\alpha(s) \tag{23}$$

Finally, introducing

$$a = \frac{T_1 + T_\alpha}{T_1}\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}C_{L_\eta} \quad\text{and}\quad b_1 = T_1^{-1}$$

equation (23) will have the same form as equation (12). The preceding developments indicate that, for the indicial function given by equation (11a) and the internal variable given by equation (22), the model

$$C_L(t) = C_{L_\alpha}\alpha(t) - a\int_0^t e^{-b_1(t-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau + \frac{\ell}{V}C_{L_q}q(t)$$

is equivalent to the model

$$C_L(t) = C_{L_\alpha}\alpha(t) + \frac{\ell}{V}C_{L_q}q(t) + C_{L_\eta}\eta(t)$$

$$T_1\dot{\eta} + \eta = -\left(T_1 + T_\alpha\right)\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}\dot{\alpha}$$

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Models for an aircraft performing one d.o.f. oscillatory motion in roll and yaw can be developed in a similar way to that for the pitching oscillations. The rolling-moment coefficient is a function of the roll angle and rolling velocity

$$C_l(t) = C_l\big(\phi(t), p(t)\big) \tag{24}$$

where the roll angle is related to the sideslip angle by the equation

$$\beta = \phi \sin(\alpha) \tag{25}$$

For the indicial function

$$C_{l_\beta}(t) = C_{l_\beta}(\infty) - ae^{-b_l t}$$

the rolling-moment coefficient can be formulated as

$$C_l(t) = C_{l_\beta}(\infty)\beta(t) - a\int_0^t e^{-b_l(t-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\beta(\tau)\mathrm{d}\tau + \frac{\ell}{V}C_{l_p}(\infty)\,p(t) \tag{26}$$

which leads to its steady response

$$C_l(t) = \overline{C_{l_\beta}}\phi_A\sin(\omega t) + \overline{C_{l_p}}\phi_A k\cos(\omega t) \tag{27}$$

where

$$\overline{C_{l_\beta}} = C_{l_\beta}(\infty)\sin(\alpha) - a\frac{\tau_1^2 k^2}{1 + \tau_1^2 k^2}\sin(\alpha)$$

$$\overline{C_{l_p}} = C_{l_p}(\infty) - a\frac{\tau_1}{1 + \tau_1^2 k^2}\sin(\alpha) \tag{28}$$

In the yawing oscillatory motion, the yawing-moment coefficient is a function of the yaw angle and its rate

$$C_n(t) = C_n\big(\psi(t), r(t)\big) \tag{29}$$

and the yaw angle is related to the sideslip angle as

$$\beta = -\psi\cos(\alpha) \tag{30}$$

The yawing-moment equation takes the form

$$C_n(t) = C_{n_\beta}(\infty)\beta(t) - a\int_0^t e^{-b_I(t-\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\beta(\tau)\mathrm{d}\tau + \frac{\ell}{V}C_{n_r}(\infty)r(t) \tag{31}$$

and its steady response the form

$$C_n(t) = \overline{C_{n_\beta}}\psi_A\sin(\omega t) + \overline{C_{n_r}}\psi_A k\cos(\omega t) \tag{32}$$

where

$$\overline{C_{n_\beta}} = C_{n_\beta}(\infty)\cos(\alpha) - a\frac{\tau_1^2 k^2}{1+\tau_1^2 k^2}\cos(\alpha)$$

$$\overline{C_{n_r}} = C_{n_r}(\infty) + a\frac{\tau_1}{1+\tau_1^2 k^2}\cos(\alpha) \tag{33}$$

For the interpretation of measured aerodynamic forces and moments in the forced-oscillation experiment, the model for an increment in the lift without any unsteady effect is usually postulated as (see reference 13)

$$C_L(t) = C_{L_\alpha}\alpha(t) + \frac{\ell}{V}\left(C_{L_{\dot\alpha}}\dot\alpha(t) + C_{L_q}q(t)\right) + \left(\frac{\ell}{V}\right)^2 C_{L_{\dot q}}\dot q(t) \tag{34}$$

