1. Answer the questions below for the language $L$ defined recursively as follows:

Basis Clause: $a \in L$.
Inductive Clause: For any string $x$, if $x \in L$, then $xa, xab$ and $xba \in L$.
Extremal Clause: Nothing is in $L$ unless it is obtained by the above two clauses.

Questions:
(a) Obtain all the strings of $L$ of length 5. [4]
   
   $aaaaa$  $aaraa$  $caafa$
   $aabar$  $aafab$  $qafba$
   $qraar$  $afab$  $afaba$

(b) Describe $L$ in English. The simpler the better. [8]

   $L$ is the set of strings that start with $a$, are of odd length and have no substring $bbb$ (i.e., every $b$ is either preceded or followed immediately by $a$).

(c) Find a regular expression for $L$. [8]

   $a (aa + ab + ba)^*$
2. Let $S$ and $T$ be languages over \{a, b\}.

(a) Give a recursive definition of $(S \cap T)^*$ following the definition of Kleene star $^*$. [8]

Base case: $A \in (S \cap T)^*$

Inductive case: For all $w \in (S \cap T)^*$ and for all $x \in S \cap T$

$wx \in (S \cap T)^*$

External case: As usual.
(b) Prove by General Induction (Structural Induction) that 
\((S \cap T)^* \subseteq S^* \cap T^* \). [10]

B.S. \( A \subseteq S^* \cap T^* \)?

Since \( a \in S^* \cap T^* \) by the def. of \( S^* \cap T^* \),
\( a \in S^* \cap T^* \).

I.S. \( \forall x \in (S \cap T)^* \) \( \forall x \in S^* \cap T^* \).

\( \forall x \in (S \cap T)^* \) \( x \in S \cap T \).

\( \forall x \in (S \cap T)^* \) \( x \in S^* \cap T^* \).

\( \forall x \in (S \cap T)^* \) \( x \in S^* \cap T^* \) by the def. of \( S^* \cap T^* \).

\( \forall x \in S^* \cap T^* \).
3 (a) Construct an NFA-Λ for $a(ab+ba)^*$ following Part 1 of Kleene Theorem faithfully. Do not simplify your answer. [8]
(b) Convert the NFA-$\Lambda$ of (a) to an NFA with no $\Lambda$-transitions that accepts the same language.[8]

<table>
<thead>
<tr>
<th>State</th>
<th>Symbols</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1, 7, 9, 5, 6, 12</td>
<td>$\phi$</td>
</tr>
<tr>
<td>2</td>
<td>1, 7, 9, 5</td>
<td>8, 10</td>
</tr>
<tr>
<td>3</td>
<td>1, 7, 9, 5</td>
<td>8, 10</td>
</tr>
<tr>
<td>4</td>
<td>5, 7, 9, 5</td>
<td>8, 10</td>
</tr>
<tr>
<td>5</td>
<td>3, 7, 9, 5</td>
<td>$\phi$</td>
</tr>
<tr>
<td>6</td>
<td>$\phi$</td>
<td>8, 10</td>
</tr>
<tr>
<td>7</td>
<td>$\phi$</td>
<td>8, 10</td>
</tr>
<tr>
<td>8</td>
<td>3, 4, 5, 6, 12</td>
<td>$\phi$</td>
</tr>
<tr>
<td>9</td>
<td>$\phi$</td>
<td>3, 4, 5, 6, 11</td>
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<tr>
<td>10</td>
<td>3, 4, 5, 6, 12</td>
<td>$\phi$</td>
</tr>
<tr>
<td>11</td>
<td>5, 7, 9, 5</td>
<td>8, 10</td>
</tr>
<tr>
<td>12</td>
<td>5, 7, 9, 5</td>
<td>8, 10</td>
</tr>
</tbody>
</table>
4. Simplify the following regular expressions:

(a) \( a(a^* + a) + a^* \). [8]

\[ a^* \]

(b) \( (a + (b + ba)^* + baa)^* \). [8]

\[ (a + b)^* \]

5. Let \( S \) and \( T \) be sets of states of an NFA-\( A \). **Prove or disprove** that if \( \Lambda(S) \subseteq \Lambda(T) \), then \( S \subseteq T \). [10]

Let \( S = \{ q, r \} \) and \( \delta(q, A) = \{ q, r \} \).

Then \( \Lambda(S) = \{ q, r \} \) and \( \Lambda(T) = \{ q, r \} \).

Hence \( \Lambda(S) \subseteq \Lambda(T) \) but \( S \not\subseteq T \).
6. Which of the following statements are true and which are false? [20]

(a) For an NFA-$\Lambda$, $\delta^*(q, \Lambda) = \{q\}$.  

\[ \mathbb{F} \]

(b) $(xy)^r = x^r y^r$ for strings $x$ and $y$, where $x^r$ denotes the reversal of $x$.

\[ \mathbb{F} \]

(c) For an NFA, $\delta^*(q, xa) = \delta(\delta^*(q, x), a)$, where $x$ is a string and $a$ is a symbol.

\[ \mathbb{F} \]

\[ \bigcup_{p \in \delta^*(q, x)} \delta(p, a) \]

(d) A language is regular if and only if it is accepted by some DFA.

\[ \mathbb{T} \]

(e) $(a + b)^*a(a + b)^*a(a + b)^*$ is a regular expression corresponding to the language of strings with exactly two $a$'s.

\[ \text{at least two} \]

\[ \mathbb{F} \]

(f) $(a + ab)^*$ corresponds to the language of the strings over $\{a, b\}$ that have no substring $bb$.

\[ \text{Can not start with b in addition,} \]

\[ \text{"ba" has no bb but not in the language.} \]

\[ \mathbb{F} \]

(g) $aaababa$ is a string in the language corresponding to $(a + ab)^*$.

\[ \mathbb{T} \]

(h) For a set of states $S$ of an NFA-$\Lambda$, $\Lambda(\Lambda(S)) = \Lambda(S)$.

\[ \mathbb{T} \]

(i) For a language $L$, $(L^*)^+ = (L^+)^*$.  

\[ L^* \subseteq (L^+)^* \subseteq L^*. \quad (L^*)^+ = L^* L^* L^* L^* \cdots = L^* \]

\[ \mathbb{T} \]

(j) A string $w$ is accepted by an NFA if and only if $\delta^*(q_0, w) \subseteq A$, where $q_0$ is its initial state and $A$ is its set of accepting states.

\[ \mathbb{F} \]

\[ \delta^*(q_0, w) \cap A \neq \phi. \]