The unsteady version of the preceding equation will have to include two indicial functions, $C_{L_\alpha}(t)$ and $C_{L_q}(t)$. Then the lift coefficient will be formulated as

$$C_L(t) = C_{L_\alpha}\alpha(t) - \int_0^t F_\alpha(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau$$

$$+ \frac{\ell}{V}C_{L_q}q(t) - \frac{\ell}{V}\int_0^t F_q(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}q(\tau)\mathrm{d}\tau \tag{35}$$

In both cases, the steady-state solution is given by equation (14) where, for the neglected unsteady aerodynamics,

$$\overline{C_{L_\alpha}} = C_{L_\alpha} - k^2 C_{L_{\dot q}}$$

$$\overline{C_{L_q}} = C_{L_q} + C_{L_{\dot\alpha}} \tag{36}$$

and, for the deficiency functions specified as

$$F_\alpha = a_1 e^{-b_1 t} \quad \text{and} \quad F_q = a_2 e^{-b_2 t},$$

$$\overline{C_{L_\alpha}} = C_{L_\alpha} - \left(a_1 \frac{\tau_1^2}{1 + \tau_1^2 k^2} - a_2 \frac{\tau_2}{1 + \tau_2^2 k^2} \right) k^2$$

$$\overline{C_{L_q}} = C_{L_q} - \left(a_1 \frac{\tau_1}{1 + \tau_1^2 k^2} + a_2 \frac{\tau_2^2 k^2}{1 + \tau_2^2 k^2} \right)$$

(37)

From a comparison of equations (36) and (37), it can be concluded that the expressions in the parentheses are the unsteady counterparts to the derivatives $C_{L_{\dot q}}$ and $C_{L_{\dot\alpha}}$. For large values of τ and small values of k, the expressions in equations (37) can be simplified to those in equation (15). Similar comparisons can be made for the remaining aerodynamic coefficients.

Short-Period Longitudinal Motion:

The airplane short-period longitudinal motion can be described by the equations

$$\dot\alpha = q + \frac{\rho V S}{2 m} C_Z\big(\alpha(t), q(t), \delta(t) \big)$$

$$\dot q = \frac{\rho V^2 S \bar c}{2 I_Y} C_m\big(\alpha(t), q(t), \delta(t) \big)$$

(38)

In the following analysis, it will be assumed that the linear approximation to the aerodynamics contains only one unsteady term represented by the indicial function

$$C_{m_\alpha}(t) = C_{m_\alpha}(\infty) - F_\alpha(t)$$

(39)

Using simplified notation for the steady aerodynamic terms, the aerodynamic equations in (38) will have the form

$$C_Z(t) = C_{Z_\alpha} \alpha(t) + \frac{\ell}{V} C_{Z_q} q(t) + C_{Z_\delta} \delta(t)$$

$$C_m(t) = C_{m_\alpha} \alpha(t) - \int_0^t F_\alpha(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} \alpha(\tau)\mathrm{d}\tau \tag{40}$$

$$+ \frac{\ell}{V} C_{m_q} q(t) + C_{m_\delta} \delta(t)$$

Specifically, for

$$C_{m_\alpha}(t) = a\left(1 - e^{-b_1 t}\right) + c$$

the pitching-moment coefficient takes the form

$$C_m(t) = C_{m_\alpha} \alpha(t) - a\int_0^t e^{-b_1(t-\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} \alpha(\tau)\mathrm{d}\tau + \frac{\ell}{V} C_{m_q} q(t) + C_{m_\delta} \delta(t)$$

$$\tag{41}$$

$$= c\alpha(t) + ab_1\int_0^t e^{-b_1\tau} \alpha(t-\tau)\mathrm{d}\tau + \frac{\ell}{V} C_{m_q} q(t) + C_{m_\delta} \delta(t)$$

where

$$C_{m_\alpha} = a + c$$

Substituting (41) into (38) and introducing dimensional parameters, the equations of motion can be written as

$$\dot{\alpha} = Z_\alpha \alpha + Z_q q + Z_\delta \delta$$

$$\dot{q} = C\alpha + B\int_0^t e^{-b_1\tau} \alpha(t-\tau)\mathrm{d}\tau + M_q q + M_\delta \delta \tag{42}$$

where the parameters in these equations are defined in table I.

Introducing a new state variable

$$x_\alpha = \int_0^t e^{-b_1\tau} \alpha(t-\tau)\mathrm{d}\tau$$

and the corresponding state equation for this variable

$$\dot{x}_\alpha = \alpha - b_1 x_\alpha$$

equations (42) can be expressed in state-space form as

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{x}_\alpha \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_q & 0 \\ C & M_q & B \\ 1 & 0 & -b_1 \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ x_\alpha \end{bmatrix} + \begin{bmatrix} Z_\delta \\ M_\delta \\ 0 \end{bmatrix} \delta
\tag{43}
$$

The characteristic polynomial of these equations has the form

$$
\Delta = \lambda^3 + K_2\lambda^2 + K_1\lambda + K_0
$$

where

$$
\begin{aligned}
K_2 &= -Z_\alpha - M_q + b_1 \\
K_1 &= Z_\alpha\left(M_q - b_1\right) - b_1 M_q - C Z_q \\
K_0 &= b_1\left(Z_\alpha M_q - Z_q M_\alpha\right)
\end{aligned}
\tag{44}
$$

The state equations of the system under consideration can also be obtained by using the internal state variable defined by equation (20) as

$$
\begin{aligned}
\dot{\eta} &= -\left(\frac{T_1 + T_\alpha}{T_1}\right)\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}\dot{\alpha} - T_1^{-1}\eta \\
&= k_\alpha \dot{\alpha} - T_1^{-1}\eta
\end{aligned}
\tag{20a}
$$

When equation (20a) is combined with the equations of motion, the complete set of state equations is

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} Z_\alpha & Z_q & 0 \\ M_\alpha & M_q & M_\eta \\ k_\alpha Z_\alpha & k_\alpha Z_q & -T_1^{-1} \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \eta \end{bmatrix} + \begin{bmatrix} Z_\delta \\ M_\delta \\ k_\alpha Z_\delta \end{bmatrix} \delta
\tag{45}
$$

where M_α and M_η are also explained in table I. After formulating the characteristic polynomial, it is found that its coefficients are equal to those defined by equations (44) for

$$
a = \left(\frac{T_1 + T_\alpha}{T_1}\right)\frac{\mathrm{d}\eta_0}{\mathrm{d}\alpha}C_{m_\eta}
$$

and

$$
b_1 = T_1^{-1}
$$

It can, therefore, be concluded that equation (43) and (45) represent the same dynamical system. As in the previous example, the system description using either indicial functions or internal state variables can be identical for specific forms of indicial functions and equations for internal state variables.

The preceding development shows that the introduction of one indicial function of the form specified by equation (39) into the aerodynamic model equations results in the increase of the order of the characteristic polynomial from two (no unsteady aerodynamics) to three. Any further addition of indicial functions into equation (42) means an additional increase in the order of the characteristic polynomial by one. From a simple observation of equation (43) or (45), it is also evident that it is not possible to estimate all the parameters in these equations from the measurements of $\alpha(t)$, $q(t)$, and $\delta(t)$. To assure parameter identifiability, equation (43) would have to be transformed into a canonical form proposed in reference 14.

In stability and control analysis where no unsteady aerodynamics is considered, the pitching-moment coefficient is formulated as

$$C_m = C_{m_\alpha}\alpha + \frac{\ell}{V}\left(C_{m_{\dot\alpha}}\dot\alpha + C_{m_q}q \right) + C_{m_\delta}\delta$$

It is expected, therefore, that the integral in equation (40)

$$I = \int_0^t F_\alpha(t-\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha(\tau)\mathrm{d}\tau \tag{46}$$

should be a counterpart of the term $\left(\dfrac{\ell\dot\alpha}{V}\right)C_{m_{\dot\alpha}}$. The reduction of this integral to the $\dot\alpha$-term can be demonstrated by approximating $\alpha(t)$ by a Fourier series

$$\alpha(t) = A_0 + \left(A_1 - iB_1 \right)e^{i\omega t} + \left(A_2 - iB_2 \right)e^{i2\omega t} + \dots$$

which leads to

$$\dot\alpha(t) = i\omega\left(A_1 - iB_1 \right)e^{i\omega t} + i2\omega\left(A_2 - iB_2 \right)e^{i2\omega t} + \dots \tag{47}$$

Substituting (47) into (46) results in

$$I = i\omega\left(A_1 - iB_1\right)e^{i\omega t}\int_0^t F_\alpha(\tau)\,e^{-i\omega\tau}\mathrm{d}\tau$$

$$+ i2\omega\left(A_2 - iB_2\right)e^{i2\omega t}\int_0^t F_\alpha(\tau)\,e^{-i2\omega\tau}\mathrm{d}\tau + \ldots \tag{48}$$

The exponential functions in (48) can be further expanded in exponential series

$$e^{i\omega\tau} = 1 + i\omega\tau + \frac{1}{2}(i\omega\tau)^2 + \ldots$$

$$e^{i2\omega\tau} = 1 + i2\omega\tau + 2(i\omega\tau)^2 + \ldots$$

$$\vdots$$

In order to maintain the approximation of the integral to the first order in frequency, it is sufficient to consider only the first terms in the exponential series. Then, all the integrals in (48) will be the same and equation (46) can be simplified as

$$I = \dot{\alpha}\int_0^t F_\alpha(\tau)\mathrm{d}\tau \tag{49}$$

As a result of this simplification, the counterpart of $C_{m_{\dot{\alpha}}}$ is proportional to the area of the deficiency function. A similar conclusion is stated in reference 4 for simple harmonic motion of an aircraft.

For a demonstration of aircraft longitudinal motion with and without unsteady aerodynamic terms, equations (42) and their simplified version

$$\dot{\alpha} = Z_\alpha\alpha + Z_q q + Z_\delta\delta$$
$$\dot{q} = M_\alpha\alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q q + M_\delta\delta \tag{50}$$

were used. Aircraft characteristics and flight conditions are summarized in table II. The unsteady parameter b_1 was selected as $b_1 = 1\left(\sec^{-1}\right)$ which corresponds to the nondimensional time constant $\tau_1 = 51.3$. The parameter a was evaluated from the relationship between the derivative $C_{m_{\dot{\alpha}}}$ and the area of the deficiency function

$$C_{m_{\dot{\alpha}}} = -\frac{V}{\ell} a \int_0^{\infty} e^{-b_I \tau} d\tau$$

as $a = 0.05$. Because $C_{m_\alpha} = a + c$, the parameter $c = -0.23$.

In table III, the computed damping coefficients and frequencies of motion from equations (42) and (50) are presented. The values of these parameters indicate that the replacement of the terms $C_{m_\alpha} \alpha$ and $C_{m_{\dot{\alpha}}} \dot{\alpha}$ by the indicial function $C_{m_\alpha}(t)$ has a negligible effect on the damping coefficient and only a small effect on the frequency. Figure 2 shows the computed time histories $\alpha(t)$ and $q(t)$ for the given input $\delta(t)$. As could be expected from the results in table III, the output variables for both cases differ only slightly. Small differences in $\alpha(t)$ and $q(t)$ might indicate possible problems when estimation of unsteady parameters from flight data is attempted.

CONCLUDING REMARKS

A short theoretical study of aircraft aerodynamic model equations with unsteady effects is presented. First, the aerodynamic forces and moments are expressed in terms of indicial functions. This formulation can be modified by including steady values of aerodynamic coefficients, corresponding to instantaneous values of state and input variables, and the so-called deficiency functions. A deficiency function defines the difference between the indicial function and its steady value. When the concept of indicial or deficiency functions is used, the resulting aircraft model is represented by a set of integro-differential equations. In the second approach to the modeling of unsteady aerodynamics, the so-called internal state variables were used. These variables are additional states upon which the aerodynamic coefficient depends. Modeling based on internal state variables preserves the state-space representation of the aircraft equations of motion.

The formulation of unsteady aerodynamics is applied in two examples. In these examples, only linear aerodynamics are considered thus limiting the application to aircraft small-amplitude motion around trim conditions. In order to further simplify the aerodynamic model

equations, the indicial functions are postulated in a simple exponential form and the internal state variables are governed by linear, time-invariant, first-order differential equations.

In the first example, a one-degree-of-freedom harmonic motion about one of the aircraft body axes is considered. In the second example, a longitudinal short-period motion is studied. In both examples, it is shown that the formulation using either indicial functions or internal state variables leads to identical models. Further, it is shown that the unsteady terms in the models are the unsteady counterparts of the aerodynamic acceleration derivatives. From an observation of the developed longitudinal equations of motion, it is evident that it will be impossible to estimate all aerodynamic parameters from measured input/output data. In addition, a simple numerical example of the short-period motion of a fighter aircraft indicates only small differences in the output time histories with the unsteady effects being either included or ignored. These small differences might create further problems when estimation of unsteady parameters from flight data is attempted.

REFERENCES

1. Wagner, H.: Über die Enstehung des dynamishen Auftriebs von Tragflügeln. Z. angew. Math. U. Mech. 5, 1925, pp. 17-35.
2. Garick, I. E.: Nonsteady Wing Characteristics. In.: Aerodynamic Components of Aircraft at High Speeds. editors A. F. Donovan and H. R. Lawrence, Princeton University Press, 1957, pp. 658-793.
3. Jones, Robert T. and Fehlner, Leo F.: Transient Effects of the Wing Wake on the Horizontal Tail. NACA TN No. 771, 1940.
4. Tobak, Murray: On the Use of the Indicial Function Concept in the Analysis of Unsteady Motions of Wings and Wing-Tail Combinations. NACA Report 1188, 1954.
5. Tobak, Murray and Schiff, Lewis B.: On the Formulation of the Aerodynamic Characteristics in Aircraft Dynamics. NACA TR R-456, 1976.

6. Goman, M. G.; Stolyarov, G. I.; Tyrtyshnikov, S. L.; Usoltsev, S. P.; and Khrabrov, A. N.: Mathematical Description of Aircraft Longitudinal Aerodynamic Characteristics at High Angles of Attack Accounting for Dynamic Effects of Separated Flow. TsAGI Preprint No. 9, 1990 (in Russian).

7. Mereau, P.; Hirsh, R.; Coulon, G.; and Rault, A.: Identification of Unsteady Effects in Lift Build Up. AGARD-CP-235, 1978, pp. 23-1 to 23-14.

8. Gupta, Naren K. and Iliff, Kenneth W.: Identification of Unsteady Aerodynamics and Aeroelastic Integro-Differential Systems. NASA TM 86749, 1985.

9. Chin, Suei and Lan, Edward C.: Fourier Functional Analysis for Unsteady Aerodynamic Modeling. AIAA Paper 91-2867-CP, 1991.

10. Hu, Chien-Cung and Lan, Edward C.; Unsteady Aerodynamic Models for Maneuvering Aircraft. AIAA Paper 93-3626-CP, 1993.

11. Goman, M. and Khrabrov, A.: State-Space Representation of Aerodynamic Characteristics of an Aircraft at High Angles of Attack. AIAA Paper 92-4651-CP, 1992.

12. Brandon, Jay M.: Dynamic Stall Effects and Applications to High Performance Aircraft. Special Course on Aircraft Dynamics at High Angles of Attack: Experiments and Modeling, AGARD Report No. 776, 1991, pp. 2-1 to 2-15.

13. Grafton, Sue B. and Libbey, Charles E.: Dynamic Stability Derivatives of a Twin-Jet Fighter Model for Angles of Attack from -10° to 110°. NASA TN D-6091, 1971.

14. Denery, Dallas G.: Identification of System Parameters from Input-Output Data with Application to Air Vehicles. NASA TN D-6468, 1971.

Table I. - Definition of parameters in equations (38) and (39).

$$Z_\alpha = \frac{\rho S V}{2\,m} C_{Z_\alpha} \qquad\qquad M_\alpha = \frac{\rho V^2 S \bar{c}}{2\,I_Y} C_{m_\alpha}$$

$$Z_q = 1 + \frac{\rho S \bar{c}}{4\,m} C_{Z_q} \qquad\qquad M_{\dot\alpha} = \frac{\rho V S \bar{c}^2}{4\,I_Y} C_{m_{\dot\alpha}}$$

$$Z_\delta = \frac{\rho S V}{2\,m} C_{Z_\delta} \qquad\qquad M_q = \frac{\rho V S \bar{c}^2}{4\,I_Y} C_{m_q}$$

$$C = \frac{\rho V^2 S \bar{c}}{2\,I_Y} c \qquad\qquad M_\eta = \frac{\rho V^2 S \bar{c}}{2\,I_Y} C_{m_\eta}$$

$$B = \frac{\rho V^2 S \bar{c}}{2\,I_Y} ab_1 \qquad\qquad M_\delta = \frac{\rho V^2 S \bar{c}}{2\,I_Y} C_{m_\delta}$$

Table II. - Characteristics of an advanced fighter aircraft
and flight conditions.

$\bar{c} = 3.51$ m	$C_{Z_\alpha} = -2.7$
$S = 37.16$ m^2	$C_{Z_q} = -36.$
$m = 15000$ kg	$C_{Z_\delta} = -0.83$
$I_Y = 170000$ kg-m^2	$C_{m_\alpha} = -0.18$
$\rho = 0.56$ kg/m^3	$C_{m_{\dot\alpha}} = -2.5$
$V = 90$ m/sec	$C_{m_q} = -10.$
	$C_{m_\delta} = -0.88$

Table III. - Damping coefficients and frequencies from simulations with and without unsteady effects.

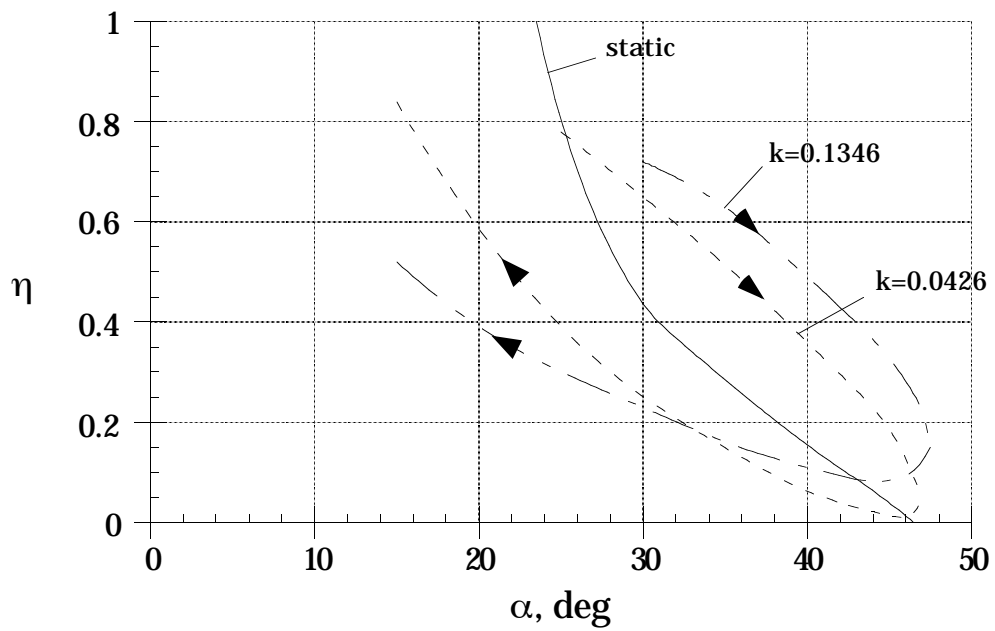|	damping coefficient	frequency
with unsteady effects	0.4859	0.6317
without unsteady effects	0.4979	0.5953

Figure 1. - Variation of internal state variable with angle of attack in static and oscillatory tests.
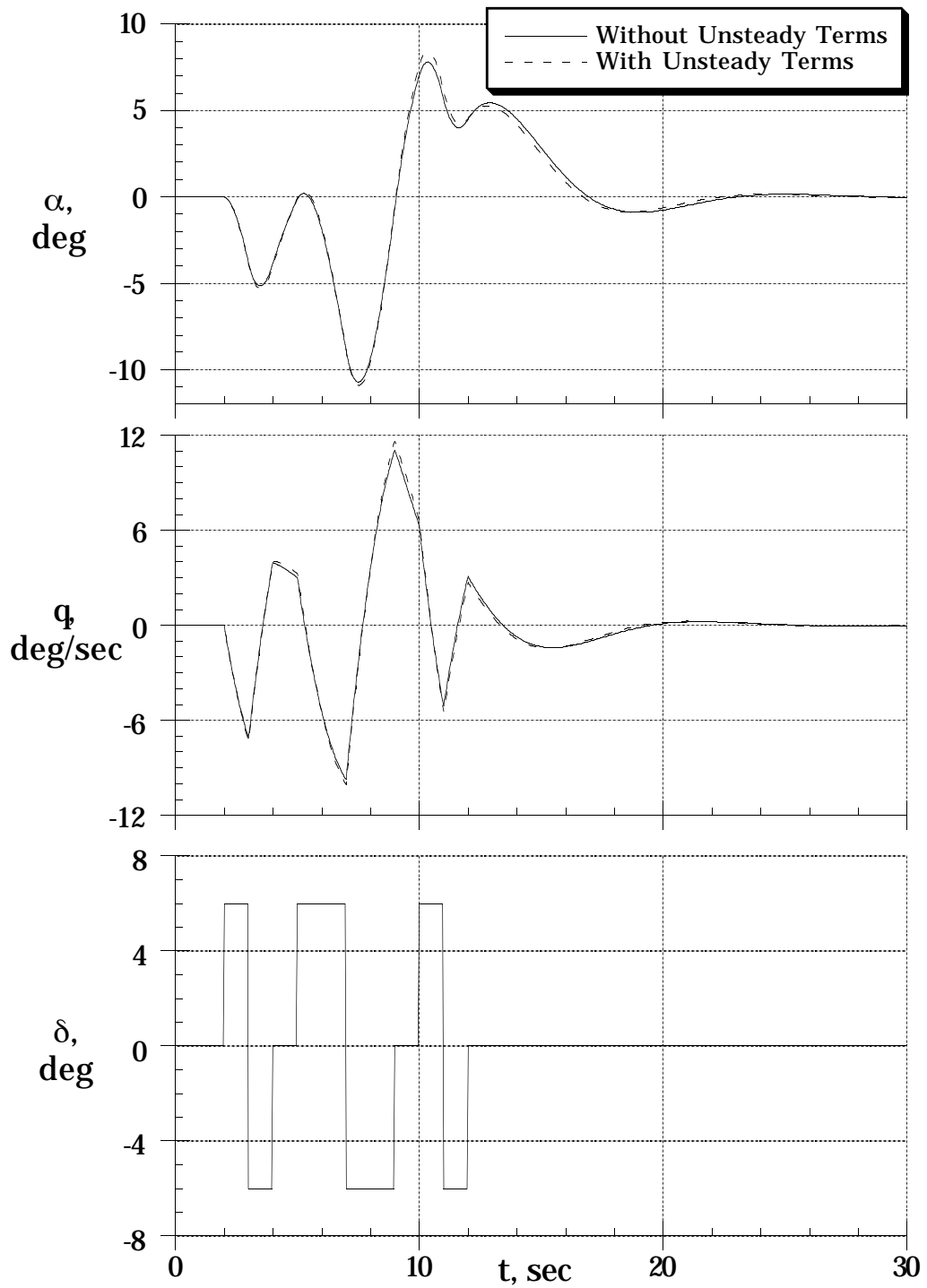
Figure 2. - Computed time histories with and without
unsteady aerodynamic terms.